Optimal Orchestration of Rewards and Punishments in Rank-Order Contests*

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Abstract

We allow negative prizes and investigate effort-maximizing prize design in rank-order contests with incomplete information. Endogenous participation arises due to less-efficient types' incentive to avoid punishments. The optimum features winner-take-all for the best performer and at most one punishment for the worst performer among all potential contestants, whenever they enter the competition. Based on this, we then (1) provide a necessary and sufficient condition for the optimality of pure winner-take-all without punishment; (2) show that the optimal entry threshold increases with the total number of contestants and converges to the Myerson cutoff in the limit.

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1 Introduction

The practice of utilizing "sticks" jointly with "carrots" in incentivizing agents has long and widely been observed. In a contest environment, a positive prize can be viewed as a carrot and a negative prize can be viewed as a stick. Negative prizes (punishments) in contests are prevalent in practice.

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As noted by Liu et al. (2018) and Hammond et al. (2019), the widely used entry fees in contests are effectively negative prizes. Many firms use tournaments with punishments to incentivize employees—"General Electric, Metlife, Microsoft, American Express, AIG, Hewlett Packard, and Yahoo! have used tournaments that incorporate both rewards and punishment, with punishments coming in such forms as job reassignment, demotion, or even firing" (Newman and Tafkov, 2014). Examples of punishments in contests also include, for instance, an F grade for students, relegation in sports leagues, among others.

Prize allocation has long been recognized as a main instrument to incentivize contestants to exert productive effort. However, despite the fact that negative prizes (sticks) are widely adopted in practice and they are a powerful tool in eliciting effort supply, a vast majority of the prize design literature has been focusing on analyzing positive prizes (carrots), with only a few exceptions. A celebrated result is established by Lazear and Rosen (1981), who show that rank-order contests with negative prizes can achieve the first best in an environment of complete information. In environments with incomplete information, however, introducing negative prizes into the analysis inevitably entails the issue of endogenous entry of contestants,[1] which would often make the analysis cumbersome, intractable, and inelegant.

A complete characterization of the optimal rank-based prize design while allowing negative prizes in all-pay contests with incomplete information has remained an open question since Moldovanu and Sela's (2001) seminal work, which largely focuses on positive prizes. Nevertheless, several studies have made important progress by considering special prize structures. Moldovanu, Sela, and Shi (2012) study both cases with exogenous and endogenous entry. For the case of endogenous entry, they assume a single positive prize for the top performer and a fixed number of uniform negative prizes for the bottom entrants.[2] Thomas and Wang (2013) and Kamijo (2016) focus on a single positive prize for the top performer and a single negative prize for the worst performer among all entrants in their analysis. Hammond et al. (2019) study prize-augmenting entry fees, which can be viewed as uniform negative prizes.

This paper aims to draw a clearer picture of this open question by investigating an important and natural class of prize structures—prize structures that only depend on ranks of contestants' performance. We provide a complete and elegant characterization of the optimum by coming up with a tractable and innovative procedure; in particular, we do not impose any further assumptions—such

[1] Please refer to Moldovanu, Sela, and Shi (2012) and Thomas and Wang (2013) for complications due to this issue.

[2] They assume that if there is a single entrant, then he gets both the positive prize and a negative prize.

as the number of positive prizes, the number of negative prizes, and uniform negative prizes—on the prize structure.

To be specific, our analysis essentially allows prizes to be negative in the incomplete information all-pay auction model of Moldovanu and Sela (2001). Each contestant is endowed with his private type, which is his marginal cost of exerting effort. The organizer, who has a fixed budget, designs a prize structure $\mathbf{v} = (v_1, \ldots, v_N)$ with $v_1 \geq v_2 \geq \cdots \geq v_N$ to maximize the total effort from N contestants, where v_n, which can be nonnegative or negative, is the prize for the participant with the nth highest effort. A nonparticipant receives no prize. Negative prizes in general lead to endogenous entry of contestants because of the participation constraint, so the actual number of entrants can be endogenous. The organizer must observe ex post budget constraints for all possible numbers of entrants, and she is allowed to value any leftover budget.

The equilibrium endogenous participation induced by a prize structure is a threshold-entry, in which only contestants with types higher than a certain threshold enter the contest. We first identify the equilibrium bidding function and entry threshold for any given prize structure. Based on these, we find that the organizer's problem, for any fixed entry threshold, is essentially a linear programming, in the sense that all of the functions in the problem—the objective function, the budget constraint, and the participation constraint—are linear in the N prizes. Nevertheless, coefficients in this linear programming problem vary in a highly intractable and nonlinear way as the induced entry threshold changes, which makes solving the problem challenging.

A typical procedure is to first identify the constrained optimum among all feasible prize structures that induce the same entry threshold, and then vary across all possible entry thresholds in the type space to pin down the universal optimum. However, this approach is rather infeasible because of the aforementioned arbitrary and intractable behavior of the coefficients in the linear programming problem, which generates complicacy in characterizing the optimum for a fixed entry threshold. We illustrate that the number of positive prizes and the number of negative prizes at the optimum for a fixed entry threshold can vary with the threshold. In particular, in general, multiple positive prizes and multiple negative prizes can arise as the optimum for fixed thresholds. Such arbitrariness in the number of prizes constitutes the main challenge of pinning down the optimal design.

We manage to tackle this difficulty by developing an innovative procedure. A first key step relies on discovering useful relations among positive coefficients of prizes in the total effort function.

This is made possible by observing a hazard rate dominance result of relevant order statistics. Specifically, these positive coefficients must be associated with higher prizes, and the ratios of these coefficients to their counterparts in the equilibrium entry condition are decreasing in ranks. Moreover, more coefficients become positive as the entry threshold gets higher. A second key step shows that when an entry threshold is sufficiently high such that all coefficients become positive, then the corresponding optimal prize structure can have only one negative prize. A third key step shows that for any fixed prize structure, we can construct an alternative prize structure to dominate it by varying only the last prize, which can induce the minimal entry threshold t^* that makes all coefficients in the total effort function positive. Therefore, the entry threshold t^* must be optimal and there can be only one negative prize at the optimum. A final step establishes that a single positive prize equal to the whole initial budget is universally optimal, relying on the key relations among ratios of coefficients in the objective function and the equilibrium entry condition, which is discovered in step one.

The optimal prize structure thus takes an elegant form of $\mathbf{v}^* = (v_1^*, 0, \ldots, 0, v_N^*)$: A single positive prize $v_1^* > 0$ equal to the budget and a single negative prize $v_N^* \leq 0$ that supports the optimal entry threshold t^*. In other words, the optimum features winner-take-all for the best performer and at most one punishment for the worst performer among all N potential contestants, whenever they enter the competition. We find that the opposite of the coefficient of v_N in the objective function can be interpreted as the marginal revenue gained by imposing an extra unit of punishment through the last negative prize. Therefore, the optimum is achieved when this marginal revenue is precisely zero. Based on this observation, we (1) further provide a necessary and sufficient condition for the optimality of pure winner-take-all (i.e., no negative prize); (2) find that the optimal entry threshold increases with N and converges to the Myerson cutoff (Myerson, 1981) in the limit.

Our analysis addresses the open question of optimal prize design when negative prizes are introduced in Moldovanu and Sela's (2001) setting. In the literature of rank-order contests with incomplete information, to the best of our knowledge, only Moldovanu, Sela, and Shi (2012), Thomas and Wang (2013), Kamijo (2016), and Hammond et al. (2019) study negative prizes in contests. As mentioned above, differing from these papers, we do not impose assumptions on the number of positive prizes, the number of negative prizes, or uniform negative prizes. Moreover, the class of prize structures in our paper depends only on rank information of contestants' performance, which is simple to implement in practice and naturally extends Moldovanu and Sela (2001). On the other hand, Moldovanu and Sela (2001) conjecture (in their Section IV) that the effort level of Myerson's optimal seller revenue should be implementable by charging an appropriate entry fee while maintaining a single prize for the highest bidder among entrants, which equals the organizer's initial prize budget. Since their prize design with entry fees is a special case of our prize structures, our analysis reveals that their prize design with entry fees is generally suboptimal, unless the optimal entry fee is zero.

In a closely related paper, Liu et al. (2018) adopt a mechanism design approach to investigate the role of negative prizes. We would like to first emphasize that since they assume that negative prizes can top up the prize budget—which is not the case in our setting—the driving forces of the optimality of having a negative prize in our setting is dramatically different from those in their setting. They find that imposing an arbitrarily high entry fee and a minimum bid can extract nearly all surplus from contestants and achieve a level of total effort that is inducible in an environment in which all contestants are of the most efficient type with certainty. In their optimal contest, all collected entry fees, together with the initial budget, are awarded to the highest bidder if the bid is above the minimum bid. If everyone bids zero, the initial budget and collected entry fees are randomly allocated to each contestant to maintain the entry incentive of all types. Therefore, their optimal design crucially relies on budget-augmenting negative prizes and the information on the level of bids, which are exactly the key differences between their model and ours—they allow information on the level of contestants' performance to be used in the mechanism, which is not feasible in rank-order contests; moreover, negative prizes cannot augment the prize budget in our setting, as the organizer has to observe ex post budget constraints for all possible numbers of entrants. As such, in their paper, all types of contestants enter the contest at the optimum, so their analysis does not involve the issue of endogenous entry.

Our paper contributes to the literature on optimal prize allocation in all-pay auctions with incomplete information. In addition to the pioneering work of Moldovanu and Sela (2001) and the papers mentioned above, other important contributions include the investigation of a two-stage all-pay auction framework (Moldovanu and Sela, 2006), the environment in which contestants care about their relative status (Moldovanu, Sela, and Shi, 2007), endogenous contest success functions (Polishchuk and Tonis, 2013), the analysis of optimal crowdsourcing contests (Chawla, Hartline, and Sivan, 2019), innovation contests (Erkal and Xiao, 2019), contests with entry costs (Liu and

Lu, 2019), and large contests (Olszewski and Siegel, 2020). All of these papers assume positive prizes.

Our paper is also related to the literature on prize design in all-pay contests with complete information. Recently, Ghosh and Hummel (2018) introduce cardinal information on contestants' performance to rank-order contests; Xiao (2019) studies ability grouping and characterizes the optimal prize structure; Fang, Noe, and Strack (2020) find that more unequal prize structures lead to lower effort provision, when contestants are symmetric and the effort cost is convex; Letina, Liu, and Netzer (2020) provide a general approach to study the contest design when the organizer can choose both the prize profile and the contest success function.

The rest of the paper is organized as follows. In Section 2, we set up the model. Section 3 presents the equilibrium analysis. We provide the analysis of non-contingent optimal design with negative prizes in Section 4. Section 5 concludes. Technical proofs are relegated to the appendix.

2 The Model

A risk-neutral contest organizer has a fixed budget $V(>0)$ to elicit effort from N (≥ 2) risk-neutral potential contestants. For contestant i, his cost of exerting effort $e_i(\geq 0)$ is e_i/t_i, where t_i is his private information. We assume that t_i's are independently and identically distributed with cumulative distribution function $F(\cdot)$ on the support $[a,b]$ with $a>0$. The corresponding probability density function $f(\cdot)$ is strictly positive everywhere on $[a,b]$.

We consider prize allocation rules that only depend on ranks of contestants' performance. Formally, a *prize structure* is an N-vector $\mathbf{v}=(v_1,v_2,\ldots,v_N)$ with $v_1\geq v_2\geq\cdots\geq v_N$, where v_n is the prize for the contestant with the nth highest effort. A key feature is that, similar to Liu et al. (2018), prizes can be negative. The possibility of ending up with negative prizes would, in general, result in endogenous entry of contestants. Throughout the paper, we use *scenario* n to refer to the situation that n entrants participate in the competition. The prize allocation rule goes as follows: When there are n entrants, they win the respective first n prizes $v_1,v_2,\ldots,v_n$ from $\mathbf{v}$ according to their performance—that is, the highest entrant wins v_1, the second highest entrant wins v_2, ... , and the last entrant wins v_n; ties are broken randomly and fairly.

Since prize allocation rules only rely on ranks, the prize allocation rule should be a fixed, N-dimensional vector, as formulated above. Further note that the number of entrants is endogenous.

Thus, the organizer's budget constraint requires that the sum of prizes cannot exceed V for any possible number of entrants. In other words, the budget constraint requires that $\sum_{i=1}^{n} v_i \leq V$ for any $n = 1, 2, \ldots, N$. Note that unlike Liu et al. (2018) in which negative prizes can augment the prize budget, the (ex post) budget constraints in our setting exclude this possibility.

