Information Revelation in Relational Contracts*

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Abstract

This paper shows that the efficiency of relational contracting can be increased by reducing the public information through a novel intertemporal-garbling process of signals. A distinctive and essential feature of our intertemporal-garbling process is that past outputs have enduring effects on future signals. This process reduces the principal's maximal reneging temptation by linking together the principal's non-reneging constraints both across states and over time.

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1 Introduction

This paper shows that partial and delayed revelation of information can sometimes increase the efficiency of relational contracting. Models of relational contracting are repeated principal-agent games with noncontractible side payments. They capture environments with high transaction costs for drafting and implementing formal contracts. The prevalence and importance of relational contracting have been emphasized both by economics and non-economics literature; see, for example, Bull (1987), Baker, Gibbons, and Murphy (1994, 2002), Chassang (2010), Fuchs (2007), Levin (2002, 2003), Macauley (1963), MacLeod (2007), MacLeod and Malcomson (1989, 1998), MacNeil (1978), and Malcomson (2008), and Rayo (2007).

It is well-recognized that relational contracting is harder to sustain when the underlying environment is more volatile. This is because sustaining a relational contract requires the principal's maximal reneging temptation not to exceed the amount of discounted future surplus of the relationship. A productive relationship can be prevented from starting when the reneging temptations are destructively high in some rare instances even if most of the time the principal's reneging temptations are not of concern.

The main contribution of our paper is to show that, through a novel intertemporal signal-garbling process, the slackness of the non-reneging constraint in some states can be exploited to enhance the efficiency of relational contracting. Intertemporal garbling repartitions and smooths the principal's reneging temptations both across states (information sets) and over time. This reduces the principal's maximal reneging temptation.[1]

A distinctive feature of the intertemporal garbling process considered here is that the original, ungarbled signals have enduring effects on future, intertemporally garbled signals. To give an example of intertemporal garbling processes with this property, suppose the original signals are equal to the output levels in each period. One intertemporal garbling process with enduring effects is to garble the output levels to two signals: "Good" and "Bad." Signal "Good" is realized if the average past outputs exceeds a threshold; signal "Bad" is realized otherwise. In this example, since the average output level up until any time depends on past outputs, the output level at time t affects all garbled signals from time t on.

[1] To simplify the exposition, we normalize the minimal reneging temptation to 0.

Specifically, the baseline model in this paper builds on Levin (2003), where the agent's efforts are his private information but the outputs are publicly observed. The key departure from Levin (2003) is that the public signal is not necessarily equal to the output in each period. Instead, public signals are realized through a signal-generating function that maps the entire past history of outputs into probability distributions over the set of possible signals. To focus on the effect of the information structure on efficiency, we study binary effort levels (effort or no effort) and binary output levels (high or low). An efficient equilibrium requires the agent to exert effort in each period.

We report three results. First, when players are restricted to using public strategies, for any $p \in (0,1)$, where p is the probability of high output given effort, there exists an intertemporal garbling process (and an associated Perfect Public Equilibrium (PPE)) that improves efficiency over perfect observability, i.e., the signal is equal to the output in each period, as long as $p \neq \frac{1}{2}$. Second, when $p = \frac{1}{2}$, perfect observability is the optimal information structure. In other words, no information structure can sustain efficiency unless it can be sustained under perfect observability of signals.

Third, and most importantly, when players can use private strategies, there exists an intertemporal garbling process and an associated Perfect Bayesian Equilibrium (PBE) that improves efficiency for sufficiently small p. We focus on PBE instead of PPE, which is the more common equilibrium concept in this literature because the agent's past effort, through affecting past outputs, influences the distribution of future signals under intertemporal garbling. This implies that the agent can use private information of his past actions to his advantage in the future, and an appropriate equilibrium concept should allow the agent to use this information.

Allowing for private strategies, however, introduces additional technical difficulties. The one-stage-deviation-principle can no longer be used to verify whether a strategy profile forms a PBE. Specifically, once the agent deviates from the equilibrium, he will have a different belief about the distribution of future signals from that of the principal (even if they share the same belief about future actions). This difference in beliefs implies that the agent may benefit from multi-stage deviations. Checking multi-stage deviations is generally difficult, and our technique of verifying whether a strategy profile forms PBE may be of independent interest.

Our paper contributes to two strands of literature. First, it contributes to the theoretical works that explores the relationship between the information structure and efficiency. In this literature, Kandori (1992) shows that garbling signals within periods weakly decreases efficiency in repeated games with imperfect public monitoring. Applying Kandori's analysis to our setting of relational contracting allows us to show that garbling within periods also weakly decreases efficiency. When signals are garbled within-period, the agent is rewarded for low outputs with some probability in each period, leading to a weaker incentive for the agent to exert effort. This extra incentive cost is reduced by intertemporal garbling because whenever an agent is rewarded for a low output, his continuation value suffers.

In addition, Abreu, Milgrom, and Pearce (1992) (AMP hereafter), in the context of repeated games, and Fuchs (2007), in relational contracting with subjective evaluation, show that efficiency can be increased by bundling signals across a fixed number of periods. Such bundling of signals increases efficiency because it allows punishment (for low outputs) to be reused, and, thus, reduces the surplus destruction in the relationship. The AMP-Fuchs type of bundling does not help in our setting, where the baseline model is one of imperfect public monitoring. Bundling increases the maximal bonus required to incentivize the agent since the bonus are paid out less frequently. In fact, any garbling process that has a fixed "restart date" will not help. To increase the efficiency in our setting, it is essential that the bygones are never completely bygones, i.e., future signals are always affected by past outputs.

In two other related papers in this literature, Kandori and Obara (2006) show that when signals do not have full support, then the use of private (mixed) strategies can give rise to equilibria that are more efficient. In our baseline model, signals have full support so there is no loss of generality in restricting to public strategies. Instead we show that efficiency can still be enhanced through persistent intertemporal signal garbling. Ekmekci (forthcoming) examines a product choice game between a long-run player and a sequence of short-run players. He defines a rating system as a mapping from past outputs to signals, and this is similar to our intertemporal-garbling process. In his setting, rating system does not help efficiency when there is no reputation effects. However, the rating system can help when there exists commitment types.

Second, this paper contributes to the literature that studies the use of external instruments to increase the efficiency of relational contracting. Baker, Gibbons, and Murphy (1994) show that the use of explicit contract can help increase the efficiency of the relationship via reducing the gain from reneging.[2] Rayo (2007) examines the role of ownership structure in sustaining relationship. He shows that when actions of the players are unobservable (and that the First Order Approach is valid), the optimal ownership shares should be concentrated. When the actions are observable (so that the First Order Approach is invalid), the optimal ownership shares should be diffused. The external instrument explored in our paper is the use of information flows. Our result implies that the efficiency of the relationship can be enhanced with less information (in the sense that the signals are intertemporally-garbled). This suggests that, via strategic manipulating of information, intermediaries can increase the efficiency of the relationship.

The rest of the paper is organized as follows: we set up the model in Section 2; we present our main results in Section 3. Section 4 concludes.

2 Setup

Time is discrete and indexed by $t \in \{1, 2, ..., \infty\}$.

2.1 Players

There is one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount factor δ. The agent's per-period outside option is $\underline{u}$; the principal's per-period outside option is $\underline{\pi}$.

2.2 Production

If the principal and the agent engage in production together in period t, the agent chooses effort $e_t \in \{0, 1\}$. The cost of effort is given by $c(0) = 0$ and $c(1) = c$. The output is binary: $Y_t \in \{0, y\}$. We assume that

$$\begin{aligned} \Pr\{Y_t &= y | e_t = 1\} = p; \\ \Pr\{Y_t &= y | e_t = 0\} = q, \end{aligned}$$

where $1 > p > q \geq 0$.

[2] They also show, however, that explicit contracts can crowd out relational contracts via improving the outside options of the players.

