Repurchase Agreements

or, Why Rent When You Can Buy?*

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Abstract

In a model with matching frictions, we provide conditions under which repurchase agreements (or repos) co-exist with asset sales. In a repo, the seller agrees to repurchase the asset at a later date at the agreed price. Absent matching frictions, repos have no role. Introducing pairwise meetings, we show that agents prefer to sell asset whenever they face little uncertainty regarding the future use of the asset. As agents become more uncertain of the value of holding the asset, repos become more prevalent. We show that while the total volume of repos is always increasing with the uncertainty, the total sales volume is hump-shaped. In other words, pairwise matching alone is sufficient to explain why repo markets exist and there is no need to introduce information asymmetries or other market frictions.

1. Introduction

Many assets, including cars and houses, are traded on a repo market where the seller agrees to repurchase the asset at a later date at a given price, and repo markets coexist with market for ownership (numbers for repos for a given class of assets). But this begs the questions: Why do agents engage in repos, i.e. rent the asset, rather than just buying the asset to resale it later? Also, what is the impact of the repo market on asset sales? It is natural to expect

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that repo and sales volumes co-move negatively for a given set of traders, as these might be seen as two substitute activities.

Some have argued that private information on the quality of assets can explain the existence of a repos market: The very fact that the seller is willing to repurchase the asset is a guarantee that the asset is of good quality (see, e.g. Koeppl and Chiu, 2011). However, in a dynamic setting, this result may fail to hold as there is learning on the quality of the asset. Others have argued that repos are useful in order to cover counterparty risk, but there is then no difference between repos and collateralized loans (see, e.g. Mills and Reed). Finally, others have argued that selling an asset involves different costs than conducting repos (see, e.g. Duffie 1996). In some sense, we would like to have a deeper understanding of these costs, without necessarily resorting to different fiscal treatments or the inability of some agents to own some class of assets (such as market mutual funds).

In this paper, we show that repos can co-exist with assets sales even when the quality of the asset is known and when there is no risk exposure. The essential elements are 1) agents trade in pair[1] and 2) their personal valuation of the asset is uncertain. We assume that agents receive some preference shocks on the current utility from holding the asset, which may be more or less persistent. Once the shock hits, agents meet in pair. We concentrate on properties of allocations that are in the pairwise core, so that our result does not depend on the game played in each pairwise meeting. Once agents trade, they have to wait until the next trading session (say the next day) to change their position and they can only trade with one agent. This is the extent to which the trading frictions prevents the emergence of a Walrasian market outcome.

With pairwise matching, the analysis is complicated by the fact that agents' asset holdings depend on their history of match. To simplify the analysis, we consider the case with two valuation shocks and we assume directed matching in the sense of Corbae, Temzelides and Wright (2003): Those agents who switched valuation are matched with each other. While this simplifies the analysis, it does not eliminate the distribution of asset holdings. We develop the model for random matching as well. With directed matching we show that an invariant distribution of assets has a two point support. The difference in these two points is increasing in the persistence of valuation shocks: As the probability to change valuation is increasing, agents tend to equate their asset holdings. Inversely, as the persistence increases, agents tend to hold different amount of the asset.

[1] More generally, they are unable to trade within a group which aggregate valuation is identical to the Walrasian market valuation.

Interestingly, the total volume of asset sales is directly linked with the range of the support: As the difference in asset holdings increases, the sale of asset in a match is also increasing. This is intuitive: With directed matching, agents who just switched their valuation from high to low are matched with those agents who switched valuation from low to high. Therefore, as the difference in their asset holdings grow, also does the gains from trade, so that they trade a larger amount of the asset. However, since types are more persistent, fewer agents switch types so that the total volume of sales can either increase or decrease. We show that it is hump-shaped. Similarly, as the future valuation becomes uncertain, i.e. types are not persistent, agents are unwilling to change their position through asset sales, but they are willing to engage in repos. Therefore, the total volume of repos is decreasing with persistence and it is higher than total sales volume when the uncertainty is high (or persistence is low).

This has interesting implications for the organization of the repo market. In particular our theory predict that the repos market will be thinner when there is little uncertainty about one's future preferences. This implies that monetary policy (which is operated in the repo market) will have a higher impact then, as a lower quantity of repos can affect the market. Similarly, starting from a situation where agents know their future preferences, as uncertainty is growing, so is the volume of asset sales. Therefore, more sales have to be conducted in order to move the market. If we associate "normal times" with times when agents have a good idea about their future preferences, then monetary policy should be conducted with repos. However, with uncertainty growing overly large, monetary policy will be more effective in moving market if it is conducted via asset sales/purchases.

The paper proceeds as follows. In Section 2 we describe the model. In Section 3 we define and characterize pairwise core allocations. In Section 4, we solve for the equilibrium when there is random matching, in the extreme cases when there is full persistence of the preference shocks and no persistence at all. Section 5 analyzes the case with directed matching and solve for the equilibrium distribution and volumes in general. Section 6 presents an example with Nash bargaining rather than pairwise core allocations.

2. The Model

This is a version of Koeppl, Monnet, and Temzelides (2008). Time is discrete and the horizon is infinite. Each period has two sub-periods: A trading stage, followed by a settlement stage. There is a continuum of agents. In each period, there is a measure $1/2$ of two types of agents, type h and type ℓ. The type of an agent switches randomly and with probability $1-\pi \in [1/2, 1]$ at the start of the transaction stage. The law of large numbers then guarantees that there is the same measure of each type in each period. Agents are anonymous in the trading stage and their type is private information. A clearinghouse records the transactions of an agent.

There is a long-lived asset in fixed supply A. As in Lagos and Rocheteau (2009), we associate this asset to a Lucas-tree: One unit of the asset yields one unit of some fruit in the settlement stage. Agents of type $i \in \{h, \ell\}$ derive utility $u_i(a)$ from holding a units of the asset.[2] For simplicity, we impose the following condition,

Assumption 1. *$u'_h(a) \geq u'_\ell(a)$ for all a.*

Therefore, for a given level of asset holding, the agent with the high type has a higher marginal utility than the agent with the low type. To be concise, we will refer to agents of type h as agents h and to agents of type ℓ as agents ℓ.

In the trading stage, agents can agree to trade the asset, in which case the seller transfers the assets *and* the fruits in the settlement stage. Or agents can only agree to trade the fruit of the asset: Then the seller only transfers the fruits that it yields, while he maintains ownership over the asset. We interpret this second trade as a repo trade, as the buyer surrenders the asset back to the seller once he enjoyed the benefits of holding it this period.

While the trading stage can be seen as a market, there is no market in the settlement stage. In the settlement stage, agents are endowed with a production technology for the numeraire good m. It costs them one unit of disutility to produce one unit of this good so that the numeraire good is akin to transferable utility. Agents also consume the numeraire good and derive one unit of utility for each unit they consume. If utility is transferable, the settlement stage does not generate any net utility gains. The numeraire good will be the settlement asset. In the settlement stage, agents settle the terms of the trade that were agreed upon in the previous trading stage.

[2] There are several interpretations for this formulation: Lagos and Rocheteau argue that this is the utility derived from the tree's fruit. Duffie, Garleanu and Pedersen (2009) explain that these are preferences from liquidity, hedging or other benefits that holding the assets may yield.

3. Benchmark Walrasian Market

We first consider the case where the trading stage is a Walrasian market. A repo trades at price p^r while the asset sales at price p. We consider only stationary equilibrium so that these prices are the same in each period. An agent $i = h, \ell$ with asset holdings a has a value $W_i(a)$ of holding the asset, where $W_i(a)$ is defined recursively as

$$\begin{aligned} W_i(a) &= \max_{a_i, q_i^r} u_i(a_i + q_i^r) - d + \beta E_{k|i} W_k(a_i) \\ &\quad \text{s.t. } d + pa = pa_i + p^r q_i^r \end{aligned}$$

where the agent repos q_i^r and purchases an amount a_i of the asset. Naturally the quantity of repos q_i^r does not enter in the continuation valuation but only in the momentary utility $u_i(.)$. The first order and envelope conditions yield

$$\begin{aligned} u_i'(a_i + q_i^r) + \beta E_{k|i} W_k'(a_i) &= p \\ u_i'(a_i + q_i^r) &= p^r \\ W_i'(a) &= p \end{aligned}$$

Notice that all agents value an additional unit of the asset in the same way when they enter the Walrasian market, independent of their type or of their asset holdings. There two reasons for this: First, the fact that the utility is linear in the numeraire good such that there is no wealth effect in this model and, second, the fact that agents are playing against the whole market. In the next section, we will modify the latter. For the time being, the equilibrium prices and quantities satisfy,

$$\begin{aligned} (1-\beta)p &= p^r \\ u_h'(a_h + q_h^r) &= p^r \\ u_\ell'(a_\ell + q_\ell^r) &= p^r \\ (a_h + q_h^r) + (a_\ell + q_\ell^r) &= 2A \end{aligned}$$

The first equation is a no-arbirtrage condition: Agents have to be indifferent between conducting a repo, in which case they have to pay the price p^r in terms of the numeraire good, and buying the asset at price p and reselling it in the next period at price βp. These two schemes are payoff equivalent and so should be their cost. However, then anything goes for repos, and in particular $q_h^r = q_\ell^r = 0$. In other words, in a Walrasian market, there is no difference between conducting a repos or buying and selling the asset. Therefore, absent any additional frictions, the Walrasian benchmark is not helpful to study the structure of the repo and other rental markets. In the following section, we depart from the Walrasian benchmark by assuming that agents can only meet in pair.