A participant's payoff is equal to the prize he receives minus the cost of exerting effort; if a contestant does not enter the contest, he receives his outside option, which is normalized as 0. The contest organizer's goal is to design a prize structure to maximize the expected total effort using her budget; at the same time, if there is money left in the budget, she values that money as well. Assume that there is a linear relationship between effort and money: 1 dollar is worth t_0 (≥ 0) units of effort for the organizer.[3] Note that the cost of 1 unit of effort for the maximum ability (b) type is $1/b$, which needs to be less than $1/t_0$; otherwise, it is optimal for the organizer not to spend any of the prize budget. Therefore, we assume that $t_0 < b$.

We call the organizer's benefit from the leftover budget *effort-equivalent*. Thus, her goal is to maximize the sum of the expected total effort induced and the effort-equivalent of the leftover budget. For simplicity and clarity, we will just say that the organizer maximizes the *expected overall effort*, which covers these two components.

$F(\cdot)$, t_0, and N are public information. The timing of the game is as follows.

Time 0: Each contestant privately learns his type.

Time 1: The organizer chooses $\mathbf{v}$ and commits to it. The prize structure $\mathbf{v}$ is announced.

Time 2: All potential contestants decide whether to participate in the contest. If a contestant decides to enter, he observes the number of rival(s) and then exerts effort.

Time 3: The contest rule is implemented according to the one announced at *time* 1.

As a special case of our model setup, the prize structure $\mathbf{v} = (V - E, -E, \ldots, -E)$ corresponds to the discussion of using an entry fee $E \geq 0$ to screen contestants in Section IV of Moldovanu and Sela (2001): That is, the organizer charges an entry fee E from every entrant and all prize budget V is awarded to the highest bidder of all entrants. Our formulation of t_0 captures the organizer's payoff from entry fees collected in this case. In fact, suppose that there are n entrants, then the leftover budget is $V - (V - nE) = nE$, which is precisely the amount of entry fees collected and

[3] All of our results hold even when $t_0 = 0$—i.e., when the organizer does not value any leftover budget. In particular, we will show that at the optimum, a negative prize can arise even when $t_0 = 0$. Note that, alternatively, $1/t_0$ can be understood as the organizer's marginal cost of exerting effort.

generates t_0nE units of effort for the organizer.

Define $J(t) = t - \frac{1-F(t)}{f(t)}$. We make the following standard assumption.[4]

Assumption 1 (Regularity)**.** *$J(t)$ is strictly increasing in t.*

Since entry is generally endogenous, the analysis of the optimal design can be complicated. We tackle the problem in the following way: We first characterize the symmetric monotone bidding equilibrium for any given prize structure. We then characterize several important features of the problem for any *fixed* entry threshold. Finally, we pin down the optimal entry threshold and thus the optimal prize structure.

3 Equilibrium Analysis

As a first step, we characterize the entry and bidding equilibrium for any given prize structure $\mathbf{v}$, which forms the base for the search of the optimal prize structure. We focus on symmetric monotone bidding equilibria in our analysis.

3.1 Equilibrium Entry

A given prize structure $\mathbf{v}$ induces an entry threshold t^c such that types higher than t^c enter with probability one, while types lower than it enter with probability zero. It is clear that the threshold type must bid zero at equilibrium and enjoys a zero expected payoff.[5] Note that type t^c obtains the lowest prize v_n for sure in scenario n for any n in any monotone bidding equilibrium. Therefore, t^c is characterized by

$$\sum_{n=1}^{N} p_n(t^c)v_n = 0, \tag{1}$$

where $p_n(t^c) = \binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)$ is the probability that there are $n-1$ rival(s) from an entrant's perspective.

There are two special cases of equilibrium entry, which are full entry ($t^c = a$) and no entry ($t^c = b$). If the prize structure induces full entry, it must be the case that $v_N \geq 0$, so $v_1 \geq \ldots \geq v_N \geq 0$—i.e., all prizes are nonnegative. This full-entry (with nonnegative prizes) case is

[4] This assumption is used to establish Lemma 5.

[5] The threshold type may enjoy a positive payoff when the prize structure induces full entry (i.e., $t^c = a$). However, this is clearly not optimal. See the discussion after Lemma 1.

studied in Moldovanu and Sela (2001). If the prize structure induces no entry, then this means that the organizer cancels the contest and enjoys a t_0V overall effort. In general, when the entry is stochastic—i.e., the prize structure $\mathbf{v}$ induces a threshold $t^c \in (a,b)$—the induced entry threshold must be unique. The following result provides details about these. (All proofs are relegated to the Appendix.)

Lemma 1. *For any prize structure* $\mathbf{v}$*, it induces a unique entry threshold, which can be full entry, no entry, or a unique interior threshold* $t^c \in (a,b)$*; more specifically, it induces: (i) full entry if and only if* $v_N \geq 0$*; (ii) no entry if and only if* $v_1 \leq 0$*; (iii) a unique interior threshold if and only if* $v_1 > 0$ *and* $v_N < 0$*.*

Notice that when the prize structure $\mathbf{v}$ induces an interior entry threshold, (1) must hold, as the threshold type must enjoy a zero expected payoff. However, when $\mathbf{v}$ induces full entry, (1) may not hold because the threshold type, type a, may enjoy a strictly positive payoff—i.e., $\sum_{n=1}^{N} p_n(a)v_n > 0$. However, this means that $v_N > 0$, which is obviously suboptimal, as the organizer can induce a strictly higher expected overall effort by reducing v_N by $\varepsilon = v_N$ and increasing v_1 by ε, which still induces full entry.[6] Therefore, whenever the prize structure induces entry (full entry or an interior entry threshold), there is no loss of generality to assume that (1) holds.

3.2 Scenario Bidding Functions and The Overall Effort

For any given prize structure $\mathbf{v}$, suppose that it induces an entry threshold $t^c \in [a,b)$ determined by (1). An entrant knows that his rivals' types are independently drawn from the truncated cumulative distribution function $G(t,t^c) = \frac{F(t)-F(t^c)}{1-F(t^c)}$, with density function $g(t,t^c) = \frac{f(t)}{1-F(t^c)}$, $t \in [t^c, b]$.

Suppose that there are n entrants. The following result characterizes the entrant's bidding function, given prizes $(v_1, v_2, \ldots, v_n)$ in scenario n.

Proposition 1. *Suppose that the prize structure* $\mathbf{v}$ *induces the (unique) threshold* t^c*. In scenario* $n \geq 1$ *with prizes* $(v_1, v_2, \ldots, v_n)$*,*

(i) The unique symmetric monotone bidding function $e^{(n)}(t, \mathbf{v}, t^c)$ *for type* $t \in [t^c, b]$ *is*

$$e^{(n)}(t,\mathbf{v},t^c) = tV^{(n)}(t) - \int_{t^c}^{t} V^{(n)}(s)ds - t^c v_n,$$

[6] This argument of the full-entry case follows from the analysis in Moldovanu and Sela (2001), because all N prizes must be nonnegative.

where

$$V^{(n)}(t) = \sum_{j=1}^{n} v_{n+1-j} \binom{n-1}{j-1} G^{j-1}(t, t^c)(1 - G(t, t^c))^{n-j}$$

is the expected prize an entrant with type t obtains.

(ii) The corresponding scenario-n expected overall effort is

$$TE^{(n)}(\mathbf{v}, t^c) = n \int_{t^c}^{b} J(t)V^{(n)}(t)g(t, t^c)dt - nt^c v_n + t_0(V - \sum_{j=1}^{n} v_j).$$

In scenario-0, no one enters, so the organizer's payoff is simply the benefit from her budget. Thus, the scenario-0 expected overall effort is $TE^{(0)}(\mathbf{v}, t^c) = t_0 V$.

The expected overall effort is simply the weighted average of scenario-n expected overall effort across all scenarios, which is summarized below.

Lemma 2. *Suppose that $\mathbf{v} = (v_1, \ldots, v_N)$ with the induced entry threshold $t^c \in [a, b)$, then the expected overall effort is*

$$TE(\mathbf{v}, t^c) = \sum_{n=0}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(\mathbf{v}, t^c),$$

where $TE^{(n)}(\mathbf{v}, t^c)$ is given in Proposition 1 when $n \geq 1$, and $TE^{(0)}(\mathbf{v}, t^c) = t_0 V$.

4 Analysis of the Optimal Design

4.1 The Organizer's Problem for a Fixed Threshold

Equipped with Lemma 2, we are ready to investigate the problem for any given entry threshold t^c. For any given t^c, the set of prize structures that induce it can be characterized by equation (1). Applying Lemma 2, the organizer's problem for a given $t^c \in [a, b)$ can be expressed as

$$\max_{\mathbf{v}} TE(\mathbf{v}, t^c) = \sum_{n=0}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(\mathbf{v}, t^c) \tag{2}$$

subject to

$$v_1 \geq v_2 \geq \ldots \geq v_N, \tag{3}$$

$$\sum_{j=1}^{n} v_j \leq V, \forall n, \tag{4}$$

$$\sum_{n=1}^{N} p_n(t^c) v_n = 0, \tag{5}$$

where (4) are the budget constraints for all scenarios, and (5) is the binding participation constraint for the threshold type.

4.2 Rewriting the Problem Using Order Statistics

We use $G_{(i,n)}(t, t^c)$ to denote the CDF of the ith order statistics of n independent random variables, with each following CDF $G(t, t^c)$. It is well known that the CDF of the ith order statistics is $G_{(i,n)}(t, t^c) = \sum_{j=i}^{n} \binom{n}{j} G^j(t, t^c)(1 - G(t, t^c))^{n-j}$, with density function $g_{(i,n)}(t, t^c) = n\binom{n-1}{i-1} G^{i-1}(t, t^c)(1 - G(t, t^c))^{n-i} g(t, t^c)$. Notice that when $t^c = a$, G reduces to F, so we use $F_{(i,n)}(t)$ and $f_{(i,n)}(t)$ to represent $G_{(i,n)}(t, a)$ and $g_{(i,n)}(t, a)$, respectively; that is,

$$F_{(i,n)}(t) = \sum_{j=i}^{n} \binom{n}{j} F^j(t)(1 - F(t))^{n-j}, \text{ and } f_{(i,n)}(t) = n\binom{n-1}{i-1} F^{i-1}(t)(1 - F(t))^{n-i} f(t). \tag{6}$$

Obviously, $F_{(i,n)}(\cdot)$ is the CDF of $X_{(i,n)}$, where $X_{(i,n)}$ denotes the random variable corresponding to the ith order statistics of n independent random variables, with each following CDF $F(t)$.

The following result shows that the objective function can be expressed as a linear function in all N prizes, with coefficients related to all these order statistics across different scenarios and different orders.

Lemma 3. *The objective function (2) can be rewritten as*

$$TE(\mathbf{v}, t^c) = N(1 - F(t^c)) \sum_{n=1}^{N} \frac{p_n(t^c)}{n} \sum_{j=1}^{n} v_{n+1-j} \left(\int_{t^c}^{b} J(t) g_{(j,n)}(t, t^c) dt - t_0 \right) + t_0 V.$$

Therefore, the objective function is linear in $v_1, v_2, \ldots, v_N$; denote the coefficient associated with v_k as $\beta_k(t^c)$. That is,

$$\beta_k(t^c) = N(1 - F(t^c)) \sum_{n=k}^{N} \frac{p_n(t^c)}{n} \left(\int_{t^c}^{b} J(t) g_{(n+1-k,n)}(t, t^c) dt - t_0 \right), \ k = 1, 2, \ldots, N.$$

These coefficients are quite complicated, with binomials and weighted average of various order statistics. However, it turns out that these coefficients can be rewritten in a remarkably simple way, as shown in the next lemma.