To make the analysis interesting, we assume that the relationship is valuable if and only if the agent puts in effort. In other words,

$$py - c > \underline{u} + \underline{\pi} > qy.$$

2.3 Information Structure

In each period t, the agent's effort, e_t, is his private information. In addition, the principal and the agent both observe a *public signal,* s_t, after the output is realized. We assume that s_t is generated by a signal-generating function, S_t, which maps the set of past outputs into the probability distribution on the set of possible signals, $\mathbf{S}$:

$$S_t : \prod_{j=1}^{t} Y_j \to \Delta\mathbf{S}.$$

The signal-generating function provides a framework for modelling information structures and below are some examples.

Example 1 (Perfect Observability of Outputs)

In a standard model of relational contracts with imperfect public monitoring (see, for example Levin (2003)), outputs are observed each period. In this case, $\mathbf{S} = \{0, y\}$, and in each period t the signal s_t is equal to y_t, the output in period t. More formally, the signal generating function is given by

$$\Pr(S_t(y_1, ..., y_t) = y_t) = 1, \text{ for all } \{y_1, ..., y_t\}.$$

Example 2: (T-period Revelation)

An information structure that has received considerable attention from economists is T-period revelation, i.e., the case where outputs are perfectly revealed every T periods and no information is revealed in between (see, for example, Abreu, Milgrom, Pearce (1991) and Fuchs (2007)). In this case, $\mathbf{S} = \{0, y\}^T \cup \{N\}$, where N stands for no information. When $t \neq nT$ for each $n \in N$, the signal $s_t = N$. When $t = nT$, $s_t = (y_{(n-1)T+1}, ..., y_{nT})$. More formally, the signal distribution function is given by

$$\Pr(S_t(y_1, ..., y_t) = N) = 1, \text{ when } t \neq nT,$$

$$\Pr(S_t(y_1,...,y_t) = (y_{(n-1)T+1},...,y_{nT})) = 1, \text{ when } t = nT.$$

Example 3: (Partial Information Revelation)

In the above two examples, each output is (eventually) known perfectly. In this example, only partial information about past outputs is revealed. Let the set of the signals be $S = \{Success, Failure\}$. In period t, signal $s_t = Success$ if more than half of the previous outcomes $y = Y$, and $s_t = Failure$ otherwise. More formally,

$$\begin{aligned}
\Pr(S_t(y_1,...,y_t) &= Success) = 1, \text{ if } \sum_{j=1}^{t} y_j > \frac{ty}{2}; \\
\Pr(S_t(y_1,...,y_t) &= Failure) = 1, \text{ if } \sum_{j=1}^{t} y_j \leq \frac{ty}{2}.
\end{aligned}$$

Under this signal-generating function, the signal in any period depends on the entire past history of outputs. Therefore, each signal is affected by the outputs in the distant past. Conversely, each output affects signals in the arbitrarily far future. This feature is crucial for our later results.

2.4 Timing

The timing is as follows. At the beginning of period t, the principal offers a contract that consists of a fixed wage, w_t. The agent chooses whether to accept it: $d_t \in \{0, 1\}$. If the agent rejects ($d_t = 0$), both the principal and the agent receive their outside options. If the agent accepts, he chooses e_t. The signal, s_t, is realized and the principal pays out $W_t \geq w_t$. Just as in the analysis of relational contracts with a perfect revelation of outputs (e.g., Levin (2002)), this restriction to nonnegative bonus helps simplify the exposition without affecting the set of equilibrium payoffs sustainable by relational contracts.

Unlike the case of perfectly revealed outputs, however, the timing of the bonus does affect the set of equilibrium payoffs sustainable by relational contracts. The reason has to do with the agent's possible multi-stage deviation that we will discuss in more detail in Subsection 3.3. Our timeline allows the principal either to pay the contingent bonus at the end of each period or to postpone the bonus to the beginning of the following period as part of the efficiency wage.[3] In the former case, payment

$$W_t = w_t + b_t,$$

where b_t is the contingent bonus. In the later case,

$$\begin{aligned} W_t &= w_t; \\ w_{t+1} &= \underline{w} + \frac{b_t}{\delta}, \end{aligned}$$

where $\underline{w}$ is the wage paid to the worker if the output is 0.

2.5 Strategy and Equilibrium Concept

2.5.1 History

We denote $h_t = \{w_t, d_t, s_t, W_t\}$ as public events that occur in period t and $h^t = \{h_n\}_{n=0}^{t-1}$ as the public history path at the beginning of period t. We set $h^1 = \Phi$. Let $H^t = \{h^t\}$ be the set of public history paths until time t and $H = \cup_t H^t$ be the set of public histories. The principal only observes the public history. For the agent, however, also observes his past actions $e^t = \{e_j\}_{j=1}^{t-1}$ at the beginning of period t. Denote $H_A^t = H^t \cup \{e^t\}$ as the set of the agent's private history at the beginning of period t.

2.5.2 Strategy and Payoff

In period t, the following functions capture the strategies of the players.

- The principal's wage offer is given by

$$w_t : H^{t-1} \to R.$$

- The agent's acceptance decision is given by

$$d_t : H_A^{t-1} \times \{w_t\} \to \{0, 1\},$$

  where the second component in the cross product denotes the set of wage offers.

[3] The principal can also do both, i.e., pay a part of the bonus at the end of a period and postpone the rest to the beginning of the following period.

- The agent's effort decision is given by

$$e_t : H_A^{t-1} \times \{w_t\} \rightarrow \{0, 1\}.$$

- The total compensation function is given by

$$W_t : H^{t-1} \times D_t \times S_t \rightarrow R.$$

The pure strategy of the agent is given by

$$s^A = \{d_t, e_t\}_{t=1}^{\infty}.$$

The pure strategy of the principal is given by

$$s^P = \{w_t, W_t\}_{t=1}^{\infty}.$$

We can also allow the principal and the agent to play mixed strategies. We restrict our attention to pure strategies partly to distinguish this paper from the literature of repeated games with private monitoring, where mixed strategies play a crucial role.

Take a strategy profile (s^A, s^P). The expected payoff of the agent following a private history h_A^t and w_t is given by

$$U(h_A^t, w_t, s^A, s^P) = E[\sum_{\tau=t}^{\infty} \delta^{\tau-t}\{\underline{u} + 1_{\{d_\tau=1\}}(-ce_\tau + W_\tau - \underline{u})\}|h_A^t, s^A, s^P].$$

We can define $U(h_A^t, w_t, d_t, s^A, s^P)$, the expected payoff of the agent following his acceptance decision in period t in similar fashion.

The principal's expected payoff following the agent's private history h_A^t is given by

$$\pi(h_A^t, s^A, s^P) = E[\sum_{\tau=t}^{\infty} \delta^{\tau-t}\{\underline{\pi} + 1_{\{d_\tau=1\}}(y(q + (p-q)e_\tau)$$
$$-W_\tau - \underline{\pi})\}|h_A^t, s^A, s^P].$$

Since the principal does not observe the agent's private history, we define

$$\Pi(h^t, s^A, s^P) = E_{\mu^P}[\pi(h_A^t, s^A, s^P)|h^t]$$

as his expected payoff following public history h^t. Here, the expectation is taken over all of the agent's possible private histories (h_A^t) according to the principal's belief (μ^P) conditional on observing public history h^t.

We also denote $\pi(h_A^t, w_t, d_t, s_t, s^A, s^P)$ as the principal's expected payoff in period t following the agent's private history h_A^t, the principal's wage offer w_t, the agent acceptance decision d_t, and the signal s_t. We define $\Pi(h^t, w_t, d_t, s_t, s^A, s^P)$ in a similar way.

2.5.3 Equilibrium Concept

When outputs are publicly observed, the standard equilibrium concept is Perfect Public Equilibrium (PPE). A strategy profile forms a PPE if the strategies of the players only depend on the public history. Moreover, following any public history, the strategies of the players form a Nash Equilibrium. The advantage of having PPE as the equilibrium concept is that the set of PPE payoffs is stationary following any history, and this allows for a recursive formulation in characterizing the PPE payoffs. This recursive formulation keeps the analysis tractable. In Section 3, we show that if the players are restricted to using public strategies, intertemporal garbling can help improve efficiency for all $p \neq \frac{1}{2}$ when the discount factor approaches 1.