4. Pairwise core allocations

We now assume that each agent h is matched with exactly one agent ℓ in the trading stage. We describe the matching technology later. We consider allocations in the pairwise core. To define an allocation, we will consider a generic meeting between an agent h holding a generic amount of the asset a_h and an agent ℓ holding a generic amount of the asset a_ℓ. An allocation is a triple $\{q^s(a_h, a_\ell), q^r(a_h, a_\ell), d(a_h, a_\ell)\}$ where q^s denotes the quantity of the asset that the agent h buys from the agent ℓ (sells if negative), q^r is the quantity of the asset that the agent of h buys or repo from the agent ℓ (sells or reverse repo if negative) and d is the numeraire transfer that the agent h makes in the settlement stage to the agent ℓ (receives if negative). We only focus on stationary and symmetric allocations. An allocation is feasible if

$$\begin{aligned} q^s(a_h, a_\ell) &\in [-a_h, a_\ell] \\ q^r(a_h, a_\ell) + q^s(a_h, a_\ell) &\in [-a_h, a_\ell] \end{aligned}$$

We will denote by $(\mathbf{q^s}, \mathbf{q^r}, \mathbf{d})$ the feasible allocations for all possible matches such that $(\mathbf{q^s}, \mathbf{q^r}, \mathbf{d})$ defines invariant distributions of asset holdings for agents h and ℓ. We denote these distributions by $\mu_i(a)$ for $i \in \{h, \ell\}$, where we have dropped the reference to the allocation for convenience. If they exist, a property of any invariant distribution is that

$$\frac{1}{2}\int a d\mu_h(a) + \frac{1}{2}\int a d\mu_\ell(a) = A$$

Then we can define recursively the expected value for agent $i \in \{h, \ell\}$ of holding asset a, before entering the trading stage, $V_i(a)$, as

$$
\begin{aligned}
V_h(a) &= \pi \int [u_h(a+q^s(a,a_\ell)+q^r(a,a_\ell)) - d(a,a_\ell) + \beta V_h(a+q^s(a,a_\ell))] d\mu_\ell(a_\ell) \qquad (1) \\
&+ (1-\pi) \int [u_\ell(a-q^s(a_h,a)-q^r(a_h,a)) + d(a_h,a) + \beta V_\ell(a-q^s(a_h,a))] d\mu_h(a_h)
\end{aligned}
$$

With probability π the agent h remains an agent h. Then he meets an agent ℓ with asset a_ℓ according to the distribution μ_ℓ. Since he remains an agent h, he enjoys instant utility $u_h(.)$ from his asset holdings $a+q^s(a,a_\ell)+q^r(a,a_\ell)$ at the end of the settlement stage. However, he only carries $a+q^s(a,a_\ell)$ over to the next period since repos do not involve the transfer of the asset but only of fruits. The agent values this portfolio according to $\beta V_h(a+q^s(a,a_\ell))$. With probability $1-\pi$ the agent h becomes an agent ℓ. In this case, he meets an agent h according to the distribution μ_h and he enjoys instant utility $u_\ell(.)$ from his asset holdings $a-q^s(a_h,a)-q^r(a_h,a)$. He values his remaining portfolio according to $\beta V_\ell(a-q^s(a_h,a))$. Similarly for agents ℓ,

$$
\begin{aligned}
V_\ell(a) &= \pi \int [u_\ell(a-q^s(a_h,a)-q^r(a_h,a)) + d(a_h,a) + \beta V_\ell(a-q^s(a_h,a))] d\mu_h(a_h) \qquad (2) \\
&+ (1-\pi) \int [u_h(a+q^s(a,a_\ell)+q^r(a,a_\ell)) - d(a,a_\ell) + \beta V_h(a+q^s(a,a_\ell))] d\mu_\ell(a_\ell)
\end{aligned}
$$

We assume that there is limited commitment, and an allocation $(\mathbf{q^s}, \mathbf{q^r}, \mathbf{d})$ is individually rational if all agents prefer the allocation to being in autarky this period. That is, for any porfolio a, an agent h matched with an agent ℓ with a portfolio a_ℓ prefers the allocation than not trading today, i.e.

$$
u_h(a+q^s(a,a_\ell)+q^r(a,a_\ell)) - d(a,a_\ell) + \beta V_h(a+q^s(a,a_\ell)) \geq u_h(a) + \beta V_h(a),
$$

and similarly for an agent ℓ matched with an agent h with portfolio a_h,

$$
u_\ell(a-q^s(a_h,a)-q^r(a_h,a)) + d(a_h,a) + \beta V_\ell(a-q^s(a_h,a)) \geq u_\ell(a) + \beta V_\ell(a).
$$

From now on, for concision and whenever there is no risk of confusion, we will drop references to the agents' portfolios in an allocation. We are interested in allocations from which no agent in a match or the pair of agents, has an interest in deviating. That is, as in Zhu, Wallace and Kennan (2009) and Rocheteau (2011), we concentrate on allocations that are in the pairwise core. Given an increasing and concave utility function $U(.)$, we define the pairwise core in a match where agent h and ℓ hold portfolio a_h and a_ℓ respectively, as the set of allocations that satisfy

$$\begin{aligned}
(q^s, q^r, d) &= argmax[u_h(a_h + q^s + q^r) - d + \beta V_h(a_h + q^s)] \\
s.t. &\quad q^s \in [-a_h, a_\ell],\; q^s + q^r \in [-a_h, a_\ell] \\
&\quad u_\ell(a_\ell - q^s - q^r) + d + \beta V_\ell(a_\ell - q^s) \geq U(a_\ell) \\
&\quad u_h(a_h + q^s + q^r) - d + \beta V_h(a_h + q^s) \geq u_h(a_h) + \beta V_h(a_h)
\end{aligned} \tag{3}$$

The first set of constraint is the two feasibility constraints. The second constraint is the participation constraint for type 2 agents, while the first constraint is the participation constraint for type 1 agents. While we do not take a stance on what the function $U(.)$ is, it has to satisfies some simple properties. For example, to insure the participation of agents ℓ, it must be that $U(a_\ell) \geq u_\ell(a_\ell) + \beta V_\ell(a_\ell)$ for all a_ℓ. Also, given a_ℓ, $U(a_\ell)$ should be increasing in β. To fix ideas, we could set $U(a_\ell) = \lambda[u_\ell(a_\ell) + \beta V_\ell(a_\ell)]$ where $\lambda \geq 1$. Finally, notice that the participation constraint of type 2 agents is always binding: Otherwise, it would be possible to lower d and raise the utility of the type 1 agent.

Given a pair (a_h, a_ℓ) a pairwise core allocation with $q^s \in (-a_h, a_\ell)$ satisfies the following first order conditions,

$$\begin{aligned}
q^s : (1+\mu)[u_h'(a_h + q^s + q^r) + \beta V_h'(a_h + q^s)] & \\
-\lambda[u_\ell'(a_\ell - q^s - q^r) + \beta V_\ell'(a_\ell - q^s)] + \xi_h - \xi_\ell &= 0 \\
q^r : (1+\mu) u_h'(a_h + q^s + q^r) - \lambda u_\ell'(a_\ell - q^s - q^r) + \xi_h - \xi_\ell &= 0 \\
d : -1 + \lambda - \mu &= 0
\end{aligned}$$

These three conditions give us[3]

$$V_h'(a_h + q^s) = V_\ell'(a_\ell - q^s) \tag{4}$$

[3] In the case where $q^s = a_\ell$, (4) becomes $V_h'(a_h + q^s) > V_\ell'(a_\ell - q^s)$.

and

$$
\begin{aligned}
u'_h(a_h + q^s + q^r) = u'_\ell(a_\ell - q^s - q^r) &\quad \text{if} \quad \xi_i = 0, i = h, \ell \\
q^s + q^r = a_\ell &\quad \text{if} \quad \xi_\ell > 0 \\
q^s + q^r = -a_h &\quad \text{if} \quad \xi_h > 0
\end{aligned}
\tag{5}
$$

Equations (4) and (5) characterize the pairwise core allocations $q^s(a_h, a_\ell)$ and $q^r(a_h, a_\ell)$, while $d(a_h, a_\ell)$ is given by (3) holding with equality. Notice that the allocation depends on the distributions of asset holdings μ_i for $i = h, \ell$ as they affect the value functions V_i. To fully characterize the equilibrium with an invariant distribution, we need to specify how agents are matched. In the next section, we assume that agents are randomly matched. Then we assume that agents are matched according to a pre-specified matching rule.

5. Random Matching: Special Cases

Here, we study two extreme cases when an agent h is randomly matched with an agent ℓ. The two extreme cases of interest are the ones for which $\pi = 1/2$ and $\pi = 1$. In the case with $\pi = 1/2$, types have no persistence and each types are as likely for an agent independently of his history of type. In the case with $\pi = 1$ types are fully persistent as types are fixed for ever.

With no persistence and random matching, we obtain the following result.

Proposition 2. *With random matching and $\pi = 1/2$, the pairwise core allocations defines a unique equilibrium characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a} = A/2$ with $q^s(\bar{a}, \bar{a}) = 0$, and $q^r(\bar{a}, \bar{a}) > 0$.*

In the case without persistence, (1) and (2) imply that $V_h(a) = V_\ell(a)$ for all a, such that agents h and ℓ value future payoff of holding the asset in the same way. In this case, (4) implies that $a_h + q^s(a_h, a_\ell) = a_\ell - q^s(a_h, a_\ell)$ with $q^s(a_h, a_\ell) > 0$ if and only if $a_\ell > a_h$ and $q^s(a_h, a_\ell) < 0$ otherwise. Therefore, the unique equilibrium is one where the distribution of asset holding is degenerate at $\bar{a} = A$ and $q^s(\bar{a}, \bar{a}) = 0$. This is very intuitive: Since all agents give the same value to future returns, they extinguish all surplus from trading the asset by averaging their asset holding (i.e. once an agent holding a_h trade with an agent hoding a_ℓ, they both end up with $(a_h + a_\ell)/2$) and in equilibrium they hold the same amount of the asset. Then (5) together with assumption 1 imply that $q^r(\bar{a}, \bar{a}) > 0$: While agents value future asset returns the same way, they differ in their valuation of current return. Therefore, there is a benefit from repos, where only the current return is traded.

With full persistence, the result is that there is neither asset sales nor repo in equilibrium.

Proposition 3. *With random matching and $\pi = 1$, the pairwise core allocations defines an equilibrium characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a}_h$ and $\bar{a}_\ell$ with $\bar{a}_h > \bar{a}_\ell$ where $q^s(\bar{a}_h, \bar{a}_\ell) = 0$ and $q^r(\bar{a}_h, \bar{a}_\ell) = 0$.*

We will first verify that the proposed allocation is an equilibrium. Since $q^s(\bar{a}_h, \bar{a}_\ell) = 0$ and $q^r(\bar{a}_h, \bar{a}_\ell) = 0$, the allocation must be such that $d(\bar{a}_h, \bar{a}_\ell) = 0$ as agents would otherwise prefer autarky. Using (1) and (2), we then have for $i = h, \ell$,

$$V_i(\bar{a}_i) \quad = \quad \frac{u_i(\bar{a}_i)}{1-\beta} \tag{6}$$

and (4) and (5) imply that $\bar{a}_h$ and $\bar{a}_\ell$ are uniquely given by

$$u'_h(\bar{a}_h) = u'_\ell(\bar{a}_\ell)$$

with $\bar{a}_h = 2A - \bar{a}_\ell$. This verifies that there is no asset sales or repos in equilibrium.[4] Also, this verifies that $\bar{a}_h > \bar{a}_\ell$. This equilibrium is unique whenever endowments are symmetric across all agents (and no constraint bind – which may happen if some agents ℓ are endowed with too many securities in the first place) so that all agents ℓ hold the same amount a_ℓ and all agents h holds a_h. To see this notice that if an agent h endowed with a_h meets an agent ℓ endowed with a_ℓ, then the pairwise core dictates that they trade so that (5) holds. But the unique solution is that $a_h + q^s(a_h, a_\ell) = \bar{a}_h$ and $a_\ell - q^s(a_h, a_\ell) = \bar{a}_\ell$. Since $a_h + a_\ell = 2A$, such a q^s exists and takes the agents directly to the equilibrium distribution of asset holdings. [is this unique even if original distribution is not symmetric?].

For general levels of persistence $\pi \in (0, 1)$ and random matching, we are unable to determine analytically the total volume of sales and repos as we cannot solve analytically for the invariant equilibrium distribution of asset holdings. However, we can partially characterize sales and repos within one match.