Lemma 4. *The coefficients can be rewritten as*

$$\beta_k(t^c) = \int_{t^c}^{b} [J(t) - t_0]\, f_{(N-k+1,N)}(t)dt,\ \ k = 1, 2, \ldots, N,$$

where $f_{(N-k+1,N)}(t)$ *is the density function of the* $(N-k+1)$*th order statistics of* N *random draws that follow CDF* $F(\cdot)$*, as defined in (6).*

For any random variable Z and an event A, we use $[Z|A]$ to denote any random variable that has its distribution the conditional distribution of Z given A. The above Lemma says that $\beta_k(t^c)$—the coefficient associated with the kth prize when the induced threshold is t^c—is precisely the product of the mean of the random variable $[J(X_{(N-k+1,N)}) - t_0|X_{(N-k+1,N)} \geq t^c]$ (conditional on $X_{(N-k+1,N)} \geq t^c$) and the probability that $X_{(N-k+1,N)} \geq t^c$; that is:

$$\beta_k(t^c) = \left[1 - F_{(N-k+1,N)}(t^c)\right] \cdot E\left[J(X_{(N-k+1,N)}) - t_0|X_{(N-k+1,N)} \geq t^c\right]. \tag{7}$$

Observing the structure of $\beta_k(t^c)$ as relating to the random variable $X_{(N-k+1,N)}$ revealed in (7) is crucial, as the stochastic orders among $X_{(N-k+1,N)}$ across k (hazard rate dominance) will greatly facilitate the analysis of the comparison among $\beta_k(t^c)$'s. This will be clearer in Lemma 5 below.

Now the organizer's problem for a fixed entry threshold t^c can be conveniently restated as

$$\max_{\mathbf{v}} TE(\mathbf{v}, t^c) = \sum_{n=1}^{N} \beta_n(t^c)v_n + t_0 V \tag{8}$$

subject to

$$v_1 \geq v_2 \geq \ldots \geq v_N, \tag{9}$$

$$\sum_{j=1}^{n} v_j \leq V, \forall n, \tag{10}$$

$$\sum_{n=1}^{N} p_n(t^c)v_n = 0. \tag{11}$$

4.3 Features of Coefficients

As a linear programming problem, it is critical to investigate the relation among coefficients $\beta_k(t^c)$ and $\beta_k(t^c)/p_k(t^c)$. The following definition would be helpful for the presentation.

Definition 1. *$n(t^c) = \max\{n \in \{1, \ldots, N\} : \beta_n(t^c) \geq 0\}$; that is, $n(t^c)$ is the largest integer in $\{1, \ldots, N\}$ such that $\beta_{n(t^c)}(t^c) \geq 0$. If $\beta_n(t^c) < 0$ for all integers $n \in \{1, \ldots, N\}$, define $n(t^c) = 0$.*

We have the following important observation regarding the comparison among these coefficients.

Lemma 5. *(i) For any $t^c \in [a, b)$ with $n(t^c) \geq 1$,*

$$\beta_1(t^c) > \beta_2(t^c) > \cdots > \beta_{n(t^c)}(t^c) \geq 0.$$

(ii) For any $t^c \in (a, b)$ with $n(t^c) \geq 1$,

$$\frac{\beta_1(t^c)}{p_1(t^c)} > \frac{\beta_2(t^c)}{p_2(t^c)} > \cdots > \frac{\beta_{n(t^c)}(t^c)}{p_{n(t^c)}(t^c)} \geq 0.$$

*(iii) **Single crossing**: On the interval $[a, b)$, $\beta_n(t^c)$ crosses 0 at most once; and if it does, it crosses 0 from below. Moreover, $n(t^c)$ is weakly increasing in $t^c \in [a, b)$ and $n(t^c) = N$ when $t^c (< b)$ is large enough—i.e., all β coefficients are positive when t^c is large enough.*

By Theorem 1.B.26 (page 31) in Shaked and Shanthikumar (2007), $X_{(N-n+1,N)} \leq_{hr} X_{(N-n+2,N)}$, which further implies that $[X_{(N-n+1,N)} | X_{(N-n+1,N)} \geq t^c] \leq_{st} [X_{(N-n+2,N)} | X_{(N-n+2,N)} \geq t^c]$ for any t^c by (1.B.7) on page 17 of Shaked and Shanthikumar (2007). Here, $\leq_{hr}$ and $\leq_{st}$ refer to hazard rate dominance and first-order stochastic dominance, respectively.[7] With an additional observation that $Np_n(t^c) = f_{(N-n+1,N)}(t^c)$, Part (i) and (ii) then readily follow from these observations. The details can be found in the proof of Lemma 5 in the Appendix.

From the objective function (8), $\beta_k(t^c)$ can be interpreted as the marginal revenue in terms of units of effort gained by increasing the kth prize by 1 dollar, taking into account the cost of losing 1 dollar (as the organizer values it as t_0 units of effort); from (11), $p_k(t^c)$ can be regarded as the shadow price of the kth prize. Lemma 5 reveals that the coefficients $\beta_k(t^c)$ and $\frac{\beta_k(t^c)}{p_k(t^c)}$ are decreasing in k whenever these coefficients are nonnegative—i.e., $\beta_k(t^c) < \beta_{k-1}(t^c)$ and $\frac{\beta_k(t^c)}{p_k(t^c)} < \frac{\beta_{k-1}(t^c)}{p_{k-1}(t^c)}$ if

[7] For any two nonnegative random variables X and Y with absolutely continuous distribution functions, $X \leq_{hr} Y$ if $\frac{f_X(t)}{1-F_X(t)} \geq \frac{f_Y(t)}{1-F_Y(t)}$, for any $t \in \mathbb{R}$, where $F_X(\cdot)$ and $F_Y(\cdot)$ are the CDF of X any Y, respectively, with $f_X(\cdot)$ and $f_Y(\cdot)$ the corresponding density function. On the other hand, $X \leq_{st} Y$ if $F_X(t) \geq F_Y(t)$ for any $t \in \mathbb{R}$.

$\beta_k(t^c) \geq 0$—however, it is silent on the case when the coefficients are negative. In fact, when $\beta_k(t^c) < 0$ for some integer $k \leq N-1$, all possibilities can arise: The signs of $\beta_k(t^c) - \beta_{k+1}(t^c)$ and $\frac{\beta_k(t^c)}{p_k(t^c)} - \frac{\beta_{k+1}(t^c)}{p_{k+1}(t^c)}$ can be arbitrary. The following example illustrates this.

Example 1. *Suppose that $N = 3$, $F(\cdot)$ is the uniform distribution on $[1, 2]$, and $t_0 = 1.5$. Then*

t^c	$\beta_2(t^c)$	$\beta_3(t^c)$	$\beta_2(t^c) - \beta_3(t^c)$	$\frac{\beta_2(t^c)}{p_2(t^c)} - \frac{\beta_3(t^c)}{p_3(t^c)}$
1.1	−0.46	−0.62	0.16	−1.8
1.2	−0.37	−0.36	−0.01	−0.6
1.55	−0.029	−0.016	−0.013	0.02
1.6	−0.003	−0.006	0.003	0.03

As can be seen from the last two columns of the table, when $\beta_2(t^c) < 0$, all four possibilities regarding the signs of $\beta_2(t^c) - \beta_3(t^c)$ and $\frac{\beta_2(t^c)}{p_2(t^c)} - \frac{\beta_3(t^c)}{p_3(t^c)}$ can arise. Moreover, one can easily verify numerically that when $t^c = 1.1$ or 1.2—i.e., when $\frac{\beta_2(t^c)}{p_2(t^c)} < \frac{\beta_3(t^c)}{p_3(t^c)}$—awarding a single negative prize cannot be optimal. In fact, this is not a coincidence, as shown in Lemma 6 below.

The subtlety of relations among coefficients when they are negative can lead to the optimality of both multiple positive prizes and multiple negative prizes for certain entry thresholds. The following result provides a set of sufficient conditions under which multiple positive prizes and/or multiple negative prizes must arise at the optimum for certain thresholds.

Lemma 6. *Fix $t^c \in (a, b)$. Suppose that $\hat{\mathbf{v}} = (\hat{v}_1, \cdots, \hat{v}_N)$ is the optimum among all prize structures that induce t^c.*
(i) If $\frac{\beta_{N-1}(t^c)}{p_{N-1}(t^c)} < \frac{\beta_N(t^c)}{p_N(t^c)}$, then multiple negative prizes must arise at the optimum—i.e., $\hat{v}_{N-1}, \hat{v}_N < 0$.
(ii) If $\beta_1(t^c) - \beta_2(t^c) + \frac{(p_2(t^c) - p_1(t^c))\beta_N(t^c)}{p_N(t^c)} < 0$ and $p_2(t^c) > p_1(t^c)$, then multiple positive prizes must arise at the optimum—i.e., $\hat{v}_1, \hat{v}_2 > 0$.
(iii) If both assumptions in (i) and (ii) are satisfied and $N \geq 4$, then multiple positive prizes and multiple negative prizes arise at the optimum.

Therefore, Example 1 is an example where the condition in part (i) of Lemma 6 is satisfied. Here, we provide another example where all conditions in Lemma 6 are satisfied.

Example 2. *Suppose that $N = 4$, $F(\cdot)$ is the uniform distribution on $[1, 2]$, and $t_0 = 1.85$. Then* *when* $t^c = 1.65$,

$\beta_1(t^c)$	$\beta_2(t^c)$	$\beta_3(t^c)$	$\beta_4(t^c)$	$\frac{\beta_3(t^c)}{p_3(t^c)} - \frac{\beta_4(t^c)}{p_4(t^c)}$
-0.105	-0.12	-0.046	-0.006	-0.048
$\frac{\beta_1(t^c)}{p_1(t^c)}$	$\frac{\beta_2(t^c)}{p_2(t^c)}$	$\frac{\beta_3(t^c)}{p_3(t^c)}$	$\frac{\beta_4(t^c)}{p_4(t^c)}$	$\beta_1(t^c) - \beta_2(t^c) + \frac{(p_2(t^c)-p_1(t^c))\beta_4(t^c)}{p_4(t^c)}$
-0.383	-0.28	-0.19	-0.14	-0.007

Note that in this example, $\beta_2(t^c) < \beta_1(t^c) < \beta_3(t^c) < \beta_4(t^c)$ *and* $\frac{\beta_1(t^c)}{p_1(t^c)} < \frac{\beta_2(t^c)}{p_2(t^c)} < \frac{\beta_3(t^c)}{p_3(t^c)} < \frac{\beta_4(t^c)}{p_4(t^c)}$*, which further suggests that the behavior of* β *coefficients can be arbitrary when they are negative. More importantly, this example satisfies all conditions in Lemma 6. Therefore, at the optimum, there must be two positive prizes and two negative prizes for* $t^c = 1.65$*.*

4.4 The Optimal Contest Structure

Lemma 5 implies that for any given $t^c \in [a, b)$, $\beta_k(t^c) > \beta_{k+1}(t^c)$ whenever $\beta_{k+1}(t^c) \geq 0$ and $\beta_k(t^c)$ must be positive when t^c is large enough. A particular threshold is t^*, defined as the smallest threshold such that all β coefficients are nonnegative—i.e.,

$$t^* = \begin{cases} \text{the unique threshold such that } \beta_N(t^*) = 0, \text{ if } \beta_N(a) \leq 0 \\ a, \text{ otherwise} \end{cases}.$$

Alternatively, by Definition 1, t^* is the smallest threshold t^c such that $n(t^c) = N$.

When $t^c > t^*$—i.e., $n(t^c) = N$—we have the following observation that there is only one negative prize at the optimum.

Proposition 2. *Fix any* $t^c \in (t^*, b)$*. Suppose that* $\hat{\mathbf{v}} = (\hat{v}_1, \cdots, \hat{v}_N)$ *is the optimum among all prize structures that induce* t^c*. Then* $\hat{v}_{N-1} \geq 0$*, so that at the optimum, there can be only one negative prize.*

Nevertheless, Examples 1 and 2 illustrate the complexity of characterizing the optimum for an arbitrary entry threshold, as they show that the optimality in general demands multiple positive and negative prizes for entry thresholds below t^*. It is thus technically challenging to fully pin down the optimum for every t^c. It follows that a procedure of first identifying the optimum for each fixed t^c and then pinning down the optimal entry threshold by comparing across all thresholds would be quite cumbersome if not infeasible.

We overcome the above-mentioned difficulties by first pinning down the optimal entry threshold without first fully characterizing the optimal prizes for every entry threshold. This is made feasible by an innovative procedure detailed as follows. Consider an arbitrary entry threshold $t_0^c \in [a, b)$ and a (non-zero) prize structure $\mathbf{v}$ which induces it. We construct a particular class of prize structures by fixing the first $N-1$ prizes of $\mathbf{v}$ and varying only the last prize. Such class of prize structures can induce any entry threshold in the interval $[a, b)$ when $t_0^c > t^*$; and it can induce any entry threshold in the interval $[t_0^c, b)$ when $t_0^c \leq t^*$. Moreover, for each threshold in these two intervals, the corresponding prize structure satisfies the required monotonicity constraint.

