1) **Infinite Horizon Economy with Durables, Money, and Taxes (Total 40 points)**

Consider a small open economy. Assume that the representative household can hold its financial wealth in foreign bonds with gross real interest rate \( R \). The household derives utility from consumption of a durable good and a non-durable good, and from real money holdings. The household chooses to save by investing in the durable good, foreign bonds, and money to maximize its lifetime utility

\[
\sum_{t=0}^{\infty} \beta^t \left\{ \eta \ln C_t + (1-\eta) \ln D_t + \ln \frac{M_{t+1}}{p_t} \right\}
\]

subject to the budget constraint

\[
B_{t+1} + \frac{M_{t+1}}{p_t} = RB_t + \frac{M_t}{p_t} + Y_t - q_t C^d_t - C_t + \tau_t
\]

and to the law of accumulation of the durable stock

\[
D_t = (1-\delta)D_{t-1} + C^d_t
\]

where in period \( s \)

- \( B_t \) is the stock of foreign bonds;
- \( M_t \) is the stock of nominal money holdings;
- \( D_t \) is the stock of durables;
- \( C_t \) is consumption of the non-durable;
- \( C^d_t \) is investment in the durable;
- \( Y_t \) is the real income of the household;
- \( q_t \) is the relative price of the durable in terms of the non-durable;
- \( p_t \) is the general price level (i.e., the price of the non-durable in terms of money);
- \( \tau_t \) is the lump-sum transfer operated by the government;
- \( \eta \) and \( \delta \) are parameters between zero and one.

i. a) (5 points) Momentarily take \( q_t \) as exogenously determined in the world market. Derive the Euler conditions for the consumption of the durable and non-durable good.

b) (5 points) Derive the marginal rate of substitution between the two goods as a function of the user cost of durables. Carefully comment on the relationship you obtain.

ii. (10 points) Now assume that money is injected by the government in the economy by means of a transfer as follows

\[
\frac{M_{t+1} - M_t}{p_t} = \tau_t
\]

Moreover, real income is constant over time and equals \( Y \) in each period. Consider a steady state where the money growth rate \( \mu \) is constant. Derive and write down all the
conditions that define the steady state equilibrium (make opportune assumptions on the parameters if needed). Discuss neutrality and super-neutrality of money in this economy.

iii. (10 points) Consider again the setup in (ii) but now assume that the government levies a proportional tax \( \lambda \) on the investment in the durable good. Thus, the modified budget constraint of the government is

\[
\frac{M_{t+1} - M_t}{p_s} + \lambda q_t C^d_t = \tau_s
\]

while the private sector budget constraint becomes

\[
B_{t+1} + \frac{M_{t+1}}{p_s} = RB_t + \frac{M_t}{p_s} + Y_t - q_t C^d_t (1 + \lambda) - C_s + \tau_s
\]

Assume that the government always keeps \( \tau_s \) fixed over time at a level \( \tau \). Analyze the effects on the steady state of a reduction in the taxation rate \( \lambda \).

iv. (10 points) Consider again the setup in (ii) but now assume that the household can purchase durables only inside the economy (i.e. the durable is non-tradable). The stock of durables in the economy is fixed at \( D \). Endogenously derive the relative price of the durable good in terms of the non-durable.

2) Implications of Euler Investment Condition under Uncertainty (Total 8 points)

Carefully discuss the relationship between the gross real interest rate \( R \) and the marginal product of capital \( \psi_{x,t} F'(K_{x,t}) \) in the standard infinite horizon economy with uncertainty studied in the first term of the course.

3) Real Interest Rates in a Markov Model (Total 26 points)

Consider a model with discrete time \( t = 0,1,\ldots \) and two possible events in every period \( s_t \in \{H,L\} \). The evolution of the state follows a Markov process. There is only one type of agent in the economy, and his/her endowment also follows a Markov process. At any time \( t \), given \( s_t \), securities \( j = 0,1,2 \) are traded. Security \( j \)'s price in terms of the time-\( t \) state-\( s_t \) consumption good is \( p^j(s_t) \), and it entitles its owner to a payoff \( x^t = (x^t_0, x^t_1(H), x^t_1(L)) \) where \( x^t_0 \) is the payoff at time \( t \), \( x^t_1(H) \) the payoff at time \( t +1 \) if \( s_{t+1} = H \) and \( x^t_1(L) \) the payoff at time \( t +1 \) if \( s_{t+1} = L \). Concretely, our three securities have payoff vectors \( x^0 = (1,0,0) \), \( x^1 = (0,1,2) \), and \( x^2 = (0,2,0) \).

i. (4 points) Suppose that whenever \( s_t = H \), prices of the three securities are \( p^0(H) = 1 \), \( p^1(H) = 1.44 \), and \( p^2(H) = 1.44 \). What are the prices of the two Arrow securities paying \((0,1,0)\) and \((0,0,1)\)? The one-period risk-free gross interest rate, \( R(H) \) is the inverse of
the price of a security with payoff vector (0,1,1). Find the price of this security when \( s_i = H \).

ii. (4 points) Suppose that whenever \( s_i = L \), prices of the three securities are \( p^0(L) = 1 \), \( p^1(L) = 1.26 \), and \( p^2(L) = 0.36 \). What are the prices of the two Arrow securities paying (0,1,0) and (0,0,1)? Again, find the inverse of risk-free gross interest rate \( R(L) \), i.e. find the price of a security with payoff vector (0,1,1) when \( s_i = L \).