Proposition 4. *With random matching and $\pi \in (0, 1)$, if V_i is concave and differentiable, any pairwise core allocation is such that $q^s(a_h, a_\ell)$ is increasing in a_ℓ and decreasing in a_h. Also $q^s(a_h, a_\ell) + q^r(a_h, a_\ell)$ is increasing in a_ℓ and decreasing in a_h.*

[4] Notice that $\bar{a}_\ell = 0$ is only an equilibrium if $\bar{a}_\ell = 0$ was agents ℓ initial endowment (since they would prefer autarky otherwise) and $U(\bar{a}_\ell) = 0$.

We leave the proof in the Appendix. Proposition 4 gives us some indication of how the distribution of asset holding moves in equilibrium. As an agent ℓ is better endowed, he will sell more to agent h, and as agent h is less endowed, he will buy more form agent ℓ. This hints to more trade as agents valuation differ and we expect that the distributions of asset holdings become more spiked around their respective mean $\bar{a}_h$ and $\bar{a}_\ell$ as π increases, where the means are diverging as π increases. However, since agents can switch randomly form one type to the other, it is difficult to fully characterize the equilibrium. We provide some examples that rationalize this intuition below [TODO]. Also, in the next section we show that this claim is true when matching is directed.

6. Directed search

We now describe the matching technology. Following Corbae, Temzelides and Wright (2003), we use directed search and the matching function specifies that those agents who just switched to new types meet with each other. Below we verify that matching function is an equilibrium matching rule (where such a term is precisely defined). The rest of this section is devoted to proving the following result,

Proposition 5. *With directed search, the pairwise core allocations defines an equilibrium characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a}_i$ with $i = h, \ell$ with $q^s(\bar{a}_h, \bar{a}_\ell) = 0$, $q^s(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell$ and $q^r(\bar{a}_h, \bar{a}_\ell) = q^r(\bar{a}_\ell, \bar{a}_h) = q^r$ where q^r solves $u_h'(\bar{a}_h + q^r) \geq u_\ell'(\bar{a}_\ell - q^r)$ (with equality if $q^r < \bar{a}_\ell$).*

[is this equilibrium unique?] We proceed by guessing and verifying that the two distributions of assets are degenerate at $\bar{a}_h$ and $\bar{a}_\ell$ for high and low types respectively. As agents who did *not* switch are matched together and the distribution of asset holdings is invariant, the pairwise core allocation is $q^s(\bar{a}_h, \bar{a}_\ell) = 0$ and $\bar{a}_h$ and $\bar{a}_\ell$ satisfy

$$V_h'(\bar{a}_h) = V_\ell'(\bar{a}_\ell)$$

with $q^r(\bar{a}_h, \bar{a}_\ell)$ given by

$$u_h'(\bar{a}_h + q^r) = u_\ell'(\bar{a}_\ell - q^r) \quad \text{if} \quad \xi_i = 0, i = 1, 2 \tag{7}$$

$$q^r = \bar{a}_\ell \quad \text{if} \quad \xi_2 > 0 \tag{8}$$

For those agents who switched we have that a "new" agent ℓ is holding $\bar{a}_h$ while a "new" agent h is holding $\bar{a}_\ell$. Therefore, the pairwise core allocation for those agents who switched is given by

$$V_h'(\bar{a}_\ell + q^s(\bar{a}_\ell, \bar{a}_h)) = V_\ell'(\bar{a}_h - q^s(\bar{a}_\ell, \bar{a}_h))$$

so that

$$q^s(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell \tag{9}$$

We will guess and later verify that $q^s \geq 0$. A repo trade satisfies

$$u_h'(\bar{a}_\ell + q^s + q^r) = u_\ell'(\bar{a}_h - q^s - q^r) \quad \text{if} \quad \xi_i = 0, i = 1,2 \tag{10}$$

$$q^s + q^r = \bar{a}_h \quad \text{if} \quad \xi_\ell > 0 \tag{11}$$

Hence, using (9) to replace $q^s(\bar{a}_\ell, \bar{a}_h)$ in (10) and (11) and comparing the result with (7) and (8), we obtain

$$q^r(\bar{a}_\ell, \bar{a}_h) = q^r(\bar{a}_h, \bar{a}_\ell).$$

In words, the property of any core allocations with direct matching is that agents who just switched type adjust their asset holdings so that they hold their type's portfolio. Then they conduct repo as if they never switched. Agents who did not switch type just engage in repo. Loosely speaking, there is a sense in which agents first access the asset market and then engage in repo. To verify that this is an equilibrium we need to verify that an agent would not prefer to be matched with a different agent than the one he is assigned to, or that no agent would prefer to interact with him to trading with his assigned agent. In the terminology of Corbae, Temzelides and Wright (2003), the proposed matching rule is an equilibrium matching if no coalition consisting of 1 or 2 agents can do better (in the sense that $u_i(q^s, q^r, d) + \beta V_i(q^s, q^r, d)$ increases for all i in the coalition) by deviating in the following sense: An individual can deviate by matching with himself (i.e. being in autarky this period) rather than as prescribed by the matching rule; and a pair can deviate by matching with each other rather than as prescribed by the matching rule.

It should be clear that pairwise core allocations are always better than autarky. Therefore we only need to check deviations by a coalition of 2 agents. However, we can rule out deviations that involve agents ℓ as they always obtain utility $U(a)$ whenever they hold assets a (be it on or off equilibrium) and so they are unable to do better by deviating. The only relevant deviation we need to check is one where an agent h holding $\bar{a}_h$ (an agent h who did not switch) is matched with an agent h holding $\bar{a}_\ell$ (an agent who was ℓ and just switched).

However, both agents like to hold the asset this period and so the agent h holding $\bar{a}_h$ is indifferent trading with an agent ℓ holding $\bar{a}_\ell$ or an agent h holding $\bar{a}_\ell$ as the allocation would give him the same payoff:

$$\begin{aligned} & u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) \\ = \;& \bar{u}_h - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) \\ = \;& \bar{u}_h + \bar{u}_\ell + \beta V_\ell(\bar{a}_\ell) + \beta V_h(\bar{a}_h) - U(\bar{a}_\ell) \geq u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \end{aligned}$$

where $\bar{u}_h = u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell))$ and $\bar{u}_\ell = u_\ell(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell))$. However, the agent h holding $\bar{a}_\ell$ would have to give at least $u_h(\bar{a}_h) + \beta V_h(\bar{a}_h)$ to the other agent h and he is worse off if $u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h)$. We show that this is the case in the Appendix, so that an agent h holding $\bar{a}_\ell$ would never want to meet an agent h with $\bar{a}_h$. Therefore, the proposed directed matching rule together with the pairwise core allocation define an equilibrium where the distribution of asset holding for each type is degenerate. The following proposition characterizes the support of the distributions.

Proposition 6. *The degenerate supports $\bar{a}_h$ and $\bar{a}_\ell$ of the two distributions are fully characterized by the following equations,*

$$\bar{a}_h + \bar{a}_\ell = 2A \tag{12}$$

$$u_h'(\bar{a}_h + q^r) = u_\ell'(\bar{a}_\ell - q^r) \tag{13}$$

$$u_h'(\bar{a}_h + q^r) = \frac{1}{(2\pi - 1)} \{(1 - \beta\pi)\pi U'(\bar{a}_\ell) - (1 - \beta(1-\pi))(1-\pi)U'(\bar{a}_h)\} \tag{14}$$

The proof is in the Appendix. There we also show that the equilibrium asset holdings are diverging in the persistence of the shock, i.e. as π increases.

Corollary 7. $\bar{a}_h - \bar{a}_\ell \geq 0$ *is increasing in π. Sales volume is hump-shaped in π while repos volume is strictly decreasing in π.*

The intuition for this result is straightforward. When $\pi = 1/2$, agents future type is independent of their current type. Therefore, two agents' future value of the asset is the same. Since the core pairwise allocation equates the marginal benefit of holding the asset, all agents hold the same quantity of assets. Therefore, $\bar{a}_h(1/2) = \bar{a}_\ell(1/2) = A$ and there are no asset sales, but only repos that allocate the fruits to those agents h who like it most.[5]

[5] Note that (14) is always satisfied at $\pi = 1/2$ and $\bar{a}_h(1/2) = \bar{a}_\ell(1/2) = A$ since cross multiplying by $2\pi - 1$ we obtain that both sides are zero.

As π increases, it is more likely that an agent h becomes once again an agent h next period. Therefore his valuation for the asset increases, and starting from $a_h = a_\ell$, there are gains from trade when an agent h meets an agent ℓ. In this case, $\bar{a}_h > A > \bar{a}_\ell$. In equilibrium only those agents who switch types have gains to trade the asset and so the volume of asset sales is increasing. Also repos are decreasing as agents hold more of the asset they like. Finally, when $\pi = 1$, agents know their type for sure. Hence in equilibrium, all gains from trades (be it asset trade or fruit trade) are extinguished, so that there is neither sales nor repos.

Directed search is very powerful in the sense that there are many allocations that are payoff equivalent to the distribution we just described. However, not all allocations belong to the core. For example, consider the case with full persistence where $\pi = 1$. It is easy to see that we can only use repos to achieve the same payoff as the allocation described above. Here is how: Let us consider the case where the core allocation $(\bar{a}_h, \bar{a}_\ell)$ as defined in Proposition 6 is such that $\bar{a}_\ell < A < \bar{a}_h$. Let us endow all agents with A. Then match each agent h with an agent ℓ. By specifying $q^s = 0$ and $\tilde{q}^r \leq A$ such that $A + \tilde{q}^r = \bar{a}_h + q^r$, we obviously satisfy the intratemporal condition for a core allocation $u_h'(A + \tilde{q}^r) = u_\ell'(A - \tilde{q}^r)$, and we achieve the same payoff as the allocation in Proposition 6. However, interestingly, it does not necessarily satisfy the intertemporal condition for a core allocation. Indeed, suppose it is in the core. Then we must have $V_h'(A) = V_\ell'(A)$. Then with $\pi = 1$, (14) becomes

$$u_h'(A + \tilde{q}^r) = (1 - \beta)U'(A)$$

and replacing $A + \tilde{q}^r = \bar{a}_h + q^r$ we obtain $u_h'(\bar{a}_h + q^r) = (1 - \beta)U'(A)$. Since $\bar{a}_\ell < A$ we have $U'(A) < U'(\bar{a}_\ell)$ so that $u_h'(\bar{a}_h + q^r) < U'(\bar{a}_\ell)$ which contradicts the fact that our original allocation was a core allocation. Therefore, we cannot replicate the allocation in Proposition (6) with $a_i = A$ and with repo only: We need the distribution of assets to be different across types.