It is well-known, however, that the restriction to public strategies is not without loss of generality. One reason is that it prevents the players from exploiting the efficiency gain of using mixed strategies. In the current setting, PPE is restrictive even if all players use pure strategies. When future signals depend on past outputs, the agent can (and should) use his past private actions in forming expectations regarding the distribution of future signals. Therefore, we believe that the more appropriate equilibrium concept is Perfect Bayesian Equilibrium (PBE).

A PBE in this model consists of the principal's strategy (s^{*P}), the agent's strategy (s^{*A}), the principal's belief (μ^P), and the agent's belief (μ^A), such that:

- following any history $\{h_A^t, w_t\}$ and $\{h_A^t, w_t, d_t\}$,

$$\begin{aligned} U(h_A^t, w_t, s^{*A}, s^{*P}) &\geq U(h_A^t, w_t, \widetilde{s}^A, s^{*P}); \\ U(h_A^t, w_t, d_t, s^{*A}, s^{*P}) &\geq U(h_A^t, w_t, d_t, \widetilde{s}^A, s^{*P}); \end{aligned}$$

- following any history h^t and $\{h^t, w_t, d_t, s_t\}$,

$$\begin{aligned} \Pi(h^t, s^{*A}, s^{*P}) &\geq \Pi(h^t, s^{*A}, \widetilde{s}^P); \\ \Pi(h^t, w_t, d_t, s_t, s^{*A}, s^{*P}) &\geq \Pi(h^t, w_t, d_t, s_t, s^{*A}, \widetilde{s}^P). \end{aligned}$$

- the beliefs are consistent with σ^* and are updated with Bayes rule whenever possible.

In addition, a PBE requires specifying the beliefs of the players. In this game, since only the agent has private information, the agent's belief is degenerate. The principal knows the public history, and his belief of the agent's private history is again degenerate whenever the agent's actions are consistent with the equilibrium play. When the agent's actions do not conform to the equilibrium play, we assume the principal believes that the agent has never put in effort in the past.

While PBE is the more appropriate concept, it is difficult to establish that a strategy profile forms a PBE. The difficulty arises because there is generally no recursive formulation when the agent's strategy can depend on his private actions. In particular, if the agent deviates from the equilibrium action, his belief about the distribution of future signals becomes different from that of the principal (even if they share the same belief about future actions). The difference in beliefs results from the intertemporal garbling of signals: future signals are affected by past outputs, which, in turn, are affected by the agent's past actions. Thus, if the agent deviates from the equilibrium action, he forms a different belief about the future than the principal, and this difference in beliefs implies that the set of PBE payoffs cannot be formulated recursively, making it difficult to check whether a strategy profile is a PBE.

3 Analysis

In this section, we study how the information structure affects the efficiency of the relational contract. In Section 3.1, we review the necessary and sufficient condition to sustain an efficient relational contract when outputs are perfectly observed. In Section 3.2, we present the main idea of how intertemporal garbling can help smooth bonus payments and increase efficiency, except when $p = \frac{1}{2}$. Finally, in Section 3.3, we formally construct an information structure (and an associated PBE) that obtains the efficient outcome when it is impossible to do so when outputs are perfectly observed.

3.1 Review: Perfect Observability of Outputs

Suppose the signals are perfectly informative of the outputs, i.e., $s_t = y_t$ for all t. Levin (2003) shows in such setting that the optimal relational contract is (constrained) efficient and can be implemented through a sequence of stationary contracts. Applying Levin (2003) to our setting, the stationary contract can be characterized by a base wage w and a performance bonus $b > 0$ for high output.

For this contract to induce effort from the agent, the bonus must be sufficiently big:

$$b \geq \frac{c}{p-q}. \tag{1}$$

When there is no restriction in payments, the principal can capture the entire surplus of the relationship by setting the base wage such that the agent's payoff inside the relationship is equal to his outside option:

$$w - c + pb = \underline{u}.$$

Finally, since the bonus is non-contractible, for the principal not to renege on the bonus, the following must be true:

$$b \leq \frac{\delta(py - c - \underline{u} - \underline{\pi})}{1-\delta}, \tag{2}$$

where $\frac{\delta(py-c-\underline{u}-\underline{\pi})}{1-\delta}$ is the discounted expected future surplus that is completely captured by the principal.

Combining equations (1) and (2) shows that a relational contract can induce effort if and only if

$$\frac{c}{p-q} \leq \frac{\delta(py - c - \underline{u} - \overline{\pi})}{1-\delta}. \tag{3}$$

In other words, the incentive cost should be smaller than the discounted expected future surplus. Inequality (3) implies that the sustainability of the relational contract depends on the extremes. In other words, the set of discount factors (δ) that allow for efficiency is completely determined once the value of the **maximal reneging temptation** ($\frac{c}{p-q}$) and the expected per period surplus in the relationship ($py - c - \underline{u} - \overline{\pi}$) are given.

3.2 Intertemporal Signal Garbling

When outputs are observed perfectly, the principal's non-reneging constraints are only binding in states where the outputs are high and are slack otherwise. When the probability of high outputs (p) decreases, the reneging temptations become more concentrated and larger in size ($\frac{c}{p-q}$) when high outputs are realized. This makes the efficient relational contract harder to sustain (controlling for surplus, i.e., keeping py and c the same). Also when p is small, the reneging temptations are slack most of the time.

The basic idea of intertemporal garbling is that, by making the signals less informative of the outputs, we can link the reneging temptations across different states (high or low outputs) and over time. In other words, intertemporal garbling repartitions the information sets to smooth the bonus payments and, thus, reduces the maximal reneging temptation. Specifically, when the signals are garbled intertemporally, a high output increases both the current and future payoffs of the agent. This allows the principal to reduce the current bonus (compared to the case of perfect observability of outputs) while maintaining incentive. While the idea is natural, one difficulty is to do this every period without causing the agent's future payoff to explode following some equilibrium play path.

To implement this idea, consider the following signal generating-process for $p < \frac{1}{2}$. The case of $p > \frac{1}{2}$ is its mirror image and will be described after Theorem 1. There are two public signals: good and bad. Following any public history, there are $n > \frac{1-p}{1-2p}$ "secret states," with lower states being more favorable to the agent. For an agent in state $k \geq 2$, if the output is high, then the public signal is good, and the agent transitions into secret state $k-1$ with probability $\frac{k-1}{n-1}$ and stays in secret state k with probability $\frac{n-k}{n-1}$. If the output is low, then the signal is bad, and the agent transitions into secret state $k-1$ with probability $\frac{(n+1-k)p}{(n-1)(1-p)}$ and stays in secret state k with probability $1-\frac{(n+1-k)p}{(n-1)(1-p)}$. Note that the players do not know the agent's secret state.

For the agent in secret state 1, if the output is high, then the signal is good, and the agent stays in secret state 1. If the output is low, then with probability $\frac{pn}{(1-p)(n-1)}$, the signal is good, and the agent transitions into secret state n. With probability $1-\frac{pn}{(1-p)(n-1)}$, the signal is bad, and the agent remains in secret state 1.

The following figure depicts the signal-generating process and the transition of states for $n=3$.

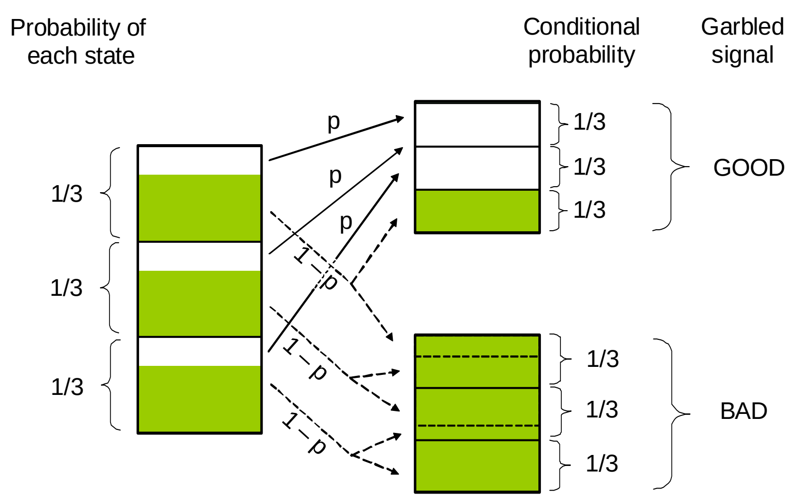

Figure 1: Intertemporal Signal Garbling w/ n = 3

A key feature of this transition function is that it helps maintain stationarity. In particular, if initially the n states are equally likely to occur and the agent puts in effort, then they are again equally likely ot occur in the next period, regardless of whether the signal is good or bad, and the probability that a good signal is announced each period is given by

$$p+\frac{1}{n}[(1-p)\frac{pn}{(1-p)(n-1)}]=\frac{np}{n-1}=\rho.$$

The stationarity is not essential for the idea of intertemporal garbling, but it helps the analysis by ensuring that the principal's maximal reneging temptation is equal to the bonus amount.