In the following Proposition, we establish that the optimal prize design for a given entry threshold $t_0^c \in [a, b)$ is strictly dominated by a particular prize structure in the class mentioned above, which induces entry threshold t^*. Note that by Proposition 2, for $t_0^c > t^*$, we must have that only the last prize is negative at the optimum.

Proposition 3. *Let $\mathbf{v} = (v_1, \cdots, v_N) \neq \mathbf{0}$ be an arbitrary non-zero prize structure with $t_0^c \in [a, b)$ being its corresponding entry threshold. Fixing the first $N-1$ prizes, for any $t^c \in [a, b)$, construct the vector*

$$\mathbf{v}(t^c) = (v_1, \ldots, v_{N-1}, -\frac{\sum_{j=1}^{N-1} p_j(t^c) v_j}{p_N(t^c)}).$$

Then:

(i) If $t_0^c \leq t^*$, then $\mathbf{v}(t^c)$ satisfies constraints (9)–(11) when $t^c \in [t_0^c, b)$, and the expected overall effort $TE(\mathbf{v}(t^c))$ induced by $\mathbf{v}(t^c)$ achieves its unique optimum over $[t_0^c, b)$ at $t^c = t^*$;

(ii) If $t_0^c > t^*$ and $v_{N-1} \geq 0$, then $\mathbf{v}(t^c)$ satisfies constraints (9)–(11) when $t^c \in [a, b)$, and the expected overall effort $TE(\mathbf{v}(t^c))$ induced by $\mathbf{v}(t^c)$ achieves its unique optimum over $[a, b)$ at $t^c = t^*$.

Propositions 2 and 3 mean that the optimal prize design for any entry threshold $t_0^c \in [a, b)$ must be strictly dominated by that for the entry threshold t^*. In the following Theorem, we fully establish the optimality of t^* by further allowing $t_0^c = b$.

Theorem 1. *The unique optimal entry threshold is t^*.*

Provided that t^* is the unique optimal threshold, we only need to identify the optimal prize design for t^* to characterize the global optimum. At the threshold t^*, by definition, all coefficients

$\beta_n(t^*)$ are nonnegative, which, by Lemma 5, further entails useful monotonicity properties. We are then ready to characterize the optimal prize design as in the following Theorem.

Theorem 2. *The unique optimal prize structure is*

$$\mathbf{v}^* = (V, \underbrace{0, \cdots, 0}_{N-2 \ times}, -\frac{p_1(t^*)V}{p_N(t^*)}).$$

The optimal design features winner-take-all for the best performer and at most one punishment for the worst performer among all N potential contestants, whenever they enter the competition; nonparticipation is regarded as the worst possible performance, when this happens. The intuitions are revealed as below.

Theorem 1 reveals that the optimal entry threshold must be t^*. In particular, Proposition 3 shows that starting from an optimal prize structure that induces an entry threshold $t_0^c \in [a, b)$, one can always adjust only the punishment v_N (< 0) to induce entry threshold t^* and at the same time generates a higher payoff for the organizer. In fact, when $t_0^c < t^*$, a harsher punishment v_N that induces t^* reduces entry but increases the effort supply of an entrant; when $t_0^c > t^*$, a more lenient punishment v_N that induces t^* increases entry but reduces the effort supply of an entrant. The optimality of t^* means that the positive effects of the above adjustment in punishment v_N always dominate its negative effects.

To further understand why t^* is the optimal entry threshold, recall that $-\beta_N(t^c)$ can be interpreted as the marginal revenue gained by using an additional unit of punishment v_N when t^c is fixed, and that the organizer's payoff can be expressed as $\sum_{n=1}^{N} \beta_n(t^c)v_n + t_0V$ by (8). We will explain how this marginal revenue links to optimality of t^*. In fact, when fixing the first $N-1$ prizes and varying the threshold t^c, the participation constraint implies that $v_N(t^c) = -\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c)$ as in Proposition 3. When increasing t^c by ε, such change leads to changes in all $\beta_n(t^c)$, $n \in \{1, \ldots, N\}$, and also in the last prize $v_N(t^c)$. As such, the marginal change in the organizer's payoff is $\varepsilon \left[\sum_{n=1}^{N-1} \beta_n'(t^c)v_n + \beta_N'(t^c)v_N(t^c) + \beta_N(t^c)v_N'(t^c)\right]$. However, notice that $\beta_n'(t^c) = -N\left[J(t^c) - t_0\right] p_n(t^c)$ for any n, so the construction of $v_N(t^c)$ (or equivalently, the participant constraint) implies that $\sum_{n=1}^{N-1} \beta_n'(t^c)v_n + \beta_N'(t^c)v_N(t^c) = 0$, which further implies that the marginal change in the organizer's payoff is simply $\varepsilon\beta_N(t^c)v_N'(t^c)$. Since $v_N'(t^c) < 0$, this marginal change has the same sign as $-\beta_N(t^c)$. Therefore, an ε increase in t^c leads to a marginal change in the organizer's payoff, which has the same sign as $-\beta_N(t^c)$—the marginal revenue gained by using an additional unit of punishment v_N. Since the marginal revenue is positive when $t^c < t^*$ and is negative when $t^c > t^*$, the optimum must stop exactly at the point when the marginal revenue is 0, which is t^*.

Given the optimality of the entry threshold t^*, the organizer's payoff can be expressed as $\sum_{n=1}^{N} \beta_n(t^*)v_n + t_0 V$ by (8), in which $\mathbf{v} = (v_n)$ is an arbitrary prize structure that induces entry t^*. Without loss of generality, we assume $t^* \in (a, b)$. Thus, we have $\beta_1(t^*) > \beta_2(t^*) > \cdots > \beta_{N-1}(t^*) > \beta_N(t^*) = 0$ by Lemma 5. Constructing $\tilde{\mathbf{v}} = (\tilde{v}_n)$ by setting $\tilde{v}_1 = V$, $\tilde{v}_n = 0$ for $n \in \{2, ..., N-1\}$ and $\tilde{v}_N = -\frac{p_1(t^*)V}{p_N(t^*)}$ (< 0). Note that $\tilde{\mathbf{v}} = (\tilde{v}_n)$ still induces entry t^*. Clearly, we have $\sum_{n=1}^{N} \beta_n(t^*)\tilde{v}_n + t_0 V > \sum_{n=1}^{N} \beta_n(t^*)v_n + t_0 V$, since the sum of all positive prizes in $\mathbf{v}$ is no greater than V and $\beta_1(t^*) > \beta_2(t^*) > \cdots > \beta_{N-1}(t^*) > \beta_N(t^*) = 0$. We thus have that $\tilde{\mathbf{v}}$ is the optimum.

Note that the punishment prevails if and only if $p_1(t^*) > 0$, i.e., $t^* > a$. When $t^* = a$, we go back to the optimal design of Moldovanu and Sela (2001) who do not allow punishments. However, our result shows that in this case, there is no loss of generality in their analysis.

4.5 Discussion

By the definition of t^* and Lemma 5, it is clear that $t^* = a$ if and only if $\beta_N(a) \geq 0$, which is the case when, for example, $J(a) \geq t_0$. In this case, the optimum induces full entry and is winner-take-all $\mathbf{v} = (V, 0, \cdots, 0)$. Thus, we have the following immediate observation.

Theorem 3. *Winner-take-all is optimal if and only if* $\beta_N(a) = \int_a^b [J(t) - t_0] f_{(1,N)}(t)dt \geq 0$.

When $\beta_N(a) < 0$, we have $t^* \in (a, b)$, which corresponds to stochastic entry. Note that this can be the case even when $t_0 = 0$; in this case, the organizer still imposes a negative prize, although she does not value it at all (as she does not value any leftover budget). We illustrate that with $t_0 = 0$, both $\beta_N(a) < 0$ and $\beta_N(a) > 0$ are possible. It is clear that if $J(a) \geq 0$, $\beta_N(a) > 0$. On the other hand, for example, if $F(\cdot)$ is the uniform distribution on $[0.25, 1.25]$ and $N = 3$, then $\beta_N(a) = \beta_3(0.25) = -0.25 < 0$. We highlight the result below.

Corollary 1. *Even when the organizer does not value any leftover budget—i.e.,* $t_0 = 0$*—at the optimum, it is still possible that a negative prize is used.*

According to well-received insights from the auction design literature, when the virtual values of less efficient types contestants are negative, excluding them would enhance the seller's revenue.

Therefore, in their Section IV, Moldovanu and Sela (2001) further investigate using entry fees to exclude less efficient types of bidders to boost contest performance while maintaining the winner-take-all prize allocation rule—i.e., awarding all prize budget V to the highest bidder of all entrants.[8] Moldovanu and Sela (2001) conjecture that such winner-take-all rule with appropriate entry fees is the optimal prize structure. As we mentioned in the model setup, such contest rule corresponds to the prize structure $\mathbf{v} = (V - E, -E, \dots, -E)$ when the entry fee is $E \geq 0$. Theorems 2 and 3 imply that the Moldovanu and Sela type of entry fees are suboptimal, unless the optimal entry fee is 0—i.e., when $\beta_N(a) \geq 0$.

Liu et al. (2018) introduce budget-augmenting negative prizes in contest design. We would like to first emphasize that since they assume that negative prizes can augment the prize budget—which is not the case in our setting—the driving forces of the optimality of having a negative prize in our setting is dramatically different from those in their setting. They show that full surplus extraction can be achieved in the limit so that the highest total effort can be arbitrarily close to an utmost level, when the bound on negative prizes approaches infinity. In their paper, optimal negative prizes do not exist (or are *infinite*). However, in our paper, the optimal negative prize is *finite*. The crucial difference between our paper and theirs is that an (endogenous) minimum effort requirement is imposed in their paper and negative prizes can augment the organizer's budget in their analysis. Thus, in their paper the prize allocation rule depends not only on the rank information of efforts but also on the information on the level of effort. In our paper, the prize allocation rule only relies on the rank information of efforts and negative prizes cannot augment the prize budget.[9] This is why we do not have full surplus extraction in our setting, and the optimal negative prizes must be finite.

Finally, it is interesting to investigate how the optimal entry threshold t^* and the corresponding punishment vary with N, the total number of contestants. To this end, define the Myerson cutoff $t^M \in [a, b)$ as: If $J(a) - t_0 \leq 0$, $t^M = J^{-1}(t_0)$; otherwise, $t^M = a$. As defined in the mechanism design literature, t^M is the smallest type in $[a, b]$ such that the *virtual efficiency function* $J(t)$ is nonnegative. To emphasize the dependence on N, we use t^*_N and P^*_N to denote the optimal entry threshold and the absolute value of the punishment (the last prize), respectively, when the total number of contestants is N. We have the following observation.

[8] Note that in rank-order contests, a minimum bid is not an eligible design instrument since its implementation requires information on the level of bids.

[9] Full entry is optimal in Liu et al. (2018), so endogenous entry is not an issue there.

Proposition 4. *(i) t_N^* is increasing in N and* $\lim_{N\to\infty} t_N^* = t^M$*.*

(ii) If $F(t^M) < \frac{1}{2}$, $\lim_{N\to\infty} P_N^* = 0$; if $F(t^M) > \frac{1}{2}$, P_N^* explodes to infinity: $\lim_{N\to\infty} P_N^* = +\infty$.

Combined with Theorem 3, which provides an equivalent condition for the optimality of no punishment (winner-take-all) that the optimal entry threshold is a, we obtain the following immediate result.

Corollary 2. *There exists a unique extended integer $n^* \in [1, +\infty) \cup \{+\infty\}$ such that the punishment is used if and only if the total number of contestants $N \geq n^*$.*[10]

The above proposition says that the optimal entry threshold increases with N and converges to the Myerson cutoff. As a direct implication, the larger N is, the more likely a punishment is used (conditional on full entry). This is quite intuitive: With a larger N, the organizer is more confident that the worst performer among all N entrants has a low ability, so that such a contestant should be punished. The above proposition also reveals the size of punishment in the limit. When t^M is relatively low ($F(t^M) < \frac{1}{2}$), the punishment vanishes; but when t^M is relatively high, the punishment explodes. The intuition is clear: Before entry, the threshold type t_N^* is comparing the probability that he wins the first prize, which is $p_1(t_N^*)$ when there is only one entrant, and the probability that he is penalized, which is $p_N(t_N^*)$ when there are N entrants. When t^M is relatively low, the latter probability dominates in the limit, so the participation constraint implies that the punishment should not be used; the opposite happens when t^M is relatively high.