7. Directed Search With Nash Bargaining

In this section, we consider that the allocation is set by Nash bargaining rather than considering an arbitrary pairwise core allocation. We still assume that agents who did not switch are matched together while those agents who just switched are matched with each other. With Nash bargaining, the allocation of an agent h with portfolio a_h matched with an agent

ℓ with portfolio a_ℓ solves the following problem:

$$\begin{aligned}\max_{q^s,q^r,d}&[u_h(a_h+q^s+q^r)-d+\beta V_h(a_h+q^s)-u_h(a_h)-\beta V_h(a_h)]^\theta\\ &\times[u_\ell(a_\ell-q^s-q^r)+d+\beta V_\ell(a_\ell-q^s)-u_\ell(a_\ell)-\beta V_\ell(a_\ell)]^{1-\theta}\end{aligned}$$

with first order conditions

$$\begin{aligned}V_h'(a_h+q^s) &= V_\ell'(a_\ell-q^s)\\ u_h'(a_h+q^s+q^r) &= u_\ell'(a_h-q^s-q^r)\\ d(a_h,a_\ell) &= (1-\theta)[u_h(a_h+q^s+q^r)-u_h(a_h)+\beta V_h(a_h+q^s)-\beta V_h(a_h)]\\ &\quad -\theta[u_\ell(a_\ell-q^s-q^r)-u_\ell(a_\ell)+\beta V_\ell(a_\ell-q^s)-\beta V_\ell(a_\ell)]\end{aligned}$$

We still assume that agents who did not switch types are matched together while those agents who just switched are matched together. In the Appendix, we show

Proposition 8. *With directed search, an equilibrium with bargaining is characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a}_i$ with $i=h,\ell$ with $q^s(\bar{a}_h,\bar{a}_\ell)=0$, $q^s(\bar{a}_\ell,\bar{a}_h)=\bar{a}_h-\bar{a}_\ell$ and $q^r(\bar{a}_h,\bar{a}_\ell)=q^r(\bar{a}_\ell,\bar{a}_h)=q^r$ where q^r solves $u_h'(\bar{a}_h+q^r)\geq u_\ell'(\bar{a}_\ell-q^r)$ (with equality if $q^r<\bar{a}_\ell$).*

Notice that $d(\bar{a}_h,\bar{a}_\ell)$ is the price of repos while $d(\bar{a}_\ell,\bar{a}_h)$ is the price of an asset sale and a repo, so that we should expect $d(\bar{a}_\ell,\bar{a}_h)>d(\bar{a}_h,\bar{a}_\ell)$. In the Appendix, we show that

$$\begin{aligned}d(\bar{a}_h,\bar{a}_\ell) &= (1-\theta)[u_h(\bar{a}_h+q^r)-u_h(\bar{a}_h)]+\theta[u_\ell(\bar{a}_\ell)-u_\ell(\bar{a}_\ell-q^r)]\\ d(\bar{a}_\ell,\bar{a}_h) &= d(\bar{a}_h,\bar{a}_\ell)+\bar{u}+\beta(1-\theta)[V_h(\bar{a}_h)-V_h(\bar{a}_\ell)]+\beta\theta[V_\ell(\bar{a}_h)-V_\ell(\bar{a}_\ell)]\end{aligned}$$

where

$$\bar{u}=(1-\theta)[u_h(\bar{a}_h)-u_h(\bar{a}_\ell)]+\theta[u_\ell(\bar{a}_h)-u_\ell(\bar{a}_\ell)]$$

We consider the following matching technology: An agent h with $\bar{a}_\ell$ meets an agent ℓ with $\bar{a}_h$ and an agent h with $\bar{a}_h$ meets an agent ℓ with $\bar{a}_\ell$. Given $q^s(\bar{a}_h,\bar{a}_\ell)=0$, $q^s(\bar{a}_\ell,\bar{a}_h)=\bar{a}_h-\bar{a}_\ell$ and $q^r(\bar{a}_h,\bar{a}_\ell)=q^r(\bar{a}_\ell,\bar{a}_h)=q^r$, we obtain the following value functions,

$$\begin{aligned}V_h(\bar{a}_h) &= \pi[u_h(\bar{a}_h+q^r)-d(\bar{a}_h,\bar{a}_\ell)+\beta V_h(\bar{a}_h)]+(1-\pi)[u_\ell(\bar{a}_\ell-q^r)+d(\bar{a}_\ell,\bar{a}_h)+\beta V_\ell(\bar{a}_\ell)]\\ V_\ell(\bar{a}_\ell) &= \pi[u_\ell(\bar{a}_\ell-q^r)+d(\bar{a}_h,\bar{a}_\ell)+\beta V_\ell(\bar{a}_\ell)]+(1-\pi)[u_h(\bar{a}_h+q^r)-d(\bar{a}_\ell,\bar{a}_h)+\beta V_h(\bar{a}_h)]\end{aligned}$$

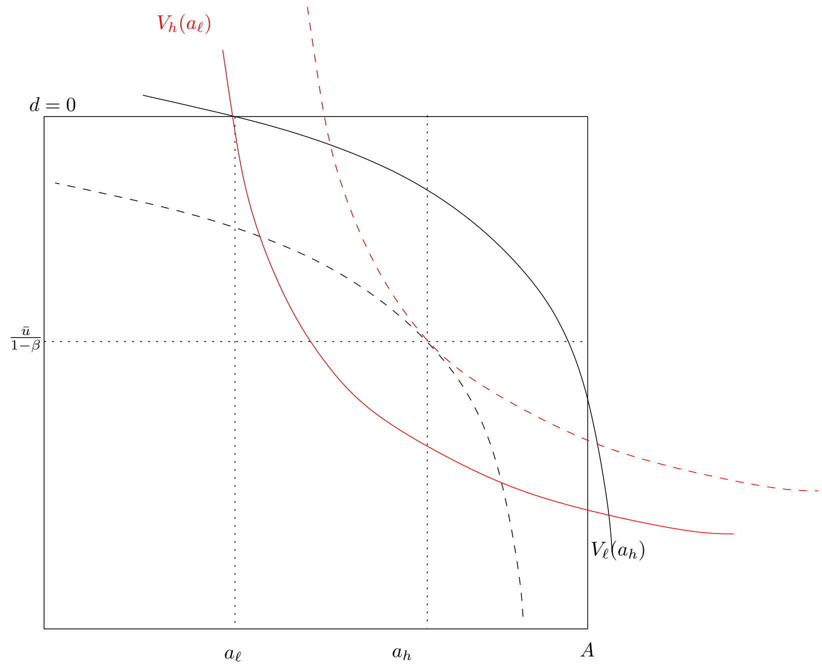


Figure 1: Intertemporal gains from trade

Notice that we need to specify the value of the outside option in order to solve for the bargaining solution in equilibrium, i.e. $V_h(\bar{a}_\ell)$ and $V_\ell(\bar{a}_h)$. A moment reflection should convince the reader that, given our matching technology,

$$\begin{aligned} V_h(\bar{a}_\ell) &= \pi[u_h(\bar{a}_h + q^r) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h)] + (1-\pi)[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] \\ V_\ell(\bar{a}_h) &= \pi[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] + (1-\pi)[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] \end{aligned}$$

Using these value functions we find that

$$d(\bar{a}_\ell, \bar{a}_h) = d(\bar{a}_h, \bar{a}_\ell) + \frac{\bar{u}}{1-\beta}$$

so that the value of selling the asset is just $\bar{u}/(1-\beta)$, the lifetime discounted surplus from adjusting portfolios. Notice that as $\pi \to 1/2$ we have $\bar{a}_h \to \bar{a}_\ell$ so that $\bar{u} \to 0$ and there is no value of selling the asset. Figures 1 and 2 illustrate this equilibrium. This is illustrated in the two following figures.

Figure 1 shows the indifference curves for $V_i(a) + d$ for two agents: One agent h endowed

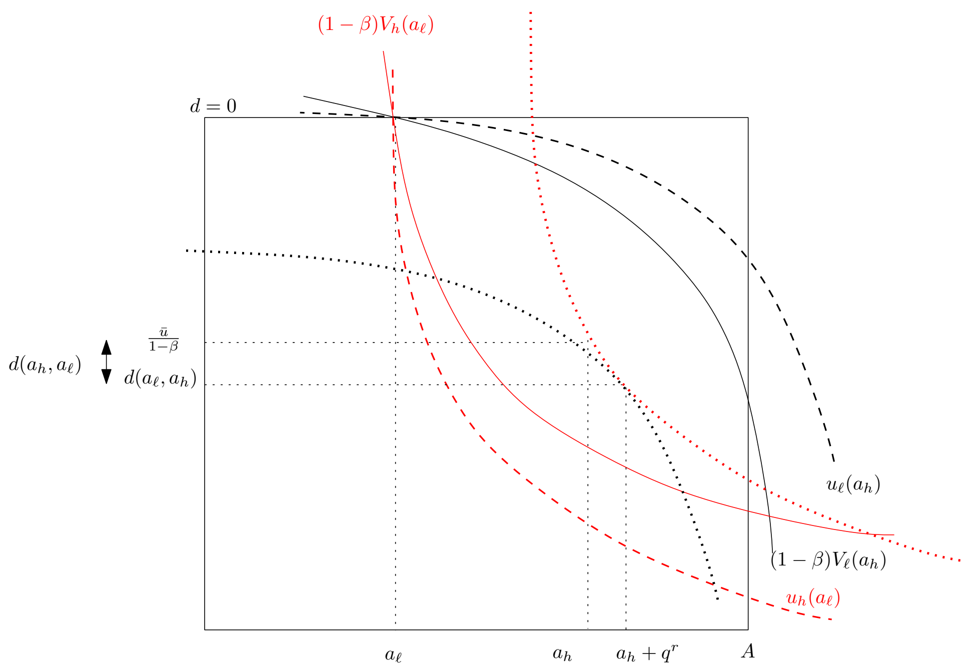


Figure 2: Intratemporal gains from trade

with a_ℓ and one agent ℓ endowed with a_h. Indifference curves are tangent at the stationary distribution points $(\bar{a}_h, \bar{a}_\ell)$ (the axis for d is reversed). Scaling the continuation utility by $1-\beta$ to compare it with present utility, notice that $(1-\beta)V_h'(a) \leq u_h'(a)$ by assumption 1 and since there is chance that an agent h reverts to being an agent ℓ in the future. By the same argument, notice that $(1-\beta)V_\ell'(a) \leq u_\ell'(a)$. Therefore, as illustrated in Figure 2, the indifference curves for $u_i(a) + d$ for both agents will be tangent at a point south-east of the $(\bar{a}_h, \bar{a}_\ell)$. This explains why repos are useful: They exploit intratemporal gains from trade.

In the Appendix, we solve for the solution to the bargaining problem with directed matching q^r, $\bar{a}_h$ and $\bar{a}_\ell$.