Note that the signal-generating process above repartitions the information sets *across time*: if the signals are only garbled within each period, then they become less informative of effort, which exacerbates the incentive problem. Such insight is due to Kandori (1992), who formalizes it in the context of repeated games without transfers. Also note that intertemporal garbling will not help in our environment if it is conducted across a fixed number of periods because once the last period is reached, the agent can no longer be rewarded with a higher continuation payoff. This helps explain why garbling using AMP's T-period revelation will not improve efficiency in the current setting.

Now suppose the players use public strategies.[4] In particular, consider strategies characterized by the following two-state automaton. Initially, all n states are equally likely to occur. On the equilibrium path, a) the principal always offers the same stationary contract with a base wage (w) and a bonus (b) and pays out the bonus for good signal, and b) the agent always accepts the offer and puts in effort. Off the equilibrium path (if either party publicly deviates), the principal and the agent take their outside options.[5] In addition, let the principal set the base wage $w = \underline{u}+c-\frac{pn}{n-1}b$ so that she captures the entire surplus of the relationship.

Denote v_k as the agent's equilibrium payoff of being in state k. When the states are equally likely to occur, the agent's expected payoff is given by

$$EV \equiv \frac{\sum_{i=1}^{n} v_i}{n} = \frac{w-c+\rho b}{1-\delta}.$$

Denote EV_H and EV_L as the agent's expected payoff for a high output and a low output, respectively.[6] Note that the difference in EV_H and EV_L is proportional to b,

[4] As mentioned in Section 2, the restriction to public strategy is not without loss of generality. Nevertheless, such restriction simplifies the analysis and helps illustrate why intertemporal garbling helps improve efficiency. The restriction to public strategies is lifted in the next section where, focusing on the case of small p, we show that intertemporal garbling can also help improve efficiency when the players can use private strategy.

[5] Public deviations by the principal include offering a different contract and not paying a bonus for a good signal. Public deviation by the agent includes not accepting the contract. More formally, if a public deviation ever occurs, the principal offers a base wage $w = \underline{u} - 1$ and does not pay out the bonus in all future periods. The agent rejects all contracts with base wage $w < \underline{u}$ and does not put in effort.

[6] Specifically, $EV_H = w - c + b + \delta \frac{\sum_{i=1}^{n-1} v_i}{n-1}$.

and define

$$K(\delta, n) = \frac{EV_H - EV_L}{b},$$

as a measure of the effectiveness of payment smoothing.

With this strategy profile, the agent is willing to put in effort if and only if

$$\frac{c}{p-q} \le EV_H - EV_L = K(\delta, n)b.$$

The principal will not renege the bonus if and only if

$$b \le \frac{\delta(py - c - \underline{u} - \pi)}{1-\delta}.$$

Combining these two conditions shows that this strategy profile can be a PPE if

$$\frac{c}{(p-q)K(\delta, n)} \le \frac{\delta(py - c - \underline{u} - \pi)}{1-\delta}.$$

Recall that the necessary condition to sustain an efficient relational contract with perfect observability of outputs is given by

$$\frac{c}{(p-q)} \le \frac{\delta(py - c - \underline{u} - \pi)}{1-\delta}.$$

Therefore, if $K(\delta, n) > 1$, the signal-generating process under the n-state transition rule above helps sustain efficiency for a larger range of discount factors. Theorem 1 shows that as the discount factor approaches 1, $K(\delta, n) > 1$ for a sufficiently large n.

Theorem 1: Let $p < \frac{1}{2}$. For $n > \frac{1}{1-2p}$, under the n-state transition rule,

$$\lim_{\delta \to 1} K(\delta, n) = 1 + \frac{(1-2p)n - 1}{2(1-p)n(n-1)}.$$

Theorem 1 implies that if the signals can be intertemporally garbled, then as the players become more patient, the efficiency of the game can be improved if the players use public strategies and $p < \frac{1}{2}$. As discussed above, the source of the gain comes from linking the principal's reneging constraints across states and over time. In particular, a high output not only leads to a good signal today but also moves the agent into more-favorable states, making future good signals more likely.

While intertemporal-garbling helps bonus smoothing, it is ex ante unclear that it helps efficiency for two reasons. First, once the signals are garbled, the agent can sometimes receive a bonus even if the output is low. (In our n-state construction, this can happen when the agent is in state 1.) Paying a bonus for low output reduces the agent's incentive to work, and this makes the relational contract harder to sustain. Second, intertemporal garbling implies that part of the reward paid to the agent is postponed. When the bonus is postponed and the agent is impatient, the total expected bonus increases, and this makes the principal's non-reneging constraint harder to satisfy. However, as the discount factor goes to 1 and as the number of states become large, however, both types of cost of intertemporal-garbling becomes small relative to the gain it generates.

In Theorem 1, the success probability (p) is less than $\frac{1}{2}$. When $p \in (\frac{1}{2}, 1)$, we can construct a signal-generating process that is the mirror image of the signal-generating process in the $p < \frac{1}{2}$ case. In essence, we turn the graph in Figure 1 upside down. We again have two signals, good and bad, and each signal (information set) contains n states. But the states are ranked in reverse order relative to the $p < \frac{1}{2}$ case. In addition, a high output corresponds to a low output in the $p < \frac{1}{2}$ case, and a good signal corresponds to a bad signal when $p < \frac{1}{2}$. Under the new information structure, the agent pays the principal a bonus if and only if the signal is good. Using this transition function, we can establish the following corollary.

Corollary 1: Let $p > \frac{1}{2}$. For $n > \frac{1}{2p-1}$, under the reversed n-state transition function,

$$\lim_{\delta \to 1} K(\delta, n) = (1 + \frac{(2p-1)n - 1}{2pn(n-1)}).$$

Theorem 1 and Corollary 1 show that for sufficiently patient players intertemporal garbling can help increase efficiency for all $p \in (0, 1)$ except at $p = \frac{1}{2}$. When $p = \frac{1}{2}$, perfect observability of output is in fact the optimal information structure. In other words, if the efficient relational contract cannot be sustained when the outputs are perfectly observed at $p = \frac{1}{2}$, no information structure can obtain efficiency.

Theorem 2: *When $p = \frac{1}{2}$, the optimal information structure is given by $s_t = Y_t$ for all t.*

Note that Theorem 2 is a general result: it holds regardless of whether the players can use private strategies. Our proof shows that if there were an information structure that sustains the efficient relational contract, the variance of the expected payoff of the agent following some public history must goes to infinity, and this is a contradiction. Below we give an alternative intuition for why intertemporal garbling does not help at $p = \frac{1}{2}$ by examining stationary strategies.

Consider the benchmark case of imperfect public monitoring in Subsection 3.1. Suppose inequality (3) holds so there is an efficient relational contract. In this case, one can construct a stationary equilibrium in which the agent receives a bonus of $\frac{c}{p-q}$ for each high output. Since high output occurs with probability p, the average reneging temptation of the principal is $\frac{pc}{p-q}$. Alternatively, we can adjust the base wage properly and have the agent pay back a bonus of $\frac{c}{p-q}$ for each low output. Since low output occurs with probability $1-p$, the average reneging temptation is $\frac{(1-p)c}{p-q}$. One can consider more general bonus schemes, but it can be shown that to induce effort, the **average reneging temptation** must be at least

$$\min\{p, 1-p\}\frac{c}{p-q},$$

regardless of the information structure.