5 Concluding Remarks

Understanding how to optimally orchestrate carrots and sticks in contests is important in both theory and practice. The optimal use of rewards and punishments in rank-order contests with incomplete information has remained an open question. In this paper, we provide a complete and elegant characterization of the optimum for a natural and important class of prize structures, using a workhorse model of all-pay auction. We find that the optimal design must feature a single positive prize equal to the budget, and potentially a single negative prize that determines the optimal endogenous entry threshold. The optimal negative prize is set at a level such that the marginal revenue of imposing an extra unit of punishment through the last prize vanishes. We

[10] If $n^* = 1$, it means that the punishment is always used regardless of N (recall that $N \geq 2$); if $n^* = +\infty$, it means that the punishment is never used regardless of N.

find that this marginal revenue is precisely captured by the coefficient of the last prize in the total effort function. As direct implications, a necessary and sufficient condition for the optimality of pure winner-take-all is established, and the limiting behavior of the optimum is characterized.

Our current study assumes that it is costless for the organizer to punish entrants. If a punishment (negative prize) is costly like in Moldovanu, Sela, and Shi (2012),[11] we expect that: When the marginal cost of punishment is high enough, it is optimal not to use any negative prize; When the marginal cost is low, negative prizes still arise at the optimum. However, the optimal entry threshold and the optimal number of negative prizes are more involved and unclear. In fact, though the organizer's problem is still a linear programming, the important intermediate result, Proposition 3, no longer generally holds with costly punishments, because in the construction the last prize is decreasing in the entry threshold, which then may violate the budget constraint with costly punishments. In particular, when our current optimum violates the budget constraint with costly punishments, it is nasty and complicated to pin down the optimal entry threshold and the corresponding negative prizes with binding budget constraints.

Finally, our current study assumes that the contestants are risk neutral and their bidding cost function is linear. Further investigating the impacts of nonlinearity of bidding cost function on the optimal design is a highly meaningful but nontrivial task. With convex cost functions, the optimality of a single positive prize no longer prevails, because the marginal cost increases as a contestant exerts more effort. Similarly, such effect may also lead to the optimality of multiple negative prizes. We leave this study to future work.

6 Appendix

Proof of Lemma 1: We first show that $\sum_{n=1}^{N} p_n(t^c)v_n$ is strictly increasing in $t^c \in [a, b]$ for any $\mathbf{v} = (v_1, \cdots, v_N) \neq v\mathbf{e}$ with $v_1 \geq \cdots \geq v_N$, where $v \in \mathbb{R}$ and $\mathbf{e} = (1, \ldots, 1)$ is the N-vector with all its elements being 1. To see this, notice first that for any integer $n \in \{1, \ldots, N-1\}$ and any

[11] When considering endogenous entry (like in our setting), Moldovanu, Sela, and Shi (2012) assume that punishments are costless and uniform.

$t^c \in (a,b)$,[12]

$$\begin{aligned}
\frac{d}{dt^c}\sum_{k=1}^{n} p_k(t^c) &= \frac{d}{dt^c}\sum_{k=1}^{n}\binom{N-1}{k-1}(1-F(t^c))^{k-1}F^{N-k}(t^c)\\
&= f(t^c)\sum_{k=1}^{n}\left[\begin{array}{c}(N-k)\binom{N-1}{k-1}(1-F(t^c))^{k-1}F^{N-k-1}(t^c)\\ -(k-1)\binom{N-1}{k-1}(1-F(t^c))^{k-2}F^{N-k}(t^c)\end{array}\right]\\
&= (N-1)f(t^c)\left[\begin{array}{c}\sum_{k=1}^{n}\binom{N-2}{k-1}(1-F(t^c))^{k-1}F^{N-k-1}(t^c)\\ -\sum_{k=2}^{n}\binom{N-2}{k-2}(1-F(t^c))^{k-2}F^{N-k}(t^c)\end{array}\right]\\
&= (N-1)f(t^c)\binom{N-2}{n-1}(1-F(t^c))^{n-1}F^{N-n-1}(t^c) > 0. \qquad (12)
\end{aligned}$$

Therefore, $\sum_{k=1}^{n} p_k(t^c)$ is strictly increasing in $t^c \in [a,b]$, for any integer $n \in \{1,\ldots,N-1\}$. Define $v_{N+1} = 0$ and notice that

$$\begin{aligned}
\sum_{n=1}^{N} p_n(t^c)v_n &= \sum_{n=1}^{N} p_n(t^c)\left[\sum_{k=n}^{N}(v_k - v_{k+1})\right] = \sum_{k=1}^{N}(v_k - v_{k+1})\left[\sum_{n=1}^{k} p_n(t^c)\right]\\
&= \sum_{k=1}^{N-1}(v_k - v_{k+1})\left[\sum_{n=1}^{k} p_n(t^c)\right] + v_N, \qquad (13)
\end{aligned}$$

where the last equality uses $\sum_{n=1}^{N} p_n(t^c) = 1$. Note that $v_k - v_{k+1} \geq 0$ for any integer $k \in \{1,\ldots,N-1\}$, with strict inequality for some $k' \in \{1,\ldots,N-1\}$ (because $\mathbf{v} \neq v\mathbf{e}$). Since $\sum_{n=1}^{k} p_k(t^c)$ is strictly increasing in $t^c \in [a,b]$, it follows that $\sum_{n=1}^{N} p_n(t^c)v_n$ is strictly increasing in $t^c \in [a,b]$ for any $\mathbf{v} \neq v\mathbf{e}$.

Now, for any prize structure $\mathbf{v} \neq v\mathbf{e}$, (i) if $\sum_{n=1}^{N} p_n(a)v_n < 0$ (i.e., $V_N < 0$) and $\sum_{n=1}^{N} p_n(b)v_n > 0$ (i.e., $v_1 > 0$), then the monotonicity established above and the Intermediate Value Theorem imply that there exists a unique $t_0^c \in (a,b)$ such that $\sum_{n=1}^{N} p_n(t_0^c)v_n = 0$, which means that such prize structure induces a unique interior entry threshold; (ii) if $\sum_{n=1}^{N} p_n(a)v_n \geq 0$ (i.e., $v_N \geq 0$), then such prize structure induces full entry; (iii) if $\sum_{n=1}^{N} p_n(b)v_n \leq 0$ (i.e., $v_1 \leq 0$), then such prize structure induces no entry. If the prize structure $\mathbf{v} = v\mathbf{e}$, then $\sum_{n=1}^{N} p_n(t^c)v_n = v$, so such prize structure induces either full entry (when $v \geq 0$) or no entry (when $v < 0$). Note that by definition, a prize structure satisfies $v_1 \geq v_N$, so these cases exhaust all possibilities, which further implies that the reverse is also true—the characterization is indeed a necessary and sufficient one. This

[12] Define $\sum_{j=m}^{n} H_j(\cdot) = 0$ when $m > n$ for any function $H_j(\cdot)$.

completes the proof. □

Proof of Proposition 1: (i) To simplify the notation, let $e(t)$ be the bidding strategy for type $t \in [t^c, b]$. In a symmetric equilibrium with strictly increasing bidding strategies, by the revelation principle, for entrant i with type t, he chooses a type $s \in [t^c, b]$ to solve

$$\max_{s \in [t^c, b]} V^{(n)}(s) - \frac{e(s)}{t},$$

where

$$V^{(n)}(s) = \sum_{j=1}^{n} v_{n+1-j} \binom{n-1}{j-1} G^{j-1}(s, t^c)(1 - G(s, t^c))^{n-j}$$

is the expected prize an entrant with type s obtains.

In equilibrium, entrant i's problem is solved at $s = t$, so the first-order condition implies that

$$\frac{dV^{(n)}(t)}{dt} - \frac{e'(t)}{t} = 0.$$

Together with the boundary condition that $e(t^c) = 0$, we have

$$e(t) = tV^{(n)}(t) - \int_{t^c}^{t} V^{(n)}(s) ds - t^c v_n.$$

(ii) Recall that the overall effort consists of total effort from entrants and the valuation of the leftover budget. Thus, we have

$$\begin{aligned}
TE^{(n)}(\mathbf{v},t^c) &= n\int_{t^c}^{b} e^{(n)}(t,\mathbf{v},t^c)g(t,t^c)dt + t_0(V-\sum_{j=1}^{n} v_j)\\
&= n\int_{t^c}^{b}(tV^{(n)}(t) - \int_{t^c}^{t} V^{(n)}(s)ds - t^c v_n)g(t,t^c)dt + t_0(V-\sum_{j=1}^{n} v_j)\\
&= n\int_{t^c}^{b}(tV^{(n)}(t) - t^c v_n)g(t,t^c)dt - n\int_{t^c}^{b} V^{(n)}(s)[\int_{s}^{b} g(t,t^c)dt]ds + t_0(V-\sum_{j=1}^{n} v_j)\\
&= n\int_{t^c}^{b}(t - \frac{1-G(t,t^c)}{g(t,t^c)})V^{(n)}(t)g(t,t^c)dt - nt^c v_n + t_0(V-\sum_{j=1}^{n} v_j)\\
&= n\int_{t^c}^{b}(t - \frac{1-\frac{F(t)-F(t^c)}{1-F(t^c)})}{\frac{f(t)}{1-F(t^c)}})V^{(n)}(t)g(t,t^c)dt - nt^c v_n + t_0(V-\sum_{j=1}^{n} v_j)\\
&= n\int_{t^c}^{b} J(t)V^{(n)}(t)g(t,t^c)dt - nt^c v_n + t_0(V-\sum_{j=1}^{n} v_j).
\end{aligned}$$

□

Proof of Lemma 3: By Proposition 1,

$$\begin{aligned}
TE^{(n)}(\mathbf{v},t^c) &= n\int_{t^c}^{b} J(t)V^{(n)}(t)g(t,t^c)dt - nt^c v_n + t_0\left(V-\sum_{j=1}^{n} v_j\right)\\
&= n\int_{t^c}^{b} J(t)[\sum_{j=1}^{n} v_{n+1-j}\binom{n-1}{j-1}G^{j-1}(t,t^c)(1-G(t,t^c))^{n-j}]g(t,t^c)dt\\
&\quad -nt^c v_n + t_0\left(V-\sum_{j=1}^{n} v_j\right)\\
&= \sum_{j=1}^{n} v_{n+1-j}\left(\int_{t^c}^{b} J(t)g_{(j,n)}(t,t^c)dt - t_0\right) - nt^c v_n + t_0 V. \qquad (14)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
TE(\mathbf{v},t^c) &= \sum_{n=0}^{N}\binom{N}{n}(1-F(t^c))^n F^{N-n}(t^c)TE^{(n)}(\mathbf{v},t^c) \\
&= \sum_{n=1}^{N}\binom{N}{n}(1-F(t^c))^n F^{N-n}(t^c)\left[\begin{array}{c}\sum_{j=1}^{n} v_{n+1-j}\left(\int_{t^c}^{b} J(t)g_{(j,n)}(t,t^c)dt-t_0\right)\\ -nt^c v_n+t_0V\end{array}\right]+F^N(t^c)t_0V \\
&= (1-F(t^c))\sum_{n=1}^{N}\frac{Np_n(t^c)}{n}\left[\sum_{j=1}^{n} v_{n+1-j}\left(\int_{t^c}^{b} J(t)g_{(j,n)}(t,t^c)dt-t_0\right)-nt^c v_n\right]+t_0V \\
&= N(1-F(t^c))\sum_{n=1}^{N}p_n(t^c)\left[\frac{\sum_{j=1}^{n} v_{n+1-j}}{n}\left(\int_{t^c}^{b} J(t)g_{(j,n)}(t,t^c)dt-t_0\right)-t^c v_n\right]+t_0V \\
&= N(1-F(t^c))\sum_{n=1}^{N}\frac{p_n(t^c)}{n}\sum_{j=1}^{n} v_{n+1-j}\left(\int_{t^c}^{b} J(t)g_{(j,n)}(t,t^c)dt-t_0\right)+t_0V,
\end{aligned}
$$

where the last equality uses (5). □

Proof of Lemma 4: Recall that

$$
\begin{aligned}
\beta_k(t^c) &= N(1-F(t^c))\sum_{n=k}^{N}\frac{p_n(t^c)}{n}\left(\int_{t^c}^{b} J(t)g_{(n+1-k,n)}(t,t^c)dt-t_0\right) \\
&= N(1-F(t^c))\sum_{n=k}^{N}\binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c) \\
&\quad\cdot\left(\int_{t^c}^{b} J(t)\binom{n-1}{n-k}G^{n-k}(t,t^c)(1-G(t,t^c))^{k-1}g(t,t^c)dt-\frac{t_0}{n}\right).
\end{aligned}
$$