Proposition 9. *The degenerate supports $\bar{a}_h$ and $\bar{a}_\ell$ of the two distributions with bargaining are fully characterized by the following equations,*

$$\begin{aligned}
u_h'(\bar{a}_h + q^r) &= u_\ell'(\bar{a}_\ell - q^r) \\
u_h'(\bar{a}_h + q^r) &= \frac{[\pi - (2\pi-1)\beta][\theta u_\ell'(\bar{a}_\ell) - (1-\theta)u_h'(\bar{a}_h)] - (1-\pi)[\theta u_\ell'(\bar{a}_h) - (1-\theta)u_h'(\bar{a}_\ell)]}{(2\pi-1)(1-\beta)(2\theta-1)} \\
\bar{a}_h + \bar{a}_\ell &= 2A
\end{aligned}$$

Notice from the second equation that when $\pi = 1/2$, we must have

$$\theta[u'_\ell(\bar{a}_\ell) - u'_\ell(\bar{a}_h)] = (1-\theta)[u'_h(\bar{a}_h) - u'_h(\bar{a}_\ell)]$$

and the unique solution is $\bar{a}_\ell = \bar{a}_h = A$. In this case, $q^s = 0$ and $q^r > 0$. Also, if $\pi = 1$ we have

$$u'_h(\bar{a}_h + q^r) = \frac{\theta u'_\ell(\bar{a}_\ell) - (1-\theta)u'_h(\bar{a}_h)}{(2\theta - 1)}$$

with solution $q^r = 0$ and $u'_\ell(\bar{a}_\ell) = u'_h(\bar{a}_h)$. In this case as well, $q^s = 0$. We can also find the price for repo and asset sales. This is our final result.

Corollary 10. *Let p^r be the price of a repo and p^s the price of a sale. Then*

$$\begin{aligned} p^r &= \frac{(1-\theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] + \theta[u_\ell(\bar{a}_\ell) - u_\ell(\bar{a}_\ell - q^r)]}{q^r} \\ p^s &= \frac{(1-\theta)[u_h(\bar{a}_h) - u_h(\bar{a}_\ell)] + \theta[u_\ell(\bar{a}_h) - u_\ell(\bar{a}_\ell)]}{(1-\beta)(\bar{a}_h - \bar{a}_\ell)} \end{aligned}$$

It is clear that in general those prices are different from their Walrasian equivalent, and in particular that $(1-\beta)p^s$ is different from p^r. However, an interesting case to consider is when agents becomes very patient. Then it is legitimate to guess that the allocation will converge to the Walrasian one, as it is, in some sense, equivalent to agents trading with each other very frequently. However, this is not the case: While it is true that agents can trade very fast and they may be able to replicate a meeting with the representative "market" agent, they are also bargaining a lot and this friction remains. Indeed, as β tends to one, the solution to the bargaining problem is characterized by $\bar{a}_h, \bar{a}_\ell \to A$, so that asset sales converge to zero. Hence, we obtain

$$\lim_{\beta \to 1}(1-\beta)p^s = (1-\theta)u'_h(A) + \theta u'_\ell(A).$$

However in the limit q^r satisfies $u'_h(A + q^r) = u'_\ell(A - q^r)$ and Assumption 1 guarantees that $q^r > 0$ is bounded away from zero. Therefore, by concavity of the utility function, $p^r < (1-\beta)p^s$ which justifies why agents only conduct repos in the first place.[6]

For illustration, we use the following utility function: $u_h(a) = \frac{a^{1-\sigma}}{1-\sigma}$ and $u_\ell(a) = \lambda u_h(a)$

[6] $p^r < (1-\beta)p^s$ follows from the fact that for a concave and differentiable function $f(.)$, $[f(x) - f(x + \Delta)]/\Delta \leq f'(x)$ for any x and $\Delta > 0$.

where $\lambda \in (0,1)$. Then we obtain

$$q^r = \frac{\lambda^{-\frac{1}{\sigma}}\bar{a}_\ell - \bar{a}_h}{1+\lambda^{-\frac{1}{\sigma}}}$$

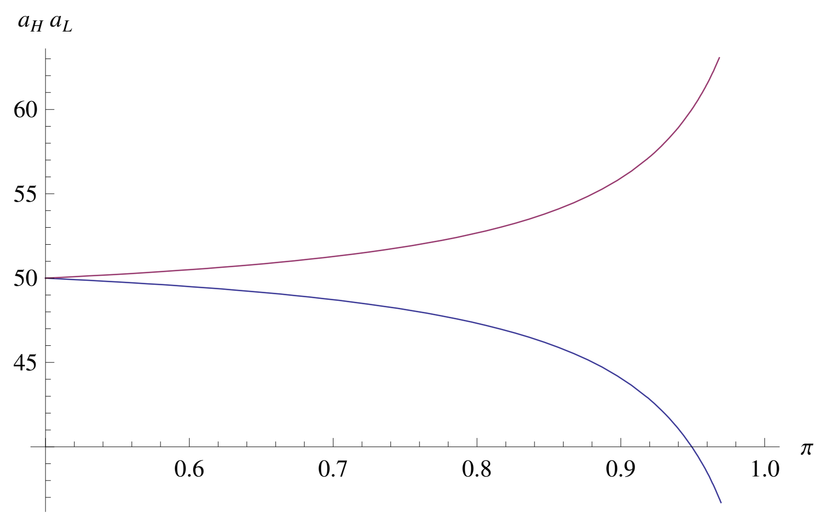
so that

$$\bar{a}_h + q^r = \lambda^{-\frac{1}{\sigma}}\frac{2A}{1+\lambda^{-\frac{1}{\sigma}}}$$

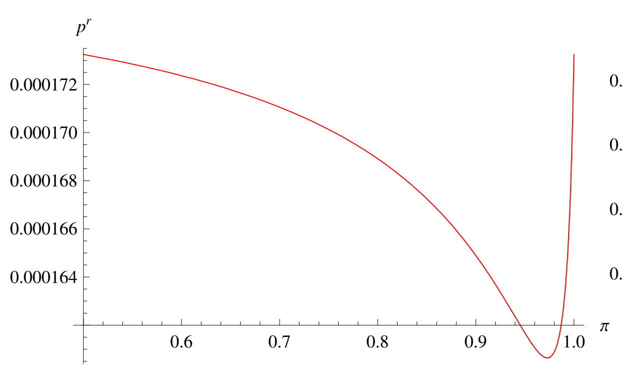
and

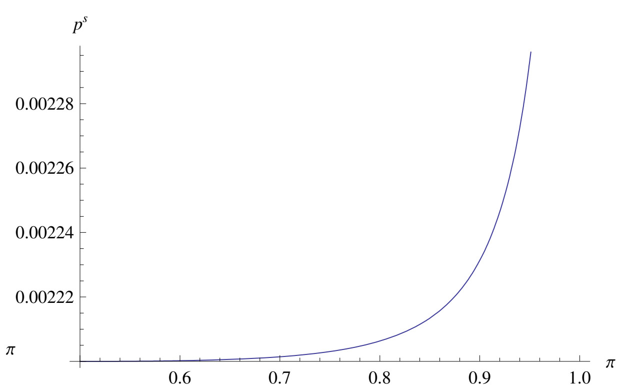
$$\bar{a}_\ell - q^r = \frac{2A}{1+\lambda^{-\frac{1}{\sigma}}}$$

Figure 1 shows how $\bar{a}_h$ (red curve) and $\bar{a}_\ell$ (blue curve) evolve as π varies from 1/2 to 1. The parameters chosen are $\theta = 0.5, \lambda = 0.1, \sigma = 2, \beta = 0.9,$ and $A = 50$. Interestingly, the rate of divergence increases as types become more persistent. Hence, as π becomes large, we should expect some wide movements in prices and quantities.

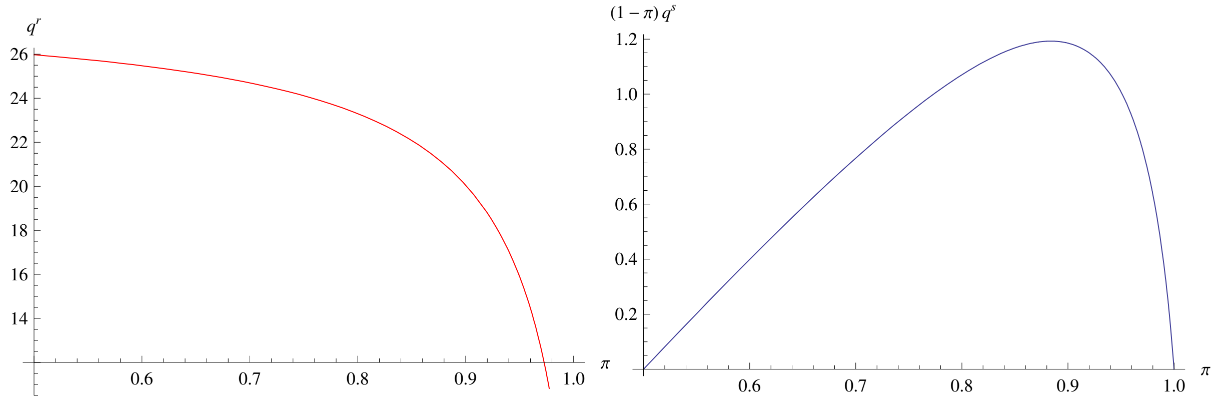


This intuition is confirmed by Figure 2 that shows prices for repo p^r and asset sales p^s.





Simimarly total repo volume q^r and total sales volume $(1-\pi)q^s$ display very different pattern, as illustrated in Figure 3. At $\pi = 0.9$, the total volume of repo is approximately 20% of the outstanding securities, while total sales are only 1% of outstanding securities.



Interestingly, the coefficient of risk aversion σ is the one with the most impact on asset volumes and values. With $A = 50$, we can match the observations on repos and sales of Treasury securities, with $\sigma = 0.5$ (close to risk neutrality) and quite high persistence, $\pi = 0.9$.[7]

8. Appendix

8.1. *Proof of Proposition 4*

To show Proposition 4 we need to know V_h' and V_ℓ'. Using the fact that the constraint on agents ℓ is always binding and the definition of value functions, we obtain

$$V_h'(a) = \pi \int [u_h'(a+q^s+q^r)(1+\frac{\partial q^s}{\partial a}+\frac{\partial q^r}{\partial a})-\frac{\partial d}{\partial a}+\beta V_h'(a+q^s)(1+\frac{\partial q^s}{\partial a})]d\mu_\ell(a_\ell)+(1-\pi)U'(a)$$

and

$$V_\ell'(a) = (1-\pi) \int [u_h'(a+q^s+q^r)(1+\frac{\partial q^s}{\partial a}+\frac{\partial q^r}{\partial a})-\frac{\partial d}{\partial a}+\beta V_h'(a+q^s)(1+\frac{\partial q^s}{\partial a})]d\mu_\ell(a_\ell)+\pi U'(a)$$

Since the participation constraint of the type 2 agents is always binding, we also know that changing the asset holding of agent of type 1 will not change the utility of type 2 agents,

[7] The average daily volume of Treasury repos is approximately twice the one for Treasury sales in the US according to ICAP, see http://www.icap.com/investor-relations/monthly-volume-data.aspx.