Now consider a stationary contract that pays out a bonus (B) with frequency ρ. To minimize the reneging temptation, we would like to minimize B. For this contract to induce effort, however, the discussion above implies that the average reneging temptation should be at least $\frac{c}{p-q}\min\{p, 1-p\}$. In other words, we would like minimize B subject to

$$B\min\{\rho, 1-\rho\} \geq \frac{c}{p-q}\min\{p, 1-p\}.$$

It is clear that B is minimized when $\rho^* = \frac{1}{2}$, which implies that

$$\frac{B^*}{c/(p-q)} = 2\min\{p, 1-p\}.$$

This expression suggests that intertemporal garbling is more effective when the success probability is uneven, i.e. for p close to 0 or 1. Moreover, when $p = \frac{1}{2}$, $B^* = \frac{c}{p-q}$ and intertemporal garbling does not help.

3.3 Perfect Bayesian Equilibrium

In the previous subsection, we showed that when the probability of success is not equal to $\frac{1}{2}$, intertemporal garbling can help increase efficiency when the players use public strategies. However, the restriction to public strategies is not particularly appropriate when the signals are intertemporally garbled. Since past outputs affect future signals, the agent can have a different belief about the distribution of future signals than the principal when he deviates (even if he follows equilibrium play in the future). Therefore, the agent's private history matters for future play, and a more appropriate equilibrium concept should allow the agent's strategy to depend on his private history.

In this subsection, we show that intertemporal garbling can also improve efficiency when Perfect Bayesian Equilibrium (PBE) is used as the equilibrium concept. While the basic idea of why intertemporal garbling helps remains the same, it is significantly harder to check whether a strategy profile forms a PBE: when the agent no longer holds the same belief as the principal after a deviation, there is typically no recursive structure in the game and all relevant multi-stage deviations need to be checked to ensure that the strategy profiles form a PBE. To keep the analysis tractable, we let $q = 0$ in this subsection, so that if the agent does not put in effort, the output must be low.[7]

Consider the following signal-generating process. Let the signals be either *good* or *bad*. Within each information set, there are two secret states: *up* or *down*. If the output is high, then regardless of the state, the signal is *good* (g) and the agent will be in the *up* state (within the good signal) next period. If the output is low, then if the agent is in the up state, the signal is good and the agent will be in the *down* state next period. If the agent is in the down state, the signal is *bad* ($\sim g$) and the state is moved to *up* with probability ρ^* and to *down* with probability $1-\rho^*$.

Probability ρ^* satisfies

$$\rho^* = \frac{p}{p + (1-p)\rho^*}$$

to maintain stationarity. In particular, if with probability ρ^* the agent is in the *up* state and he puts in effort, the he will again be in the up state next period with

[7] The game remains one of imperfect monitoring as long as $p < 1$ because when the output is low, the principal is unable to infer whether it is due to a lack of effort or bad luck.

probability ρ^* regardless of which signal is realized. The following figure illustrates how different outputs and previous states lead to different signals and states.

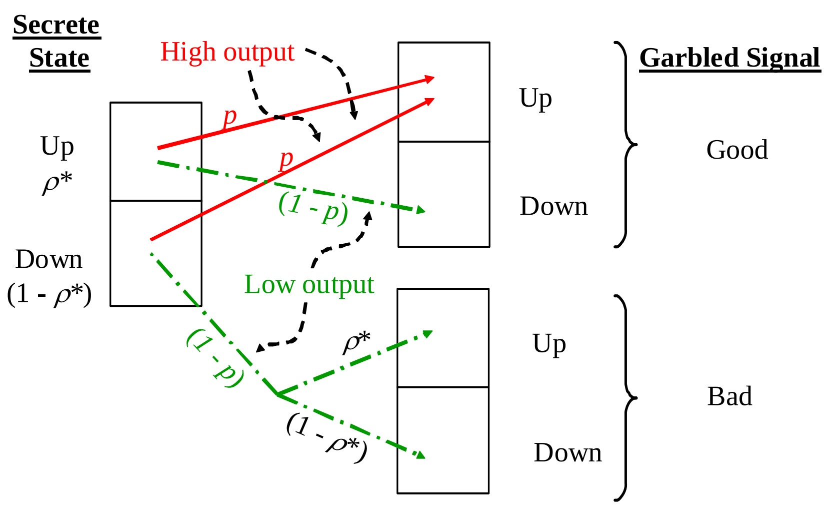

Figure 2: Signal Garbling Process with Details

This signal-generating process is similar in spirit to the two-state case in Subsection 3.2. However, this process is not a special case of that in Subsection 3.2 because the two states here do not occur with equal probability. Given the generating function (with ρ^* as the initial probability of the up state), consider the following strategies.

The principal offers

$$w_1 = w$$

in period 1. If the agent has always accepted the contract, the principal offers for $t > 1$

$$w_t = \begin{cases} w & \text{if } s_{t-1} = \sim g, \\ w + \frac{B}{\delta} & \text{if } s_{t-1} = g. \end{cases}$$

If the agent has ever rejected the principal's offer, the principal offers

$$w_t = \underline{u} - 1.$$

In addition, no discretionary bonus is given out, so

$$W_t = w_t \text{ for all } t.$$

The principal always believes that the probability of the up state is ρ^*.

The agent accepts the principal's contract offer if

$$w_t > \underline{u}$$

or if the principal has never deviated. The agent puts in effort if the principal has never deviated and the probability of the up state satisfies $\rho \le \rho^*$. *The agent calculates the probability of the up state using the Bayes rule according to his past history of efforts.*[8]

Theorem 3: *Let* $w = c + (1-\delta)\underline{u} - \frac{p}{\rho^*}B$ *and* $B = \frac{c}{A(p,\delta)p}$, *where* $A(p,\delta) = (\frac{1}{\rho^*} - \frac{\rho^*}{p} + \delta\frac{(1-\rho^* - \delta(1-\rho^*)\rho^*)}{1-p\delta^2(1-\rho^*)})$. *The strategy and belief above form a PBE if and only if*

$$B \le \frac{\delta}{1-\delta}(py - c - \underline{u} - \underline{\pi}).$$

In addition,

$$\lim_{p\to 0} A(p,\delta) = 1+\delta.$$

Theorem 3 implies that an efficient PBE exists if

$$\frac{c}{p} \le A(p,\delta)\frac{\delta\,(py - c - \underline{u} - \underline{\pi})}{1-\delta}.$$

Recall that when the outputs are publicly observed each period, the necessary condition for an efficient relational contract is given by

$$\frac{c}{p} \le \frac{\delta\,(py - c - \underline{u} - \underline{\pi})}{1-\delta}.$$

When p approaches 0, Theorem 3 states that $A(p,\delta)$ approaches $1+\delta$. Since $A(p,\delta)$ is continuous in p, intertemporal garbling helps increase efficiency for small enough p. For p close to 0, the signal-generating process above cuts the surplus required for efficiency by a factor of $\frac{1}{1+\delta}$.

[8] If the contract is not accepted in a period, no signal is generated in that period and future signals are generated as if that period did not exist.

To see why $1+\delta$ is the proportion of gain as p goes to 0, let β be the benefit of being in the up state instead of the down state. When the probability of the up state is ρ^*, the agent's benefit of exerting effort is given by

$$p(1-\rho^*)B + \delta p\left[1-(1-\rho^*)\,\rho^*\right]\beta,$$

where $p(1-\rho^*)$ is the additional probability of a good signal[9] and $p\left[1-(1-\rho^*)\,\rho^*\right]$ is the additional probability of being in the up state. When p goes to zero, ρ^* also goes to zero, and more importantly, β approaches B. The later is because as p goes to zero, the probability of receiving a bonus goes to 0, yet once the agent is in the up state he receives a bonus with probability one. In other words, as p goes to zero, the value of being in the *up* state is basically that the bonus is guaranteed for this period. Therefore, β goes to B as p goes to zero. It follows that as $p \to 0$,

$$\begin{aligned} & p(1-\rho^*)B + \delta p\left[1-(1-\rho^*)\,\rho^*\right]\beta \\ \approx\ & p(1-\rho^*)B + \delta p\left[1-(1-\rho^*)\,\rho^*\right]B \\ \approx\ & (1+\delta)B. \end{aligned}$$

This explains why the benefit of exerting effort is approximately $(1+\delta)\,pB$, compared to pB when the signal is not garbled. In other words, with intertemporal signal garbling, the same bonus amount provides a stronger incentive to put in effort.