For convenience, denote $\beta_k(t^c) = A - B$, where

$$
A = N(1-F(t^c))\sum_{n=k}^{N}\binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)\int_{t^c}^{b} J(t)\binom{n-1}{n-k}G^{n-k}(t,t^c)(1-G(t,t^c))^{k-1}g(t,t^c)dt,
$$

and

$$
B = N(1-F(t^c))\sum_{n=k}^{N}\frac{\binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)t_0}{n}.
$$

Recall that $G(t,t^c) = \frac{F(t)-F(t^c)}{1-F(t^c)}$ and $g(t,t^c) = \frac{f(t)}{1-F(t^c)}$. Then,

$$A = N\sum_{n=k}^{N}\binom{N-1}{n-1}F^{N-n}(t^c)\int_{t^c}^{b}J(t)\binom{n-1}{n-k}\left(F(t)-F(t^c)\right)^{n-k}\left(1-F(t)\right)^{k-1}f(t)dt.$$

Notice further that for any integer $n \in \{k,\ldots,N\}$,

$$\binom{N-1}{n-1}\binom{n-1}{n-k} = \frac{(N-1)!}{(N-n)!\cdot(n-k)!\cdot(k-1)!} = \binom{N-1}{N-k}\binom{N-k}{n-k}.$$

Thus,

$$\begin{aligned} A &= N\binom{N-1}{N-k}\sum_{n=k}^{N}\binom{N-k}{n-k}F^{N-n}(t^c)\int_{t^c}^{b}J(t)\left(F(t)-F(t^c)\right)^{n-k}\left(1-F(t)\right)^{k-1}f(t)dt \\ &= N\binom{N-1}{N-k}\int_{t^c}^{b}J(t)F^{N-k}(t)(1-F(t))^{k-1}f(t)dt = \int_{t^c}^{b}J(t)f_{(N-k+1,N)}(t)dt. \end{aligned}$$

On the other hand, since $F_{(N-k+1,N)}(t) = \sum_{j=N-k+1}^{N}\binom{N}{j}F^j(t)(1-F(t))^{N-j}$,

$$\begin{aligned} B &= N(1-F(t^c))\sum_{n=k}^{N}\frac{\binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)t_0}{n} = t_0\sum_{n=k}^{N}\binom{N}{n}(1-F(t^c))^nF^{N-n}(t^c) \\ &= t_0\left[1-\sum_{n=0}^{k-1}(1-F(t^c))^nF^{N-n}(t^c)\right] = t_0\left[1-\sum_{n=N-k+1}^{N}(1-F(t^c))^{N-n}F^n(t^c)\right] \\ &= t_0\int_{t^c}^{b}dF_{(N-k+1,N)}(t) = t_0\int_{t^c}^{b}f_{(N-k+1,N)}(t)dt. \end{aligned}$$

Therefore,

$$\beta_k(t^c) = A-B = \int_{t^c}^{b}J(t)f_{(N-k+1,N)}(t)dt - t_0\int_{t^c}^{b}f_{(N-k+1,N)}(t)dt = \int_{t^c}^{b}\left[J(t)-t_0\right]f_{(N-k+1,N)}(t)dt.$$

Proof of Lemma 5: (i) It suffices to show that for if $\beta_n(t^c) \geq 0$ for some integer $n \in \{2,\ldots,N\}$, then $\beta_{n-1}(t^c) > \beta_n(t^c)$. To this end, notice that from Lemma 4 and (7)

$$\beta_n(t^c) = \left[1-F_{(N-n+1,N)}(t^c)\right]\cdot E\left[J(X_{(N-n+1,N)})-t_0|X_{(N-n+1,N)} \geq t^c\right].$$

By Theorem 1.B.26 (page 31) in Shaked and Shanthikumar (2007), $X_{(N-n+1,N)} \leq_{hr} X_{(N-n+2,N)}$, which further implies that $[X_{(N-n+1,N)}|X_{(N-n+1,N)} \geq t^c] \leq_{st} [X_{(N-n+2,N)}|X_{(N-n+2,N)} \geq t^c]$ for any t^c by (1.B.7) on page 17 of Shaked and Shanthikumar (2007). Since $J(\cdot)$ is strictly increasing, the definition of first-order stochastic dominance further implies that[13]

$$E\left[J(X_{(N-n+1,N)}) - t_0|X_{(N-n+1,N)} \geq t^c\right] < E\left[J(X_{(N-n+2,N)}) - t_0|X_{(N-n+2,N)} \geq t^c\right].$$

Since $\beta_n(t^c) \geq 0$, we have

$$0 \leq E\left[J(X_{(N-n+1,N)}) - t_0|X_{(N-n+1,N)} \geq t^c\right] < E\left[J(X_{(N-n+2,N)}) - t_0|X_{(N-n+2,N)} \geq t^c\right],$$

so

$$\begin{aligned} 0 &\leq \underbrace{\left[1 - F_{(N-n+1,N)}(t^c)\right] \cdot E\left[J(X_{(N-n+1,N)}) - t_0|X_{(N-n+1,N)} \geq t^c\right]}_{\beta_n(t^c)} \\ &< \underbrace{\left[1 - F_{(N-n+2,N)}(t^c)\right] \cdot E\left[J(X_{(N-n+2,N)}) - t_0|X_{(N-n+2,N)} \geq t^c\right]}_{\beta_{n-1}(t^c)}, \end{aligned}$$

where we use the fact that $F_{(N-n+1,N)}(t^c) \geq F_{(N-n+2,N)}(t^c)$, which is due to $X_{(N-n+1,N)} \leq_{st} X_{(N-n+2,N)}$.[14]

(ii) It suffices to show that for if $\beta_n(t^c) \geq 0$ for some integer $n \in \{2, \ldots, N\}$, then $\frac{\beta_{n-1}(t^c)}{p_{n-1}(t^c)} > \frac{\beta_n(t^c)}{p_n(t^c)}$. Notice that $Np_k(t^c) = f_{(N-k+1,N)}(t^c)$ for any k. Since $X_{(N-n+1,N)} \leq_{hr} X_{(N-n+2,N)}$, we have, by definition, $\frac{f_{(N-n+1,N)}(t^c)}{1-F_{(N-n+1)}(t^c)} \geq \frac{f_{(N-n+2,N)}(t^c)}{1-F_{(N-n+2)}(t^c)}$ for any $t^c \in (a, b)$, which further implies that $\frac{p_n(t^c)}{1-F_{(N-n+1)}(t^c)} \geq \frac{p_{n-1}(t^c)}{1-F_{(N-n+2)}(t^c)}$ for any $t^c \in (a, b)$. Therefore,

$$\begin{aligned} \frac{\beta_n(t^c)}{p_n(t^c)} &= \frac{1 - F_{(N-n+1,N)}(t^c)}{p_n(t^c)} \cdot E\left[J(X_{(N-n+1,N)}) - t_0|X_{(N-n+1,N)} \geq t^c\right] \\ &< \frac{1 - F_{(N-n+1,N)}(t^c)}{p_n(t^c)} \cdot E\left[J(X_{(N-n+2,N)}) - t_0|X_{(N-n+2,N)} \geq t^c\right] \\ &\leq \frac{1 - F_{(N-n+2,N)}(t^c)}{p_{n-1}(t^c)} \cdot E\left[J(X_{(N-n+2,N)}) - t_0|X_{(N-n+2,N)} \geq t^c\right] = \frac{\beta_{n-1}(t^c)}{p_{n-1}(t^c)}, \end{aligned}$$

where the first inequality is from part (i).

[13] Here it is a strictly inequality, because the CDFs of $X_{(N-n+1,N)}$ and $X_{(N-n+2,N)}$ obviously differ at at least one point in (t^c, b) (so they differ over a set with strictly positive measure, because of continuity).

[14] As mentioned above, $[X_{(N-n+1,N)}|X_{(N-n+1,N)} \geq t^c] \leq_{st} [X_{(N-n+2,N)}|X_{(N-n+2,N)} \geq t^c]$ for any t^c; in particular, this holds when $t^c = a$.

(iii) Since $\beta_n(t^c) = \int_{t^c}^b [J(t) - t_0] f_{(N-n+1,N)}(t)dt$, $\beta'_n(t^c) = -[J(t^c) - t_0] f_{(N-n+1,N)}(t^c)$. Together with the fact that $J(\cdot)$ is strictly increasing and that $\int_{t^c}^b [J(t) - t_0] f_{(N-n+1,N)}(t)dt > 0$ for any t^c with $J(t^c) \geq t_0$ (note that $J(b) = b > t_0$), $\beta_n(t^c)$ crosses zero at most once and $\beta_n(t^c) > 0$ when t^c satisfies $J(t^c) > t_0$. Moreover, if $\beta_n(t^c)$ crosses zero, it must cross from below. $n(t^c)$ being increasing in t^c is obvious from part (i). □

Proof of Lemma 6: Notice first that Lemma 1 implies that $\hat{v}_1 > 0$ and $\hat{v}_N < 0$.

(i) Suppose, to the contrary, that $\hat{v}_{N-1} \geq 0$. Then the prize structure

$$\tilde{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_{N-2}, \hat{v}_{N-1} - \frac{\varepsilon}{p_{N-1}(t^c)}, \hat{v}_N + \frac{\varepsilon}{p_N(t^c)}),$$

which differs from $\hat{\mathbf{v}}$ only at the last two prizes, still satisfies constraints (9)–(11), when $\varepsilon > 0$ is small enough such that $\hat{v}_N + \frac{\varepsilon}{p_N(t^c)} < 0$ and $\hat{v}_N + \frac{\varepsilon}{p_N(t^c)} < \hat{v}_{N-1} - \frac{\varepsilon}{p_{N-1}(t^c)}$. However, the expected overall effort induced by $\tilde{\mathbf{v}}$ is strictly higher than that by $\hat{\mathbf{v}}$:

$$TE(\tilde{\mathbf{v}}) - TE(\hat{\mathbf{v}}) = -\varepsilon \left(\frac{\beta_{N-1}(t^c)}{p_{N-1}(t^c)} - \frac{\beta_N(t^c)}{p_N(t^c)} \right) > 0,$$

contradicting the optimality of $\hat{\mathbf{v}}$. Thus, $\hat{v}_{N-1} < 0$.

(ii) Suppose, to the contrary, that $\hat{v}_2 \leq 0$. Then, when $\varepsilon > 0$ is small enough, $\hat{v}_1 - \varepsilon > 0$, $\hat{v}_2 + \varepsilon < \hat{v}_1 - \varepsilon$ and $\hat{v}_N - \frac{(p_2(t^c) - p_1(t^c))\varepsilon}{p_N(t^c)} < \hat{v}_N$ (because $p_2(t^c) > p_1(t^c)$). As such, when $\varepsilon > 0$ is small enough, the prize structure

$$\check{\mathbf{v}} = (\hat{v}_1 - \varepsilon, \hat{v}_2 + \varepsilon, \hat{v}_3, \cdots, \hat{v}_{N-1}, \hat{v}_N - \frac{(p_2(t^c) - p_1(t^c))\,\varepsilon}{p_N(t^c)}),$$

which differs from $\hat{\mathbf{v}}$ only at the 1st, 2nd, and the last prizes, still satisfies constraints (9)–(11). However, the expected overall effort induced by $\check{\mathbf{v}}$ is strictly higher than that by $\hat{\mathbf{v}}$:

$$TE(\check{\mathbf{v}}) - TE(\hat{\mathbf{v}}) = -\varepsilon \left[\beta_1(t^c) - \beta_2(t^c) + \frac{(p_2(t^c) - p_1(t^c))\,\beta_N(t^c)}{p_N(t^c)} \right] > 0,$$

contradicting the optimality of $\hat{\mathbf{v}}$. Thus, $\hat{v}_2 > 0$.