iii. (7 points) Suppose that consumers in this economy have impatience parameter \( \beta = 0.9 \) and logarithmic utility. Also, assume that the prices of securities \( j = 0,1,2 \) are equilibrium prices. Propose a stochastic transition matrix for the state variable and an endowment process that is consistent with these prices. (Hint: Only one matrix will work, more than one endowment process will work.)

iv. (6 points) Suppose that at time \( t \) agents can purchase a security that will pay 1 unit of consumption at time \( t+2 \) regardless of the realizations of uncertainty at times \( t+1 \) and \( t+2 \). Give a formula for the price of this security for case in which \( s_i = H \) and for the case in which \( s_i = L \). (Clearly explain how to compute these two prices, you do not need to give numerical answers.)

v. (5 points) Suppose that at time \( t \) agents can purchase a security that will pay 1 unit of consumption at time \( t+5 \) if \( s_{1+5} = H \) regardless of the realizations of uncertainty at times \( t+1 \), \( t+2 \), \( t+3 \) and \( t+4 \). Give a formula for the price of this security for case in which \( s_i = H \) and for the case in which \( s_i = L \). (Again you do not need to give numerical answers, but you need to give a formula. Hint: The use of matrices will be very useful.)

4) Taxation in the Deterministic Growth Model (Total 26 points)
Consider a deterministic neoclassical growth model with endogenous labor supply, Cobb-Douglas technology

\[
Y_t = AK_t^\alpha N_t^{1-\alpha}, \quad \text{with } \alpha \in (0,1) \text{ and } A > 0,
\]

depreciation rate \( \delta \in (0,1) \), and utility

\[
U = \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{2} \ln C_t + \frac{1}{2} \ln(1 - N_t) \right], \quad \text{with } \beta \in (0,1).
\]

At the beginning of period zero, consumers own \( K_0 > 0 \) units of capital. (Consumers also have claim to the profits of the firm, but those are zero in equilibrium, so we ignore them.) Every period, consumers work for \( N_t \in (0,1) \) units of time, earning labor income
They rent capital to the firm, earning a before-tax capital income \( r_i K_i \), of which they must pay \( r_{i'} r_i K_i \) in taxes. They also pay a lump-sum tax \( \tau_{i'} \). After-tax income is split between consumption \( C_i \geq 0 \), which is itself taxed at the rate \( \tau_{c_i} \), and investment \( K_{i+1} - (1 - \delta) K_i \) (with \( K_{i+1} \geq 0 \)). In sum, the household's period-by-period budget constraint is given by

\[
(1 + \tau_{c_i}) C_i + K_{i+1} - (1 - \delta) K_i = w_i N_i + r_i (1 - \tau_{i'}) K_i - \tau_{i'}.
\]

The government must finance an exogenous sequence of expenditures \( \{G_i\}_{t=0}^{\infty} \) using a mix of consumption taxes, capital-income taxes, and lump-sum taxes. We assume that the government starts with zero debt, and that the present value of its expenditures must equal the present value of its tax revenues:

\[
\sum_{t=0}^{\infty} q_i^0 G_i = \sum_{t=0}^{\infty} q_i^0 \left[ \tau_{c_i} C_i + \tau_{r_i} r_i K_i + \tau_{i'} \right]
\]

where \( q_i^0 \) is the price of 1 unit of consumption at time \( t \) in terms of consumption in period 0. (Note that this is a closed economy, not a small open economy, and that therefore the gross interest rate \( q_i^0 / q_{i+1}^0 \) is endogenously determined.)

i. (5 points) Transform the sequence of period-by-period consumer budget constraints into a single intertemporal budget constraint.

ii. (5 points) Define a competitive equilibrium with taxes and government expenditures.

iii. (4 points) In the consumer's problem, take first-order conditions and derive an Euler equation and an equation that governs the consumption-vs-leisure trade-off. According to these equations, are capital-income taxes distortionary? Are lump-sum taxes distortionary? Are consumption taxes distortionary?

iv. (4 points) Suppose the tax and expenditure policy is such that, after some period \( T \), government spending, consumption tax rates and capital tax rates and become constant forever, i.e. \( (G_s, \tau_{c_s}, \tau_{s}) = (G, \tau_{c}, \tau_{b}) \) for all \( s \geq T \). In that case, the equilibrium quantities will also eventually converge to a steady state. Set up a system of three equations and three unknowns that, if solved, would yield the steady-state equilibrium quantities \( (C_s, K_s, N_s) \). (You do not need to actually solve for these quantities.)

v. (6 points) From now on, continue as if you had solved the above system and knew the values of the steady-state quantities \( (C_s, K_s, N_s) \). Describe an algorithm that you can implement to find the equilibrium quantities of consumption, capital, and labor during the transitional periods, starting with an arbitrary capital stock \( K_0 \) and with potentially varying tax rates for the first periods. (Hint: You may want describe the algorithm that
Ljungqvist and Sargent call the "Shooting" algorithm, only adapting it to an economy with endogenous labor choice.

vi. (2 points) Finally, you are given a chance to adjust lump-sum taxes in order to make the government's budget constraint hold with equality. Once you have made this adjustment, do you need to make another adjustment to ensure that the consumer's intertemporal budget constraint holds? (Your answer should be one or two sentences, you don't need to show anything analytically.)