One key feature of our construction is that bonus is paid at the beginning of the next period (as part of an increased base wage). The timing of the bonus is irrelvant for preventing one-stage deviation[10] but is important for preventing multi-stage deviation. In particular, a postponed bonus prevents the agent from shirking and then exiting the relationship after a good signal (and a bonus.) To see this, suppose instead the bonus is paid out at the end of each period upon a good signal, then at the beginning of each period the agent is indifferent between choosing his outside option and accepting the relational contract when the up state occurs with probability ρ^*. However, if the agent shirks and a good signal is realized, he knows that the up state occurs with probability 0, which implies that the agent's value in the relationship falls below his outside option. Thus, he would strictly prefer his outside

[9] The probability of good signal without effort is ρ^*.

[10] Under our construction, the agent is indifferent between working and shirking if he never leaves the relationship. In addition, if he always puts in effort, the agent has a weakly lower payoff by taking his outside option.

option in the next period. This strict preference implies that shirking and exiting after bonus is a profitable deviation.

Since there may be profitable multi-stage deviations, one-stage deviation principle cannot be applied in checking the equilibrium, and, thus, complicating the proof. We bypass the difficulty in our equilibrium construction by restricting the number of secret states to two. As a result, the up state probability becomes a sufficient statistic of the agent's future payoff. This allows for a recursive formulation of the agent's value function in which the probability of the up state is the state variable. Our restriction to $q = 0$ allows us to calculate the value function explicitly. Figure 3 below illustrates the value function of the agent if the bonus were paid at the end of each period (and the agent never takes the outside option). The value function is piecewise linear in the probability of the up state and has a kink at ρ^*. The observation that $V(0)$ is less than $\underline{u}$ again underscores the importance of postponing the bonus payment to the beginning of next period in the actual equilibrium.

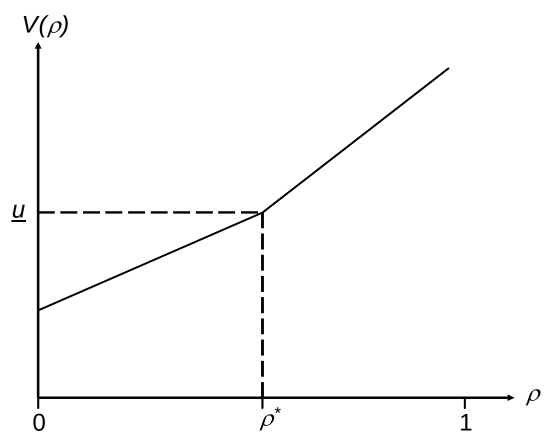

Figure 3: Agent's value function

4 Conclusion

This paper shows that intertemporal garbling of signals can sometimes help relational contracting. Through repartitioning and smoothing the reneging temptations both across states and over time, intertemporal garbling increases efficiency by reducing the principal's maximal reneging temptation. A feature of the intertemporal-garbling process essential to improve efficiency is that past outputs have enduring effects on future signals.

Our theoretical investigation has implications on how better to sustain relational contracting in practice. For example, since the full revelation of information is in general suboptimal, intermediaries can help relational contracting by controlling information flows. By reducing the transparency of the relationship, an intermediary can sometimes increase its efficiency. In addition, our analysis suggests that gain from using an intermediary is larger in relationships in which the temptation to renege is small in all but a few rare instances. Obviously, introducing an intermediary creates a host of other issues. Further research in this area is needed.

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Appendix

Proof of Theorem 1. The proof proceeds by explicitly calculating the value of $EV_H - EV_L$. Note that

$$\begin{aligned} EV_H &= w - c + b + \delta \frac{\sum_{i=1}^{n-1} v_i}{n-1} \\ &= w - c + b + \delta \frac{nEV - v_n}{n-1}. \end{aligned}$$

and

$$EV = w - c + \rho b + \delta EV.$$

Therefore,

$$\begin{aligned} & EV_H - EV_L \\ = & \frac{EV_H - EV}{1-p} \\ = & \frac{1}{1-p}[(1-\rho)b + \delta(\frac{nEV - v_n}{n-1} - EV)] \\ = & b + \frac{1}{(1-p)(n-1)}[\delta[EV - v_n] - pb]. \end{aligned}$$

To calculate $EV - v_n$, define $s_{k+1} = v_{k+1} - v_k$. For $k \geq 2$,

$$\begin{aligned} v_k &= w - c + p\{b + \delta[(k-1)yv_{k-1} + (1-(k-1)y)v_k]\} \\ &\quad + \delta(1-p)[(n+1-k)zv_{k-1} + (1-(n+1-k)z)v_k], \end{aligned}$$

and

$$\begin{aligned} v_{k+1} &= w - c + p\{b + \delta[(kyv_k + (1-ky)v_{k+1}]\} \\ &\quad + \delta(1-p)[(n-k)zv_k + (1-(n-k)z)v_{k+1}], \end{aligned}$$

we have

$$\begin{aligned}
s_{k+1} &= v_{k+1} - v_k \\
&= p\delta[kyv_k + (1-ky)v_{k+1} - (k-1)yv_{k-1} - (1-(k-1)y)v_k] \\
&\quad +\delta(1-p)[(n-k)zv_k + (1-(n-k)z)v_{k+1} - (n+1-k)zv_{k-1} - (1-(n+1-k)z)v_k] \\
&= p\delta[(k-1)y(v_k - v_{k-1}) + (1-ky)(v_{k+1} - v_k)] \\
&\quad +\delta(1-p)[(1-(n-k)z)(v_{k+1} - v_k) + (n+1-k)z(v_k - v_{k-1})] \\
&= p\delta[(k-1)ys_k + (1-ky)s_{k+1}] + \delta(1-p)[(1-(n-k)z)s_{k+1} + (n+1-k)zs_k] \\
&= \delta\{[(p(k-1)y + (1-p)(n+1-k)z)]s_k + [p(1-ky) + (1-p)(1-(n-k)z)]s_{k+1}]\}.
\end{aligned}$$

This implies that

$$s_{k+1} = \delta[\rho s_k + (1-\rho)s_{k+1}], \quad \text{for } k \geq 2,$$

and, thus, for $k \geq 2$,

$$s_k = (1 + \frac{1-\delta}{\delta\rho})s_{k+1} = C(\delta)s_{k+1},$$

where $C(\delta) = (1 + \frac{1-\delta}{\delta\rho})$ is independent of k.

In addition, note

$$\begin{aligned}
v_n &= w - c + p(b + \delta v_{n-1}) \\
&\quad +\delta\,(1-p)\,[\frac{p}{(n-1)\,(1-p)}v_{n-1} + (1 - \frac{p}{(n-1)\,(1-p)})v_n].
\end{aligned}$$

Since $v_{n-1} = v_n - s_n$, we have

$$s_n = \frac{w - c + pb - (1-\delta)v_n}{\delta\rho}.$$

Using $v_k = v_{k+1} - s_{k+1}$, it can be seen that

$$\begin{aligned}
\sum_{i=2}^{n} v_i &= (n-1)v_n - (n-2)s_n - \ldots - s_3 \\
&= (n-1)v_n - K(\delta)s_n,
\end{aligned}$$

where it can be shown that

$$K(\delta) = \frac{C^{n-1} - 1 - (C-1)(n-1)}{(C-1)^2}.$$

To see the above, note that

$$\begin{aligned} & s_3 + 2s_4 + ... + (n-2)s_n \\ = \; & s_n(C^{n-3} + 2C^{n-4} + ... + (n-3)C + (n-2)). \end{aligned}$$