or

$$-u'_\ell(a_\ell - q^s - q^r)(\frac{\partial q^s}{\partial a_h} + \frac{\partial q^r}{\partial a_h}) + \frac{\partial d}{\partial a_h} - \beta V'_\ell(a_\ell - q^s)\frac{\partial q^s}{\partial a_h} = 0$$

Therefore,

$$\frac{\partial d}{\partial a_h} = \beta V'_\ell(a_\ell - q^s)\frac{\partial q^s}{\partial a_h} + u'_\ell(a_\ell - q^s - q^r)(\frac{\partial q^s}{\partial a_h} + \frac{\partial q^r}{\partial a_h})$$

replacing this expression in V'_h and V'_ℓ and using the fact that in the pairwise core (4) holds, we obtain

$$\begin{aligned} V'_h(a) &= \pi \int (\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a})[u'_h(a + q^s + q^r) - u'_\ell(a_\ell - q^s - q^r)] + u'_h(a + q^s + q^r) + \beta V'_h(a + q^s) d\mu_\ell(a_\ell) \\ &\quad +(1-\pi)U'(a) \end{aligned}$$

Using the first order conditions we obtain

$$\begin{aligned} (\frac{\partial q^s}{\partial a_h} + \frac{\partial q^r}{\partial a_h})[u'_\ell(a_\ell - q^s - q^r) - u'_h(a_h + q^s + q^r)] &= 0 \quad \text{if } \xi_i = 0 \\ \frac{\partial q^s}{\partial a_h} + \frac{\partial q^r}{\partial a_h} &= 0 \quad \text{if } \xi_\ell > 0 \\ \frac{\partial q^s}{\partial a_h} + \frac{\partial q^r}{\partial a_h} &= -1 \quad \text{if } \xi_h > 0 \end{aligned}$$

where the last two equalities follows from the fact that when $\xi_\ell > 0$ we have $q^s + q^r = a_\ell$ while when $\xi_h > 0$ we have $q^s + q^r = -a_h$ (and $u'_h(0) \leq u'_\ell(a_\ell + a)$). Therefore,

$$V'_h(a) = \pi \int_{\{a_\ell : \xi_h(a, a_\ell) > 0\}} [u'_h(0) - u'_\ell(a_\ell + a)] d\mu_\ell(a_\ell) + \pi \int u'_h(a + q^s + q^r) + \beta V'_h(a + q^s) d\mu_\ell(a_\ell)$$

and as the set of a_ℓ such that $\xi_h > 0$ has measure zero by Assumption 1, we obtain

$$V'_h(a) = \pi \int [u'_h(a + q^s + q^r) + \beta V'_h(a + q^s)] d\mu_\ell(a_\ell) + (1-\pi)U'(a) \tag{15}$$

and using a similar argument, we obtain

$$V'_\ell(a) = (1-\pi) \int [u'_h(a + q^s + q^r) + \beta V'_h(a + q^s)] d\mu_\ell(a_\ell) + \pi U'(a) \tag{16}$$

How does (15) and (16) change as a is increasing? We have

$$\frac{\partial}{\partial a} V_h'(a+q^s) = V_h''(a+q^s)(1+\frac{\partial q^s}{\partial a}) < 0$$

and

$$\frac{\partial}{\partial a} u_h'(a+q^s+q^r) = u_h''(a+q^s+q^r)(1+\frac{\partial (q^s+q^r)}{\partial a}) < 0$$

Where the inequalities follow from the first order conditions, as we get

$$u_h''(a_h+q^s+q^r)(1+\frac{\partial (q^s+q^r)}{\partial a_h}) = u_\ell''(a_\ell-q^s-q^r)(-\frac{\partial (q^s+q^r)}{\partial a_h})$$

so that

$$\frac{\partial (q^s+q^r)}{\partial a_h} = \frac{-u_h''}{u_\ell''+u_h''} \in (-1,0)$$

and using (4) we also obtain (guessing that the value function is concave and differentiable)

$$\frac{\partial q^s}{\partial a_h} = \frac{-V_h''}{V_\ell''+V_h''} \in (-1,0)$$

Therefore, given $V_i'(a)$ is decreasing in a (verifying the initial guess). Also, increasing a_ℓ, we have

$$\frac{\partial}{\partial a_\ell} V_h'(a+q^s) = V_h''(a+q^s)\frac{\partial q^s}{\partial a_\ell} < 0$$

and

$$\frac{\partial}{\partial a_\ell} u_h'(a+q^s+q^r) = u_h''(a+q^s+q^r)\frac{\partial (q^s+q^r)}{\partial a_\ell} < 0$$

where the signs are given by the following argument: Using the properties of the core allocation we obtain that either $\partial(q^s+q^r)/\partial a_\ell = 1 > 0$ if $\xi_\ell > 0$ and otherwise

$$u_h''(a_h+q^s+q^r)\frac{\partial (q^s+q^r)}{\partial a_\ell} = u_\ell''(a_\ell-q^s-q^r)(1-\frac{\partial (q^s+q^r)}{\partial a_\ell})$$

so that

$$\frac{\partial (q^s+q^r)}{\partial a_\ell} = \frac{u_\ell''}{u_h''+u_\ell''} \in (0,1)$$

and using (4) we also obtain (guessing that the value function is concave and differentiable)

$$\frac{\partial q^s}{\partial a_\ell} = \frac{V_\ell''}{V_h''+V_\ell''} \in (0,1)$$

8.2. *Proof of* $u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h)$

First notice that the equilibrium payoff of an agent h holding $\bar{a}_\ell$ is

$$\begin{aligned}
& u_h(\bar{a}_\ell + q^s(\bar{a}_\ell, \bar{a}_h) + q^r(\bar{a}_\ell, \bar{a}_h)) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_\ell + q^s(\bar{a}_\ell, \bar{a}_h)) \\
= \; & \bar{u}_h - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h) \\
= \; & \bar{u}_h + \bar{u}_\ell + \beta V_\ell(\bar{a}_\ell) + \beta V_h(\bar{a}_h) - U(\bar{a}_h) \geq u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)
\end{aligned}$$

where $\bar{u}_h = u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell))$ and $\bar{u}_\ell = u_\ell(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell))$ and where the inequality follows from the participation constraint. Adding $u_h(\bar{a}_h)$ on both sides and rearranging terms, we must have

$$\bar{u}_h + \bar{u}_\ell + \beta V_\ell(\bar{a}_\ell) + u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h) + u_h(\bar{a}_h) + u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell) \tag{17}$$

However, notice that

$$\bar{u}_h + \bar{u}_\ell + \beta V_\ell(\bar{a}_\ell) < u_h(\bar{a}_h) + u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell) \tag{18}$$

To see this, observe that

$$\bar{u}_h + \bar{u}_\ell \leq u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) + u_h(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell))$$

(notice that we have changed the subscript from ℓ to h in the last term, thus explaining the inequality sign), and concavity of the utility function guarantees that

$$u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) + u_h(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell)) \leq u_h(\bar{a}_h) + u_h(\bar{a}_\ell)$$

Hence, $\bar{u}_h + \bar{u}_\ell \leq u_h(\bar{a}_h) + u_h(\bar{a}_\ell)$ while we also have $V_h(a) \geq V_\ell(a)$ for all a. Therefore (18) holds. Given (18), the only way that (17) can hold is that $u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h)$.

8.3. *Proof of Proposition 6*

From (15) as well as our direct matching mechanism, we obtain for any (a_h, a_ℓ)

$$V_h'(a_h) = \pi[u_h'(a_h + q^r(a_h, a_\ell)) + \beta V_h'(a_h)] + (1-\pi)U'(a_h) \tag{19}$$

as an agent h who did not switch only enters into repos with an agent ℓ who did not switch. Therefore,

$$V_h'(a_h) = \frac{\pi}{1-\beta\pi} u_h'(a_h + q^r(a_h, a_\ell)) + \frac{1-\pi}{1-\beta\pi} U'(a_h) \tag{20}$$

Also, from (16) and the direct matching rule, we have for any (a_h, a_ℓ)

$$V_\ell'(a_\ell) = (1-\pi)[u_h'(a_\ell + q^s(a_\ell, a_h) + q^r(a_\ell, a_h)) + \beta V_h'(a_\ell + q^s(a_\ell, a_h))] + \pi U'(a_\ell)$$

as a type ℓ who switched is matched with a type h who just switched.

Using the pairwise core allocations $q^s(a_\ell, a_h) = a_h - a_\ell$ and $q^r(a_\ell, a_h) = q^r(a_h, a_\ell)$ we obtain

$$V_\ell'(a_\ell) = (1-\pi)[u_h'(a_h + q^r(a_h, a_\ell)) + \beta V_h'(a_h)] + \pi U'(a_\ell)$$

and using (20) and simplifying we obtain[8]

$$V_\ell'(a_\ell) = \frac{1-\pi}{1-\beta\pi} u_h'(a_h + q^r(a_h, a_\ell)) + \frac{\pi}{1-\beta\pi} U'(a_\ell) \tag{21}$$

Simplifying, we obtain $V_h'(a_h) - V_\ell'(a_\ell) = 0$ if and only if

$$(2\pi - 1)V_h'(a_h) = \pi^2 U'(a_\ell) - (1-\pi)^2 U'(a_h)$$

Using equation (20) we simplify this expression to

$$u_h'(a_h + q^r) = \frac{1}{(2\pi - 1)} \{(1-\beta\pi)\pi U'(a_\ell) - (1-\beta(1-\pi))(1-\pi)U'(a_h)\} \tag{22}$$

as follows: Starting from

$$(2\pi - 1)V_1'(a_h) = \pi^2 U'(a_\ell) - (1-\pi)^2 U'(a_h)$$

use equation (20) to obtain

$$\frac{\pi(2\pi-1)}{1-\beta\pi} u_1'(a_h + q^r) + \frac{(1-\pi)(2\pi-1)}{1-\beta\pi} U'(a_h) = \pi^2 U'(a_\ell) - (1-\pi)^2 U'(a_h)$$

[8] or $V_2'(a_\ell) = \frac{1-\pi}{\pi}[V_1'(a_h) - (1-\pi)U'(a_h)] + \pi U'(a_\ell)$.

so that

$$\frac{\pi(2\pi-1)}{1-\beta\pi}u_1'(a_h+q^r)=\pi^2U'(a_\ell)-[(1-\pi)^2+\frac{(1-\pi)(2\pi-1)}{1-\beta\pi}]U'(a_h)$$