(iii) This part is obvious from part (i) and part (ii). □

Proof of Proposition 2: Notice first that Lemma 1 implies that $\hat{v}_1 > 0$. Suppose, to the contrary, that $\hat{v}_{N-1} < 0$. Due to monotonicity (9), there exists some integer $j \in \{2, \ldots, N-1\}$ such that

$\hat{v}_{j-1} \geq 0$ and $\hat{v}_j < 0$—that is, j is the smallest integer such that $\hat{v}_j < 0$. Consider the prize structure

$$\check{\mathbf{v}} = (\hat{v}_1, \ldots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \ldots, \hat{v}_{N-1}, \hat{v}_N + \frac{p_j(t^c)}{p_N(t^c)}\hat{v}_j),$$

which differs from $\hat{\mathbf{v}}$ only at the jth and Nth prizes. Obviously, since $\hat{v}_j < 0$, $\check{\mathbf{v}}$ satisfies constraints (9)–(11) for threshold t^c. However, since $t^c > t^*$, by Lemma 5 and the definition of t^*, the expected overall effort induced by $\check{\mathbf{v}}$ is strictly higher than that by $\hat{\mathbf{v}}$:

$$TE(\check{\mathbf{v}}) - TE(\hat{\mathbf{v}}) = \hat{v}_j \left(\beta_N(t^c)\frac{p_j(t^c)}{p_N(t^c)} - \beta_j(t^c)\right) = \underbrace{\hat{v}_j p_j(t^c)}_{<0} \underbrace{\left(\frac{\beta_N(t^c)}{p_N(t^c)} - \frac{\beta_j(t^c)}{p_j(t^c)}\right)}_{<0 \text{ by Lemma 5, as } t^c > t^*} > 0,$$

contradicting the optimality of $\hat{\mathbf{v}}$. □

Proof of Proposition 3: The following observation is useful, the proof of which is relegated to the end of the appendix.

Lemma 7. *(i) For any non-zero prize structure* $\mathbf{v} = (v_1, \ldots, v_N)$ *with induced entry threshold* $t_0^c < b$ *(i.e.,* $\sum_{n=1}^{N} p_n(t_0^c)v_n = 0$*) and* $\sum_{n=1}^{N-1} p_n(t_0^c)v_n \geq 0$*,*

$$\frac{d}{dt^c}\left(\frac{\sum_{n=1}^{N-1} p_n(t^c)v_n}{p_N(t^c)}\right) > 0 \text{ for any } t^c \in (t_0^c, b).$$

(ii) If the prize structure $\mathbf{v} = (v_1, \ldots, v_N)$ *satisfies* $v_1 > 0$ *and* $v_{N-1} \geq 0$*, then*

$$\frac{d}{dt^c}\left(\frac{\sum_{n=1}^{N-1} p_n(t^c)v_n}{p_N(t^c)}\right) > 0 \text{ for any } t^c \in [a, b).$$

Now we are ready to prove the proposition. Notice that by construction, $\mathbf{v}(t^c)$ satisfies the participation constraint (11) and

$$TE(\mathbf{v}(t^c)) = \sum_{j=1}^{N-1} \beta_j(t^c)v_j - \frac{\beta_N(t^c)\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} + t_0 V.$$

Therefore,

$$TE'(\mathbf{v}(t^c)) = \sum_{j=1}^{N-1} \beta_j'(t^c)v_j - \beta_N'(t^c)\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} - \beta_N(t^c)\frac{d}{dt^c}\left(\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}\right).$$

From Lemma 4,

$$\begin{aligned} \beta_j'(t^c) &= -\left[J(t^c)-t_0\right] f_{(N-j+1,N)}(t^c) = -\left[J(t^c)-t_0\right] N \binom{N-1}{N-j} F^{N-j}(t^c)(1-F(t^c))^{j-1} \\ &= -N\left[J(t^c)-t_0\right] p_j(t^c). \end{aligned}$$

Thus, when $t^c < b$,

$$\begin{aligned} TE'(\mathbf{v}(t^c)) &= -N\left[J(t^c)-t_0\right] \left(\underbrace{\sum_{j=1}^{N-1} p_j(t^c)v_j - p_N(t^c)\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}}_{=0} \right) - \beta_N(t^c)\frac{d}{dt^c}\left(\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}\right) \\ &= -\beta_N(t^c)\frac{d}{dt^c}\left(\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}\right). \end{aligned} \tag{15}$$

For part (i), if $\sum_{j=1}^{N-1} p_j(t_0^c)v_j \geq 0$, then Lemma 7 implies that

$$\frac{d}{dt^c}\left(\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}\right) > 0 \text{ for any } t^c \in (t_0^c, b),$$

which, combined with (15), further implies that $TE'(\mathbf{v}(t^c))$ has the opposite sign to $\beta_N(t^c)$ in (t_0^c, b). Thus, if $\sum_{j=1}^{N-1} p_j(t_0^c)v_j \geq 0$, then $TE(\mathbf{v}(t^c))$ achieves its unique optimum over $[t_0^c, b)$ at $t^c = t^*$; this is because $TE(\mathbf{v}(t^c))$ is differentiable in $[t_0^c, b)$ and $TE'(\mathbf{v}(t^c))$ has the opposite sign to $\beta_N(t^c)$ in (t_0^c, b), which, by definition, satisfies[15]

$$\beta_N(t^c) \begin{cases} < 0, \text{ if } t^c < t^* \\ = 0, \text{ if } t^c = t^* \\ > 0, \text{ if } t^c > t^* \end{cases}.$$

Thus, it remains to show that $\sum_{j=1}^{N-1} p_j(t_0^c)v_j \geq 0$. If the induced entry threshold $t_0^c = a$, $\sum_{j=1}^{N-1} p_j(t_0^c)v_j = \sum_{j=1}^{N-1} p_j(a)v_j = 0 \geq 0$. If $t_0^c > a$, then, by definition,

$$v_N = -\frac{\sum_{j=1}^{N-1} p_j(t_0^c)v_j}{p_N(t_0^c)}.$$

[15] If $t^* = a$, then $\beta_N(t^c) \geq 0$ for all $t^c \in [a, b)$ with equality only possibly when $t^c = a$.

Lemma 1 implies that $v_N < 0$, which further implies that $\sum_{j=1}^{N-1} p_j(t_0^c)v_j > 0$.

Finally, what is left is to show that $\mathbf{v}(t^c)$ satisfies constraints (9)–(11) when $t^c \in [t_0^c, b)$. To see this, note that $\mathbf{v}(t^c)$ satisfies (11) by construction; for the rest two constraints, it suffices to verify that $-\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \leq v_N$ when $t^c \in [t_0^c, b)$, because $\mathbf{v}$ is a prize structure (so $v_1 \geq \cdots \geq v_N$ and $\sum_{j=1}^{n} v_j \leq V$ for all n). Note that by definition, $-\sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c) = v_N$, so it remains to verify that $\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \geq \sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c)$. To see why this is true, notice that $\sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c) = -v_N \geq 0$ by Lemma 1. Then, Lemma 7 implies that $\sum_{j=1}^{N-1} p_j(t)v_j/p_N(t)$ is increasing in $t \in [t_0^c, b)$. $\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \geq \sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c)$ when $t^c \in [t_0^c, b)$ then follows from $t^c \geq t_0^c$.

For part (ii), if

$$\frac{d}{dt^c}\left(\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}\right) > 0 \text{ for any } t^c \in [a, b), \tag{16}$$

then, similarly, (15) directly implies the claim of optimality in this proposition. However, notice that it is assumed that $v_{N-1} \geq 0$ and $t_0^c > t^*$, which, by Lemma 1, implies that $v_1 > 0$. (16) then follows directly from part (ii) of Lemma 7. Finally, what is left is to show that $\mathbf{v}(t^c)$ satisfies constraints (9)–(11) when $t^c \in [a, b)$. To see this, note that $\mathbf{v}(t^c)$ satisfies (11) by construction; for the rest two constraints, it suffices to verify that $-\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \leq 0$, because $v_{N-1} \geq 0$. This is obvious, because $v_1 \geq \cdots \geq v_{N-1} \geq 0$ immediately implies $-\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \leq 0$.

This completes the proof of the proposition. □

Proof of Theorem 1: We show that t^* is the unique optimal entry threshold. Suppose, to the contrary, that t^{**} is an optimal entry threshold with $t^{**} \neq t^*$. Let $\mathbf{v} = (v_1, \ldots, v_N)$ be the corresponding optimal prize structure to threshold t^{**}. There are three cases.

Case 1: $t^{**} < t^*$. In this case, part (i) of Proposition 3 implies that the expected overall effort induced by $\mathbf{v}$ is strictly less than that by

$$\mathbf{v}(t^*) = (v_1, \ldots, v_{N-1}, -\frac{\sum_{j=1}^{N-1} p_j(t^*)v_j}{p_N(t^*)}),$$

which induces threshold t^*. The contradiction will arise if we can show that $\mathbf{v}(t^*)$ satisfies constraints (9)–(11), which is true by Proposition 3.

Case 2: $t^{**} \in (t^*, b)$. In this case, Proposition 2 implies that $v_{N-1} \geq 0$. Since $v_{N-1} \geq 0$, part (ii) of Proposition 3 implies that the expected overall effort induced by $\mathbf{v}$ is strictly less than that by

$$\mathbf{v}(t^*) = (v_1, \dots, v_{N-1}, -\frac{\sum_{j=1}^{N-1} p_j(t^*)v_j}{p_N(t^*)}),$$

which induces threshold t^*. The contradiction will arise if we can show that $\mathbf{v}(t^*)$ satisfies constraints (9)–(11), which is true by Proposition 3.

Case 3: $t^{**} = b$. In this case, the mechanism induces no entry, so that the expected overall effort is $t_0 V$. However, for the prize structure

$$\mathbf{v}^* = (V, \underbrace{0, \cdots, 0}_{N-2 \text{ times}}, -\frac{p_1(t^*)V}{p_N(t^*)}),$$

which induces the entry threshold t^* and satisfies constraints (9)–(11), it yields an expected overall effort of

$$TE(\mathbf{v}^*) = \underbrace{\beta_1(t^*)V - \beta_N(t^*)\frac{p_1(t^*)V}{p_N(t^*)}}_{=\beta_1(t^*)V>0} + t_0 V > t_0 V.$$

The inequality follows from Lemma 5 and the fact that if $t^* > a$, then $\beta_N(t^*) = 0$ and $\beta_1(t^*) > 0$; if $t^* = a$, then by the definition of t^*, $\beta_N(a) \geq 0$, so $\beta_1(a) > 0$ (note $p_1(a) = 0$ and $p_N(a) = 1$). Thus, the no-entry case is strictly dominated by a prize structure inducing t^*. $\square$

Proof of Theorem 2: If $t^* = a$—i.e., $\beta_N(a) \geq 0$—then $\mathbf{v}^* = (V, 0, \cdots, 0)$ is optimal. To see this, note that in this case $\beta_1(a) > \cdots > \beta_N(a) \geq 0$ by Lemma 5. Lemma 1 implies that for any prize structure $\mathbf{v} = (v_1, \cdots .v_N)$ inducing threshold a, we have $v_N \geq 0$. Now, suppose, to the contrary, that $\hat{\mathbf{v}} = (\hat{v}_1, \dots, \hat{v}_N) \neq \mathbf{v}^*$ is optimal. Since $\hat{v}_1 \geq \cdots \geq \hat{v}_N \geq 0$, there exists some integer $k \in \{2, \dots, N\}$ such that $\hat{v}_k > 0$. Let $j \in \{k, \dots, N\}$ be the largest integer such that $\hat{v}_j > 0$. Consider the prize structure

$$\check{\mathbf{v}} = (\hat{v}_1 + \hat{v}_j, \hat{v}_2, \dots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \dots, \hat{v}_N),$$

which differs from $\hat{\mathbf{v}}$ only at the 1st and jth prizes. Obviously, $\check{\mathbf{v}}$ satisfies constraints (9)–(11) for threshold $t^c = a$. However,

$$TE(\check{\mathbf{v}}) - TE(\hat{\mathbf{v}}) = \underbrace{\big(\beta_1(a) - \beta_j(a)\big)}_{>0} v_j > 0,$$

which contradicts the optimality of $\hat{\mathbf{v}}$. Therefore, when $t^* = a$, $\mathbf{v}^* = (V, 0, \cdots, 0)$ is optimal.

We proceed to the case when $t^* > a$. Suppose, to the contrary, that $\hat{\mathbf{v}} = (\hat{v}_1, \ldots, \hat{v}_N) \neq \mathbf{v}^*$ is optimal. Then there must exist some integer $k \in \{2, \ldots, N-1\}$ such that $\hat{v}_k \neq 0$. Note that $\hat{v}_N < 0$ and $\hat{v}_1 > 0$ by Lemma 1. There are two cases to consider.