Let

$$K = C^{n-3} + 2C^{n-4} + ... + (n-3)C + (n-2),$$

then

$$CK = C^{n-2} + 2C^{n-3} + ... + (n-3)C^2 + (n-2)C,$$

so

$$\begin{aligned} (C-1)K &= C^{n-2} + C^{n-3} + ... + C + 1 - (n-1) \\ &= \frac{C^{n-1} - 1}{C - 1} - (n-1), \end{aligned}$$

so

$$K = \frac{C^{n-1} - 1 - (n-1)(C-1)}{(C-1)^2}$$

Therefore,

$$\begin{aligned} nEV &= \sum_{i=1}^{n} v_i \\ &= v_1 + (n-1)v_n - K(\delta)s_n \\ &= \frac{w - c + (p+\rho)b + \delta\rho v_n}{1 - \delta(1-\rho)} + (n-1)v_n + K(\delta)\frac{(1-\delta)v_n - (w - c + pb)}{\delta\rho}. \end{aligned}$$

This implies two things. First,

$$n(EV - v_n) = \frac{w - c + (p+\rho)b - (1-\delta)v_n}{1 - \delta(1-\rho)} + K(\delta)\frac{(1-\delta)v_n - (w - c + pb)}{\delta\rho}.$$

Note that the $w - c$ term will cancel out, so we might just assume that they're equal to 0, the equation above simplifies to

$$n(EV - v_n) = \frac{(p+\rho)b - (1-\delta)v_n}{1 - \delta(1-\rho)} + K(\delta)\frac{(1-\delta)v_n - pb}{\delta\rho}.$$

This can be rewritten as

$$v_n(n - \frac{(1-\delta)}{1-\delta(1-\rho)} + K(\delta)\frac{(1-\delta)}{\delta\rho}) = nEV - \frac{(p+\rho)b}{1-\delta(1-\rho)} + K(\delta)\frac{pb}{\delta\rho},$$

and, thus,

$$v_n = \frac{n\frac{\rho b}{1-\delta} - \frac{(p+\rho)b}{1-\delta(1-\rho)} + K(\delta)\frac{pb}{\delta\rho}}{n - \frac{(1-\delta)}{1-\delta(1-\rho)} + K(\delta)\frac{(1-\delta)}{\delta\rho}},$$

Now let δ go to 1, note that $C = 1 + \frac{1-\delta}{\delta\rho} \equiv 1 + \varepsilon$, so we have $\varepsilon \to 0$ as $\delta \to 1$, and thus $C^M \simeq 1 + M\varepsilon$. This implies that

$$\begin{aligned}
\lim_{\delta\to 1} K(\delta) &= \lim_{\delta\to 1} \frac{1 + C + C^2 + ... + C^{n-2} - (n-1)}{C-1} \\
&\simeq \frac{\sum_{i=1}^{n-2} M\varepsilon}{\varepsilon} \\
&= \frac{(n-1)(n-2)}{2}.
\end{aligned}$$

Therefore, as $\delta \to 1$,

$$\begin{aligned}
\lim_{\delta\to 1} v_n(1-\delta) &= \lim_{\delta\to 1} \frac{n\rho b - \frac{(1-\delta)(p+\rho)b}{1-\delta(1-\rho)} + (1-\delta)K(\delta)\frac{pb}{\delta\rho}}{n - \frac{(1-\delta)}{1-\delta(1-\rho)} + K(\delta)\frac{(1-\delta)}{\delta\rho}} \\
&= \rho b.
\end{aligned}$$

In addition, as $\delta \to 1$,

$$\begin{aligned}
\lim_{\delta\to 1} n(EV - v_n) &= \lim_{\delta\to 1} \frac{(p+\rho)b - (1-\delta)v_n}{1-\delta(1-\rho)} + \lim_{\delta\to 1} K(\delta)\frac{(1-\delta)v_n - pb}{\delta\rho}. \\
&= \frac{pb}{\rho} + \frac{(n-1)(n-2)}{2}(\frac{\rho - p}{\rho})b \\
&= (\frac{n-1}{n} + \frac{(n-1)(n-2)}{2n})b \\
&= \frac{(n-1)}{2}b.
\end{aligned}$$

Finally,

$$
\begin{aligned}
& \lim_{\delta\to 1}(EV_H - EV_L) \\
= \quad & b + \frac{1}{(1-p)(n-1)}[\lim_{\delta\to 1}[EV - v_n] - pb] \\
= \quad & b + \frac{1}{(1-p)(n-1)}(\frac{n-1}{2n} - p)b \\
= \quad & (1 + \frac{(1-2p)n-1}{2(1-p)n(n-1)})b.
\end{aligned}
$$

■

Proof of Theorem 2. First recall that when $s_t = Y_t$ for all t, the necessary and sufficient condition for sustaining cooperation is given by equation (3):

$$\frac{c}{p-q} \leq \frac{\delta}{1-\delta}(py - c - \underline{u} - \underline{\pi}) \equiv S,$$

where without confusion in this proof, S denotes the surplus of the relationship (when the agent puts in effort each period.) We want to show that if the inequality above fails, it is impossible to construct an equilibrium in which the agent puts in effort. In particular, using standard argument as in Fuchs (2007), it suffices to show that there does not exist an equilibrium in which the agent always puts in effort (unless the relationship is terminated).

Consider an arbitrary information partition process. Pick one information set (h^t). Use x to denote the possible states within the information set. One interpretation of x is some output realizations y^t that falls into h^t.

Let $V(x)$ be the agent's continuation payoff in state x after e_{t+1} is put in but before y_{t+1} is realized and W_{t+1} is paid out. Let $V(x_i)$ be the agent's continuation payoff in state x after e_{t+1} is put in, y_{t+1} is realized but before W_{t+1} is paid out. Within each state x, we have $x_i \in \{x_y, x_0\}$, where x_y denotes that $Y_t = y$ is realized following x, and x_0 denotes that $Y_t = 0$ is realized.

Note that

$$V(x) = V(x) + p(V(x_y) - V(x)) + (1-p)(V(x_0) - V(x)).$$

And since the output Y_t is independent of the past state, we have $Cov(V(x_i) - V(x), V(x)) = 0$.

To induce effort, we need

$$E_x[V(x_y) - V(x_0)] = E_x[V(x_y) - V(x)] - E_x[V(x_0) - V(x)] \geq \frac{c}{p-q}.$$

This helps give a lower bound for $Var(V(x_i))$. In particular,

$$\begin{aligned} Var(V(x_i)) &= Var(V(x)) + Var(V(x_i) - V(x)) \\ &= Var(V(x)) + E_y[Var(V(x_i) - V(x))|Y] \\ &\quad + Var(E_x[V(x_i) - V(x)|Y]) \\ &\geq Var(V(x)) + E_y[Var(V(x_i) - V(x))|Y] \\ &\quad + p(1-p)(\frac{c}{p-q})^2 \\ &\geq Var(V(x)) + p(1-p)(\frac{c}{p-q})^2, \end{aligned}$$

where the first line follows because $Cov(V(x_i)-V(x), V(x)) = 0$, the second line uses the variance decomposition formula, the third line follows because $E_x[V(x_i)-V(x)|Y]$ is a binary value ($Y \in \{0, y\}$) such that with probability p its value is $E_x[V(x_y)-V(x)]$ and with probability $1-p$ its value is $E_x[V(x_0) - V(x)]$, and $E_x[V(x_y) - V(x)] - E_x[V(x_0) - V(x)] \geq \frac{c}{p-q}$.

Now let's provide an upper bound for $Var(V(x_i))$. Suppose a public signal $s(x_i)$ will be sent out after state x_i. Let $b(s)$ be the bonus paid out to the agent (at the end of the period) following signal s. This allows us to write

$$V(x_i) = b(s(x_i)) + \delta V_{s(x_i)}(x_i),$$

where $V_{s(x_i)}(x_i)$ is the continuation payoff of x_i, which goes to the information set by signal $s(x_i)$.

Note that for the principal to be willing to pay the bonus, we must have

$$\max_s\{b_s + \delta E_{x_i}[V_s(x_i)|s]\} - \min_s\{b_s + \delta E_{x_i}[V_s(x_i)|s]\} \leq S.$$

Because otherwise the expected payoff of the principal following some signal will be below his outside option.