And arranging,

$$\begin{aligned}
u_1'(a_h+q^r) &= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-\frac{1-\beta\pi}{\pi(2\pi-1)}[(1-\pi)^2+\frac{(1-\pi)(2\pi-1)}{1-\beta\pi}]U'(a_h)\\
&= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-[\frac{(1-\beta\pi)(1-\pi)^2}{\pi(2\pi-1)}+\frac{(1-\pi)}{\pi}]U'(a_h)\\
&= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-\frac{(1-\pi)}{(2\pi-1)}[\frac{(1-\beta\pi)(1-\pi)}{\pi}+\frac{(2\pi-1)}{\pi}]U'(a_h)\\
&= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-\frac{(1-\pi)}{(2\pi-1)}[\frac{(1-\beta\pi)(1-\pi)+(2\pi-1)}{\pi}]U'(a_h)\\
&= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-\frac{(1-\pi)}{(2\pi-1)}[\frac{(1-\pi)-(\beta\pi)(1-\pi)+(2\pi-1)}{\pi}]U'(a_h)\\
&= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-\frac{(1-\pi)}{(2\pi-1)}[\frac{-(\beta\pi)(1-\pi)+(\pi)}{\pi}]U'(a_h)\\
&= \frac{(1-\beta\pi)\pi}{(2\pi-1)}U'(a_\ell)-\frac{(1-\pi)}{(2\pi-1)}[\frac{-(\beta)(1-\pi)+1}{1}]U'(a_h)\\
&= \frac{1}{(2\pi-1)}\left\{(1-\beta\pi)\pi U'(a_\ell)-(1-\beta(1-\pi))(1-\pi)U'(a_h)\right\}
\end{aligned}$$

The conditions for the core equilibrium then become

$$\begin{aligned}
u_1'(a_h+q^r) &= u_2'(a_\ell-q^r)\\
u_1'(a_h+q^r) &= \frac{1}{(2\pi-1)}\left\{(1-\beta\pi)\pi U'(a_\ell)-(1-\beta(1-\pi))(1-\pi)U'(a_h)\right\}\\
a_h+a_\ell &= A
\end{aligned}$$

8.4. Proof of Corollary 7

Notice that we can rewrite (13) as

$$u_h'(\bar{a}_h+q^r(\bar{a}_h,\bar{a}_\ell))=\alpha_\ell(\pi)U'(\bar{a}_\ell)-\alpha_h(\pi)U'(\bar{a}_h)$$

where

$$\alpha_\ell'(\pi)=\alpha_h'(\pi)=\frac{2\beta\pi(1-\pi)-1}{(2\pi-1)^2}<0$$

Therefore, using $\bar{a}_h + \bar{a}_\ell = A$ and the implicit function theorem, we have

$$u_h''(1 + \frac{\partial q^r}{\partial a_h})d\bar{a}_h = \alpha'(\pi)[U'(\bar{a}_\ell) - U'(\bar{a}_h)]d\pi - \alpha_\ell(\pi)U_\ell''(A - \bar{a}_h) + \alpha_h(\pi)U_h''(\bar{a}_h)]d\bar{a}_h \quad (23)$$

Notice first that so that $\frac{\partial q^r}{\partial \bar{a}_h} = -1$: Since $u_h'(\bar{a}_h + q^r) = u_\ell'(A - \bar{a}_h - q^r)$ we have

$$u_h''(1 + \frac{\partial q^r}{\partial \bar{a}_h}) = -u_\ell''(1 + \frac{\partial q^r}{\partial \bar{a}_h})$$

since $u_h'' < 0$ while $-u_\ell'' > 0$ the only solution is that

$$\frac{\partial q^r}{\partial \bar{a}_h} = -1 \quad (24)$$

Therefore using (23) we obtain

$$\frac{d\bar{a}_h}{d\pi} = \frac{\alpha'(\pi)[U'(\bar{a}_\ell) - U'(\bar{a}_h)]}{\alpha_\ell(\pi)U_2''(A - \bar{a}_h) + \alpha_h(\pi)U_1''(\bar{a}_h)}$$

Since both the denominator and the numerator are negative we have $d\bar{a}_h/d\pi > 0$.

Given π the volume of repos in this economy is given by q^r (since all agents use repo) while the volume of asset sales is given by $(1 - \pi)q^s = (1 - \pi)(\bar{a}_h - \bar{a}_\ell)$. Clearly, the sales volume is hump shaped as when $\pi = 1/2$ we have $\bar{a}_h = \bar{a}_\ell$ so that $q^s = 0$ while when $\pi = 1$, $q^s = 0$ as well. However, $(1 - \pi)q^s > 0$ for all other values of π. Since the problem is continuous, sales volume is hum-shaped. Also, (24) implies that the total volume of repo is declining in π. Since there are no repo when $\pi = 1$, the volume of repo is declining to zero.

8.5. *Proof of Proposition 8*

We need to show that no 1 or 2 agent(s) wish(es) to form a coalition and be better off. It should be clear that no 1 agent wants to form a coalition (this option is already embedded in the bargaining problem).

Now, an agent ℓ with $\bar{a}_h$ could decide to form a coalition with an agent ℓ with $\bar{a}_\ell$ or an agent h with $\bar{a}_h$. It is a property of the bargaining solution that an agent ℓ will obtain a lower payoff being matched with an agent h with a higher amount of asset (he can extract less since the marginal utility of obtaining more of the asset is lower for this agent). Hence, an agent ℓ with $\bar{a}_h$ prefers to be matched with an agent h with $\bar{a}_\ell$. Also, it is a property of the bargaining solution that, given he has to meet an agent with asset holdings a, an ℓ

agents prefer to be matched with the agent with the highest marginal utility (so agent h).[9]

Also, an agent h with $\bar{a}_\ell$ could decide to form a coalition with an agent h with $\bar{a}_h$ or an agent ℓ with $\bar{a}_\ell$. As above, however, it is a property of the bargaining solution that an agent h payoff matched with an agent ℓ will get a higher utility whenever the agent ℓ is holding more asset. Hence, the agent h will not want to be matched with an agent ℓ holding $\bar{a}_\ell$. Also, an agent h with $\bar{a}_h$ prefers to be matched with the agent holding $\bar{a}_\ell$ with the lowest marginal utility, i..e with an ℓ agent.

Hence there are no 2-agents coalition where both agents would do better than under the prescribed matching technology, which shows that it, together with the distribution over $\{\bar{a}_h, \bar{a}_\ell\}$ is an equilibrium.

Proof of Proposition 9

The value functions are

$$\begin{aligned}
V_h(\bar{a}_h) &= \pi[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] + (1-\pi)[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] \\
V_\ell(\bar{a}_\ell) &= \pi[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] + (1-\pi)[u_h(\bar{a}_h + q^r) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h)]
\end{aligned}$$

Adding both equations, we obtain

$$V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) = \frac{u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r)}{1-\beta} \tag{25}$$

Also

$$\begin{aligned}
V_h(\bar{a}_\ell) &= \pi[u_h(\bar{a}_h + q^r) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h)] + (1-\pi)[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] \\
V_\ell(\bar{a}_h) &= \pi[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] + (1-\pi)[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)]
\end{aligned}$$

and adding both equations, we obtain also

$$V_h(\bar{a}_\ell) + V_\ell(\bar{a}_h) = V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) \tag{26}$$

[9] It is easy to show this with $u_h(a) = \alpha u_\ell(a)$ with $\alpha > 1$: Compute the bargainaing solution and show that $\partial[u_\ell(a_\ell - q^r - q^s) + \beta V(a - q^s) + d]/\partial\alpha > 0$.

From the bargaining first order condition, we obtain

$$d(a_h, a_\ell) = (1-\theta)[u_h(a_h+q^s+q^r)-u_h(a_h)+\beta V_h(a_h+q^s)-\beta V_h(a_h)]-\theta[u_\ell(a_\ell-q^s-q^r)-u_\ell(a_\ell)+\beta V_\ell(a_\ell-q^s)-\beta V$$

so that the transfer $d(\bar{a}_h, \bar{a}_\ell)$ is

$$\begin{aligned} d(\bar{a}_h, \bar{a}_\ell) &= (1-\theta)[u_h(\bar{a}_h + q^s + q^r) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h + q^s) - \beta V_h(\bar{a}_h)] \\ &\quad -\theta[u_\ell(\bar{a}_\ell - q^s - q^r) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell - q^s) - \beta V_\ell(\bar{a}_\ell)] \\ &= (1-\theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] - \theta[u_\ell(\bar{a}_\ell - q^r) - u_\ell(\bar{a}_\ell)] \end{aligned}$$

where we have used the fact that $q^s(\bar{a}_h, \bar{a}_\ell) = 0$. Therefore, we obtain

$$\begin{aligned} u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) &= \\ u_h(\bar{a}_h + q^r) - (1-\theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] + \theta[u_\ell(\bar{a}_\ell - q^r) - u_\ell(\bar{a}_\ell)] + \beta V_h(\bar{a}_h) &= \\ u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) + \theta[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)] & \end{aligned}$$

Also

$$\begin{aligned} u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) &= \\ u_\ell(\bar{a}_\ell - q^r) + (1-\theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] - \theta[u_\ell(\bar{a}_\ell - q^r) - u_\ell(\bar{a}_\ell)] + \beta V_\ell(\bar{a}_\ell) &= \\ u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) + (1-\theta)[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)] & \end{aligned}$$

In a similar fashion, we obtain (using $q^s(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell$)

$$\begin{aligned} d(\bar{a}_\ell, \bar{a}_h) &= (1-\theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_h) - \beta V_h(\bar{a}_\ell)] \\ &\quad -\theta[u_\ell(\bar{a}_\ell - q^r) - u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - \beta V_\ell(\bar{a}_h)] \end{aligned}$$

It is easy to rewrite it as

$$d(\bar{a}_\ell, \bar{a}_h) = d(\bar{a}_h, \bar{a}_\ell) + \bar{u} + \beta(1-\theta)[V_h(\bar{a}_h) - V_h(\bar{a}_\ell)] + \beta\theta[V_\ell(\bar{a}_h) - V_\ell(\bar{a}_\ell)]$$

where

$$\bar{u} = (1-\theta)[u_h(\bar{a}_h) - u_h(\bar{a}_\ell)] + \theta[u_\ell(\bar{a}_h) - u_\ell(\bar{a}_\ell)]$$

Therefore,

$$\begin{aligned}
u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell) &= \\
u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h) + (1-\theta)[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) & \\
+\beta V_h(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h) - \beta V_h(\bar{a}_\ell) - \beta V_\ell(\bar{a}_h)] &= \\
u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h) + (1-\theta)[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h)] &=
\end{aligned}$$

where we have used (26), and similarly

$$\begin{aligned}
u_h(\bar{a}_h + q^r) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h) &= \\
u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell) + \theta[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) & \\
+\beta V_h(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h) - \beta V_h(\bar{a}_\ell) - \beta V_\ell(\bar{a}_h)] &= \\
u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell) + \theta[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h)] &
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
V_h(\bar{a}_h) &= \pi[u_h(\bar{a}_h) + \beta V_h(\bar{a}_h)] + (1-\pi)[u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h)] + \pi\theta S + (1-\pi)(1-\theta)\tilde{S} \\
V_\ell(\bar{a}_\ell) &= \pi[u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] + (1-\pi)[u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)] + \pi(1-\theta)S + (1-\pi)\theta\tilde{S} \\
V_h(\bar{a}_\ell) &= \pi[u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)] + (1-\pi)[u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] + (1-\pi)(1-\theta)S + \pi\theta\tilde{S} \\
V_\ell(\bar{a}_h) &= \pi[u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h)] + (1-\pi)[u_h(\bar{a}_h) + \beta V_h(\bar{a}_h)] + (1-\pi)\theta S + \pi(1-\theta)\tilde{S}
\end{aligned}$$

where

$$\begin{aligned}
S &= u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell) \\
\tilde{S} &= u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h)
\end{aligned}$$