Case 1: $\hat{v}_k < 0$. Due to monotonicity (9), there exists some integer $j \in \{2, \ldots, k\}$ such that $\hat{v}_{j-1} \geq 0$ and $\hat{v}_j < 0$—that is, j is the smallest integer such that $\hat{v}_j < 0$. Consider the prize structure

$$\check{\mathbf{v}} = (\hat{v}_1, \ldots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \ldots, \hat{v}_{N-1}, \hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)}\hat{v}_j),$$

which differs from $\hat{\mathbf{v}}$ only at the jth and Nth prizes. Obviously, since $\hat{v}_j < 0$, $\check{\mathbf{v}}$ satisfies constraints (9)–(11) for threshold t^*. However,

$$TE(\check{\mathbf{v}}) - TE(\hat{\mathbf{v}}) = \hat{v}_j \left(\beta_N(t^*)\frac{p_j(t^*)}{p_N(t^*)} - \beta_j(t^*)\right) = \hat{v}_j p_j(t^*) \underbrace{\left(\frac{\beta_N(t^*)}{p_N(t^*)} - \frac{\beta_j(t^*)}{p_j(t^*)}\right)}_{<0 \text{ by Lemma 5}} > 0,$$

which contradicts the optimality of $\hat{\mathbf{v}}$.

Case 2: $\hat{v}_k > 0$. Due to monotonicity (9), there exists some integer $j \in \{k, \ldots, N-1\}$ such that $\hat{v}_j > 0$ and $\hat{v}_{j+1} \leq 0$—that is, j is the largest integer such that $\hat{v}_j > 0$. Note that, if $j \leq N-2$, then the argument in Case 1 implies that that $\hat{v}_i = 0$ for all $i \in \{j+1, \ldots, N-1\}$.

Consider the prize structure

$$\check{\mathbf{v}} = (\hat{v}_1 + \hat{v}_j, \hat{v}_2, \ldots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \ldots, \hat{v}_{N-1}, \hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)}\hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)}\hat{v}_j),$$

which differs from $\hat{\mathbf{v}}$ only at the 1st, jth and Nth prizes. Note that since $\hat{v}_i = 0$ for all $i \in \{j+1, \ldots, N-1\}$ (if $j \leq N-2$), to verify that $\check{\mathbf{v}}$ satisfies constraints (9)–(11) for threshold t^*, one only needs to show

$$\hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)}\hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)}\hat{v}_j < 0.$$

This is true, because with $\hat{v}_i = 0$ for all $i \in \{j+1, \ldots, N-1\}$ (if $j \le N-2$), we have

$$\begin{aligned}
\hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)}\hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)}\hat{v}_j &= \frac{1}{p_N(t^*)}\left(p_N(t^*)\hat{v}_N + p_j(t^*)\hat{v}_j - p_1(t^*)\hat{v}_j\right) \\
&= \frac{1}{p_N(t^*)}\left(-\sum_{i=1}^{N-1} p_i(t^*)\hat{v}_i + p_j(t^*)\hat{v}_j - p_1(t^*)\hat{v}_j\right) \\
&= \frac{1}{p_N(t^*)}\left(-\sum_{i=1}^{j-1} p_i(t^*)\hat{v}_i - p_1(t^*)\hat{v}_j\right) < 0,
\end{aligned}$$

(obviously, this argument also applies to the case when $j = N-1$). Thus, $\check{\mathbf{v}}$ satisfies constraints (9)–(11) for threshold t^*. However,

$$TE(\check{\mathbf{v}}) - TE(\hat{\mathbf{v}}) = \hat{v}_j \underbrace{\left(\beta_1(t^*) - \beta_j(t^*)\right)}_{>0 \text{ by Lemma 5}} + \underbrace{\beta_N(t^*)}_{=0}\left(\frac{p_j(t^*)}{p_N(t^*)}\hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)}\hat{v}_j\right) > 0,$$

which contradicts the optimality of $\hat{\mathbf{v}}$.

The proof completes. □

Proof of Proposition 4: (i) To avoid notational confusions, define a two variable function $H(\cdot,\cdot)$: $[a,b] \times \mathbb{Z}_+$ as

$$H(t^c, N) = \beta_N(t^c) = N\int_{t^c}^{b} \left[J(t) - t_0\right](1-F(t))^{N-1} f(t)dt.$$

It is easy to see that if $J(a) - t_0 \ge 0$, then $t_N^* = a$ for any $N \ge 2$. Therefore, when $J(a) - t_0 \ge 0$, $t_N^* = a = t^M$ for any $N \ge 2$, which implies that (i) of Proposition 4 trivially holds.

Now assume that $J(a) - t_0 < 0$. Then for any $N \ge 2$ and for any $t^c \in [a, t^M)$,

$$\begin{aligned}
H(t^c, N+1) &= (N+1)\int_{t^c}^{b} \left[J(t) - t_0\right](1-F(t))^{N} f(t)dt \\
&= (N+1)\left(\int_{t^c}^{t^M} + \int_{t^M}^{b}\right)\int_{t^c}^{b} \left[J(t) - t_0\right](1-F(t))^{N} f(t)dt \\
&< (N+1)(1-F(t^M))\int_{t^c}^{t^M} \left[J(t) - t_0\right](1-F(t))^{N-1} f(t)dt \\
&\quad +(N+1)(1-F(t^M))\int_{t^M}^{t^b} \left[J(t) - t_0\right](1-F(t))^{N-1} f(t)dt \\
&= \frac{N+1}{N}(1-F(t^M))H(t^c, N).
\end{aligned}$$

Therefore, if $H(t^c, N) \leq 0$ for some $N \geq 2$, then $H(t^c, N') < 0$ for any $N' > N$. In other words, for any $t^c \in [a, t^M)$, the function $H(t^c, \cdot)$ has the single-crossing property: It crosses zero at most once; and if it does, it must cross zero from above. The definition of $t^*_{N'}$ implies that: If $H(\tilde{t}^c, N') < 0$, then $t^*_{N'} > \tilde{t}^c$. Based on these observations, we have: If $J(a) - t_0 < 0$, then $t^*_{N'} \geq t^*_N$ for any $N' > N$, with strict inequality when $t^*_N > a$.

Thus, t^*_N is increasing in N. What is left is to show that $\lim_{N\to\infty} t^*_N = t^M$. By definition, $t^*_N < t^M$ for any $N \geq 2$; therefore, the sequence $\{t^*_k\}_k$ converges to some entry threshold $\bar{t} \in [a, t^M]$. We claim that $\bar{t} = t^M$.

In fact, suppose, to the contrary, that $\bar{t} < t^M$ when $t^M > a$. We show that for any given $\bar{t} \in [a, t^M)$, $H(\bar{t}, N) < 0$ when N is large enough. In fact,

$$H(\bar{t}, N) = N \int_{\bar{t}}^{b} [J(t) - t_0] (1-F(t))^{N-1} f(t)dt = N(1-F(t^M))^{N-1} \int_{\bar{t}}^{t^b} [J(t) - t_0] \frac{(1 - F(t))^{N-1}}{(1 - F(t^M))^{N-1}} f(t)dt,$$

which implies that $H(\bar{t}, N)$ has the same sign as $\int_{\bar{t}}^{t^b} [J(t) - t_0] \frac{(1-F(t))^{N-1}}{(1-F(t^M))^{N-1}} f(t)dt$. Now, notice that

$$\int_{\bar{t}}^{t^b} [J(t) - t_0] \frac{(1 - F(t))^{N-1}}{(1 - F(t^M))^{N-1}} f(t)dt = \left(\int_{\bar{t}}^{t^M} + \int_{t^M}^{b} \right) [J(t) - t_0] \left(\frac{1 - F(t)}{1 - F(t^M)} \right)^{N-1} f(t)dt.$$

Observe also that $\frac{1-F(t)}{1-F(t^M)} > 1$ for any $t \in [\bar{t}, t^M)$ and that $\frac{1-F(t)}{1-F(t^M)} < 1$ for any $t \in (t^M, b]$. Moreover, when $N \to \infty$: $\left(\frac{1-F(t)}{1-F(t^M)}\right)^{N-1} \to +\infty$ for any given $t \in [\bar{t}, t^M)$; $\left(\frac{1-F(t)}{1-F(t^M)}\right)^{N-1} \to 0$ for any given $t \in (t^M, b]$. Therefore, when N is large enough, since $J(t) - t_0 < 0$ for any $t \in [\bar{t}, t^M)$, one must have

$$\left(\int_{\bar{t}}^{t^M} + \int_{t^M}^{b} \right) [J(t) - t_0] \frac{(1 - F(t))^{N-1}}{(1 - F(t^M))^{N-1}} f(t)dt < 0,$$

which further implies that for any given $\bar{t} \in [a, t^M)$, $H(\bar{t}, N) < 0$ when N is large enough. However, this implies that when N is large enough, $t^*_N > \bar{t}$, contradicting the fact that $\{t^*_k\}_k$ is an increasing sequence that converges to $\bar{t}$.

(ii) Regarding the magnitude of negative prize, notice that it is equal to

$$\left(\frac{F(t^*_N)}{1 - F(t^*_N)} \right)^{N-1} V,$$

so its limit depends on the comparison between $\frac{F(t^M)}{1-F(t^M)}$ and 1—i.e., $F(t^M)$ and 1/2. When $F(t^M) <$

$\frac{1}{2}$, the limit is 0; when $F(t^M) > \frac{1}{2}$, obviously, the limit does not exist and the magnitude explodes to infinity. $\square$

Proof of Lemma 7: (i) Notice that

$$\begin{aligned}
\frac{d}{dt^c}\left(\frac{\sum_{n=1}^{N-1} p_n(t^c)v_n}{p_N(t^c)}\right) &= \frac{d}{dt^c}\left[\sum_{n=1}^{N-1}\binom{N-1}{n-1}\left(\frac{F(t^c)}{1-F(t^c)}\right)^{N-n} v_n\right] \\
&= \underbrace{\frac{d}{dt^c}\left(\frac{F(t^c)}{1-F(t^c)}\right)}_{>0}\cdot\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}\left(\frac{F(t^c)}{1-F(t^c)}\right)^{N-n-1} v_n,
\end{aligned}$$

so it suffices to show that

$$\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}\left(\frac{F(t^c)}{1-F(t^c)}\right)^{N-n-1} v_n > 0 \text{ for any } t^c \in (t_0^c, b). \tag{17}$$

To this end, note that since $\sum_{n=1}^{N-1} p_n(t_0^c)v_n \geq 0$ and $v_1 \geq \cdots \geq v_{N-1}$, either $v_1 > 0$ and $v_{N-1} \geq 0$ or there exists some integer $j \in \{2,\ldots,N-1\}$ such that $v_k \geq 0$ and $v_{k'} < 0$ for all $k \in \{1,\ldots,j-1\}$ and all $k' \in \{j,\ldots,N-1\}$. In the former case, (17) obviously holds. For the latter case, v_1 must be strictly positive and notice that

$$\begin{aligned}
&\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}\left(\frac{F(t^c)}{1-F(t^c)}\right)^{N-n-1} v_n \\
= \;& \frac{\sum_{n=1}^{N-1}(N-n)p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}} = \frac{\sum_{n=1}^{j-1}(N-n)p_n(t^c)v_n + \sum_{n=j}^{N-1}(N-n)p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}} \\
> \;& \frac{\sum_{n=1}^{j-1}(N-j)p_n(t^c)v_n + \sum_{n=j}^{N-1}(N-j)p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}} = \frac{(N-j)\sum_{n=1}^{N-1} p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}}.
\end{aligned}$$

Thus, in this case, one only needs to show

$$\sum_{n=1}^{N-1} p_n(t^c)v_n > 0 \text{ for any } t^c \in (t_0^c, b),$$

which must be true, because in this case, $v_N < 0$ (because $v_{k'} < 0$ for all $k' \in \{j,\ldots,N-1\}$), which implies that if $\sum_{n=1}^{N-1} p_n(t_1^c)v_n \leq 0$ for some $t_1^c \in (t_0^c, b)$, then $\sum_{n=1}^{N} p_n(t_1^c)v_n < 0$, which contradicts the fact that $\sum_{n=1}^{N} p_n(t_0^c)v_n = 0$ and $\sum_{n=1}^{N} p_n(t_1^c)v_n \geq \sum_{n=1}^{N} p_n(t_0^c)v_n$ (because $t_1^c > t_0^c$ and $\sum_{n=1}^{N} p_n(t^c)v_n$ is increasing in t^c, as established in the proof of Lemma 1).

(ii) Similar to the proof of part (i), it suffices to show that

$$\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}\left(\frac{F(t^c)}{1-F(t^c)}\right)^{N-n-1} v_n > 0 \text{ for any } t^c \in [a,b),$$

which is obviously true, because $v_1 > 0$ and $v_{N-1} \geq 0$ (so $v_2 \geq \cdots \geq v_{N-1} \geq 0$). $\square$

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