Solving for $V_h(\bar{a}_h)$ we obtain

$$(1-\beta)V_h(\bar{a}_h) = \frac{(1-\pi)[u_\ell(\bar{a}_h) + (1-\theta)\tilde{S}] + [\pi - (2\pi-1)\beta][u_h(\bar{a}_h) + \theta S]}{1-(2\pi-1)\beta}$$

And taking the derivative, we have

$$(1-\beta)V_h'(\bar{a}_h) = \frac{u_\ell'(\bar{a}_h)(1-\pi) + [\pi - (2\pi-1)\beta]u_h'(\bar{a}_h) + (1-\pi)(1-\theta)\frac{\partial\tilde{S}}{\partial\bar{a}_h} + \theta\frac{\partial S}{\partial\bar{a}_h}[\pi - (2\pi-1)\beta]}{1-(2\pi-1)\beta}$$

and using the first order condition for q^r we obtain

$$\begin{aligned}(1-\beta)(1-(2\pi-1)\beta)V_h'(\bar a_h) &= u_\ell'(\bar a_h)(1-\pi)+[\pi-(2\pi-1)\beta]u_h'(\bar a_h)\\ &\quad +(1-\pi)(1-\theta)[u_h'(\bar a_h+q^r)-u_\ell'(\bar a_h)]\\ &\quad +\theta[u_h'(\bar a_h+q^r)-u_h'(\bar a_h)][\pi-(2\pi-1)\beta]\end{aligned}$$

so that after some simplifications,

$$\begin{aligned}(1-\beta)(1-(2\pi-1)\beta)V_h'(\bar a_h) &= u_\ell'(\bar a_h)\theta(1-\pi)+u_h'(\bar a_h)(1-\theta)[\pi-(2\pi-1)\beta]\\ &\quad +u_h'(\bar a_h+q^r)[1-\pi+(2\pi-1)(1-\beta)\theta]\end{aligned}$$

Since (25) holds, and using the first order condition for q^r we obtain

$$\begin{aligned}(1-\beta)(1-(2\pi-1)\beta)V_\ell'(\bar a_\ell) &= u_\ell'(\bar a_\ell-q^r)(1-(2\pi-1)\beta)-(1-\beta)(1-(2\pi-1)\beta)\frac{\partial V_h(\bar a_h)}{\partial \bar a_\ell}\\ (1-\beta)(1-(2\pi-1)\beta)V_\ell'(\bar a_\ell) &= u_\ell'(\bar a_\ell-q^r)(1-(2\pi-1)\beta)\\ &\quad -(1-\pi)(1-\theta)\frac{\partial \tilde S}{\partial \bar a_\ell}-\theta\frac{\partial S}{\partial \bar a_\ell}[\pi-(2\pi-1)\beta]\\ (1-\beta)(1-(2\pi-1)\beta)V_\ell'(\bar a_\ell) &= u_\ell'(\bar a_\ell-q^r)(1-(2\pi-1)\beta)\\ &\quad -u_\ell'(\bar a_\ell-q^r)[1-\pi+(2\pi-1)(1-\beta)\theta]\\ &\quad +(1-\pi)(1-\theta)u_h'(\bar a_\ell)+\theta[\pi-(2\pi-1)\beta]u_\ell'(\bar a_\ell)\\ (1-\beta)(1-(2\pi-1)\beta)V_\ell'(\bar a_\ell) &= u_\ell'(\bar a_\ell-q^r)[\pi-(2\pi-1)(\beta+(1-\beta)\theta)]+(1-\pi)(1-\theta)u_h'(\bar a_\ell)\\ &\quad +\theta[\pi-(2\pi-1)\beta]u_\ell'(\bar a_\ell)\end{aligned}$$

The first condition for q^s imposes that $V_h'(\bar a_h)=V_\ell'(\bar a_\ell)$. Using the fact that $u_\ell'(\bar a_\ell-q^r)=u_h'(\bar a_h+q^r)$ and simplifying, we obtain

$$u_h'(\bar a_h+q^r)=\frac{[\pi-(2\pi-1)\beta][\theta u_\ell'(\bar a_\ell)-(1-\theta)u_h'(\bar a_h)]-(1-\pi)[\theta u_\ell'(\bar a_h)-(1-\theta)u_h'(\bar a_\ell)]}{(2\pi-1)(1-\beta)(2\theta-1)}$$

Together with the first order condition on asset sales and the feasibility constraint, this completes the proof.

$$V_h(\bar a_h) = \pi[u_h(\bar a_h)+\beta V_h(\bar a_h)]+(1-\pi)[u_\ell(\bar a_h)+\beta V_\ell(\bar a_h)]+\pi\theta S+(1-\pi)(1-\theta)\tilde S$$

$$
\begin{aligned}
V_h(\bar{a}_h) &= \pi[u_h(\bar{a}_h) + \beta V_h(\bar{a}_h)] + (1-\pi)[u_\ell(\bar{a}_h) + \beta \frac{\pi u_\ell(\bar{a}_h) + (1-\pi)u_h(\bar{a}_h) + \beta(1}{} \\
V_h(\bar{a}_h)(1 - \pi\beta - \frac{(1-\pi)^2\beta^2}{1-\pi\beta}) &= \pi u_h(\bar{a}_h) + (1-\pi)[u_\ell(\bar{a}_h) + \beta \frac{\pi u_\ell(\bar{a}_h) + (1-\pi)u_h(\bar{a}_h) + (1-\pi)\theta S + \pi(}{1-\pi\beta} \\
V_h(\bar{a}_h)(1 - \pi\beta - \frac{(1-\pi)^2\beta^2}{1-\pi\beta}) &= \pi u_h(\bar{a}_h) + (1-\pi)[u_\ell(\bar{a}_h) + \beta \frac{\pi u_\ell(\bar{a}_h) + (1-\pi)u_h(\bar{a}_h)}{1-\pi\beta} + \beta \frac{(1-\pi)\theta S +}{1-\pi} \\
&= \pi u_h(\bar{a}_h) + (1-\pi)u_\ell(\bar{a}_h) + \beta(1-\pi)\frac{\pi u_\ell(\bar{a}_h) + (1-\pi)u_h(\bar{a}_h)}{1-\pi\beta} + (1-\pi)\frac{(}{} \\
&= \pi[u_h(\bar{a}_h) + \theta S] + (1-\pi)[u_\ell(\bar{a}_h) + (1-\theta)\tilde{S} + \beta \frac{\pi[u_\ell(\bar{a}_h) + (1-\theta)\tilde{S}] + (}{1-\pi} \\
&= \pi[u_h(\bar{a}_h) + \theta S] + \beta \frac{(1-\pi)^2[u_h(\bar{a}_h) + \theta S]}{1-\pi\beta} + (1-\pi)[\frac{u_\ell(\bar{a}_h) + (1-\theta)\tilde{S}}{1-\pi\beta}] \\
&= \frac{[\pi + \beta(1-2\pi)][u_h(\bar{a}_h) + \theta S]}{1-\pi\beta} + (1-\pi)[\frac{u_\ell(\bar{a}_h) + (1-\theta)\tilde{S}}{1-\pi\beta}] \\
(1-\beta)V_h(\bar{a}_h)\frac{(1+\beta(1-2\pi))}{1-\pi\beta} &= \frac{[\pi + \beta(1-2\pi)][u_h(\bar{a}_h) + \theta S]}{1-\pi\beta} + (1-\pi)[\frac{u_\ell(\bar{a}_h) + (1-\theta)\tilde{S}}{1-\pi\beta}] \\
(1-\beta)V_h(\bar{a}_h)(1+\beta(1-2\pi)) &= \pi[u_h(\bar{a}_h) + \theta S] + (1-\pi)[u_\ell(\bar{a}_h) + (1-\theta)\tilde{S}] + \beta(1-2\pi)][u_h(\bar{a}_h) + \theta S] \\
&= \pi[u_h(\bar{a}_h) + \theta S + \beta(1-2\pi)[u_h(\bar{a}_h) + \theta S]] + (1-\pi)[u_\ell(\bar{a}_h) + (1-\theta)\tilde{S} + \\
(1-\beta)V_h(\bar{a}_h) &= u_h(\bar{a}_h) + \theta S - \frac{(1-\pi)}{1-\beta(2\pi-1)}[u_\ell(\bar{a}_h) + (1-\theta)\tilde{S} - u_h(\bar{a}_h) - \theta S]
\end{aligned}
$$

And for V_ℓ

$$
\begin{aligned}
V_\ell(\bar{a}_\ell) &= \pi[u_\ell(\bar{a}_\ell) + (1-\theta)S + \beta V_\ell(\bar{a}_\ell)] + (1-\pi)[u_h(\bar{a}_\ell) + \theta\tilde{S}] + \beta(1-\pi)V_h(\bar{a}_\ell) \\
V_h(\bar{a}_\ell) &= \frac{\pi}{1-\beta\pi}[u_h(\bar{a}_\ell) + \theta\tilde{S}] + \frac{(1-\pi)}{1-\beta\pi}[u_\ell(\bar{a}_\ell) + (1-\theta)S + \beta V_\ell(\bar{a}_\ell)]
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
V_\ell(\bar{a}_\ell) &= [\pi + \beta\frac{(1-\pi)^2}{1-\beta\pi}][u_\ell(\bar{a}_\ell) + (1-\theta)S + \beta V_\ell(\bar{a}_\ell)] + [1-\pi+\beta\frac{(1-\pi)\pi}{1-\beta\pi}][u_h(\bar{a} \\
(1-\beta)V_\ell(\bar{a}_\ell)[\frac{1+\beta(1-2\pi)}{1-\beta\pi}] &= [\pi + \beta\frac{(1-\pi)^2}{1-\beta\pi}][u_\ell(\bar{a}_\ell) + (1-\theta)S] + \frac{1-\pi}{1-\beta\pi}[u_h(\bar{a}_\ell) + \theta\tilde{S}] \\
(1-\beta)V_\ell(\bar{a}_\ell)[\frac{1+\beta(1-2\pi)}{1-\beta\pi}] &= \frac{1+\beta(1-2\pi)}{1-\beta\pi}[u_\ell(\bar{a}_\ell) + (1-\theta)S] + \frac{1-\pi}{1-\beta\pi}[u_h(\bar{a}_\ell) + \theta\tilde{S} - u_\ell(\bar{a}_\ell) - (1- \\
(1-\beta)V_\ell(\bar{a}_\ell) &= u_\ell(\bar{a}_\ell) + (1-\theta)S - \frac{(1-\pi)}{1-\beta(2\pi-1)}[u_\ell(\bar{a}_\ell) + (1-\theta)S - u_h(\bar{a}_\ell) - \theta\tilde{S}]
\end{aligned}
$$