Decomposing the variance on the signals, we have

$$\begin{aligned} Var(V(x_i)) &= Var(E[b_s + \delta V_s(x)|s]) + E[Var(b_s + \delta V_s(x_i)|s)] \\ &\le \frac{1}{4}S^2 + \delta^2 E[Var(V_s(x_i)|s)]. \end{aligned}$$

Now combining the upper and lower bound for $Var(V(x_i))$, we get that

$$\frac{1}{4}S^2 + \delta^2 E[Var(V_s(x_i)|s)] \ge Var(V(x)) + p(1-p)(\frac{c}{p-q})^2,$$

or equivalently,

$$E[Var(V_s(x_i)|s)] \ge \frac{1}{\delta^2}(Var(V(x)) + p(1-p)(\frac{c}{p-q})^2 - \frac{1}{4}S^2).$$

When $p = \frac{1}{2}$, and $\frac{c}{p-q} > S$, the inequality above implies that

$$E[Var(V_s(x_i)|s)] > \frac{1}{\delta^2}(Var(V(x)).$$

In particular, there will be one information set (associated with a signal) whose variance exceeds $\frac{1}{\delta^2}(Var(V(x))$. Now we can perform the same argument on this new information set, and we can construct a sequence of information set whose variance approaches infinity. This leads to a contradiction. ■

Proof of Theorem 3. It can be checked that given the choice of the base wage, the principal captures the entire surplus. So when $B \le \frac{\delta}{1-\delta}(py - c - \underline{u} - \underline{v})$, the principal will not renege. The key is to check that the agent will not deviate. To simplify the exposition, we set $\underline{u} = 0$ here. It follows that

$$w = c - \frac{p}{\rho^*}B.$$

Given the strategy of the principal, the agent's payoff is completely determined by his probability of being in the up-state. Let $V(\rho)$ denote the agent's value function if the previous signal is bad, i.e. the agent will receive w if he accepts the contract today.

Recognizing that the agent can choose between exerting and not exerting effort given every ρ, the value function has the following recursive representation:

$$\begin{aligned} V(\rho) &= \max\{w - c + (p + (1-p)\rho)(B + \delta V(\frac{p}{p + (1-p)\rho})) \\ &\quad +(1 - (p + (1-p)\rho))\delta V(\rho^*), w + \rho(B + \delta V(0)) + (1-\rho)\delta V(\rho^*)\}. \end{aligned}$$

In the above expression, we implicitly assume that the agent does not choose his outside option. This will be verified.

Now note that the operator on the right hand side satisfies the Blackwell Sufficiency Conditions, so it is a contraction mapping. Therefore, there is a unique value function V that satisfies this equation.

To check the strategy profile forms a PBE, we take the following steps. First, we give an explicit expression for V. Second, given the value function, we show that, if the agent accepts the contract, his optimal response is to put in effort if $\rho \leq \rho^*$ and not put in effort otherwise. Third, we check that the agent will accept the contract.

For the value function, it takes the following form:

$$V(\rho) = \begin{cases} (B + \delta V(0))(\rho - \rho^*) & \text{for } \rho > \rho^* \\ (1-p)(B - \delta\rho^*(B + \delta V(0)))(\rho - \rho^*) & \text{for } \rho \leq \rho^*, \end{cases}$$

where

$$V(0) = \frac{p(1 - \frac{1}{\rho^*} + \delta(1 - \rho^*))}{1 - p\delta^2(1-\rho^*)} B. \tag{4}$$

To see that this is the value function (given the agent's equilibrium strategy), we first note that the choice of ρ^* and w insures that $V(\rho^*) = 0$. In addition, it can be checked that the slope of V for $\rho > \rho^*$ is a constant $(B + \delta V(0))$. In other words, V is linear for $\rho > \rho^*$ although it remain to be checked that

$$V(\rho^*_+) \equiv \lim_{\rho \to \rho^*_+} V(\rho) = V(\rho^*).$$

Next, the linearity of V for $\rho > \rho^*$ implies that V is linear for $\rho < \rho^*$ as well, since for $\rho < \rho^*$,

$$
\begin{aligned}
V(\rho) &= w - c + (p + (1-p)\rho)(B + \delta V(\frac{p}{p+(1-p)\rho})) + (1 - (p + (1-p)\rho))\delta V(\rho^*) \\
&= w - c + (p + (1-p)\rho)\left[B + \delta[(\frac{p}{p+(1-p)\rho} - \rho^*)(B + \delta V(0)) + c]\right] \\
&\quad +(1 - (p + (1-p)\rho))\delta V(\rho^*) \\
&= (1-p)(B - \delta\rho^*(B + \delta V(0)))(\rho - \rho^*).
\end{aligned}
$$

The last equality follows from the linearity of $V(\rho)$. Finally, (4) follows

$$
\begin{aligned}
U(0) &= w - c + p(B + \delta U(1)) \\
&= -\frac{p}{\rho^*}B + p(B + \delta(B + \delta U(0))(1 - \rho^*)).
\end{aligned}
$$

Next, $w + \rho^*(B + \delta U(0)) = 0$ implies

$$
c - \frac{p}{\rho^*}B + \rho^*\left[B + \delta\frac{p(1 - \frac{1}{\rho^*} + \delta(1-\rho^*))}{1 - p\delta^2(1-\rho^*)}B\right] = 0
$$

or equivalently

$$
B = \frac{c/p}{(\frac{1}{\rho^*} - \frac{\rho^*}{p} + \delta\frac{(1-\rho^*-\delta(1-\rho^*)\rho^*)}{1-p\delta^2(1-\rho^*)})}. \tag{5}
$$

This choice of B ensures that at ρ^*, the agent is indifferent between putting in effort or not, so we have $V(\rho^*_+) = V(\rho^*)$. This leads to the expression the of value function derived based on the assumption that the agent puts in effort when $\rho \leq \rho^*$ and does not do so when $\rho > \rho^*$. This finishes the first step.

In the second step, we check that the effort decisions specified are optimal. To do that, we first need to make sure that for $\rho \leq \rho^*$,

$$
\begin{aligned}
& w + \rho(B + \delta V(0)) \\
\leq\ & w - c + [(p + (1-p)\rho)(B + \delta V(\frac{p}{p+(1-p)\rho})) \\
=\ & (1-p)(B - \delta\rho^*(B + \delta V(0)))(\rho - \rho^*),
\end{aligned}
$$

Note that the above is satisfied if

$$
B + \delta V(0) \geq (1-p)(B - \delta\rho^*(B + \delta V(0))).
$$

Let $x = B + \delta V(0)$, and define $T(x) = (1-p)(B - \delta\rho^* x)$, then the above can be rewritten as

$$T(x) \leq x.$$

We also want to make sure that for $\rho > \rho^*$, we have

$$\begin{aligned}
& w + \rho(B + \delta V(0)) \\
\geq\ & w - c + [(p + (1-p)\rho)(B + \delta V(\frac{p}{p + (1-p)\rho})) \\
=\ & w - c + [(p + (1-p)\rho)(B + \delta(1-p)(B - \delta\rho^*(B + \delta V(0)))(\frac{p}{p + (1-p)\rho} - \rho^*))] \\
=\ & (1-p)(B - \delta\rho^*(1-p)(B - \delta\rho^*(B + \delta V(0))))(\rho - \rho^*).
\end{aligned}$$

If we again have $x = B + \delta V(0)$ and $T(x) = (1-p)(B - \delta\rho^* x)$, then the slope of ρ in the expression above is given by $T(T(x))$, and we need

$$T(T(x)) \leq x.$$

Now note that $T(x)$ is an affine function of x with slope $-\delta\rho^*(1-p) > -1$. Let x^* be such that $T(x^*) = x^*$, then

$$\begin{aligned}
(1-p)(B - \delta\rho^* x^*) &= x^* \\
x^* &= \frac{(1-p)B}{1 + \delta\rho^*(1-p)}.
\end{aligned}$$

Now note that if $x \geq x^*$, then

$$T(x) \leq x.$$

Moreover, since the slope of $T(x)$ is equal to $-(1-p)\delta\rho^* > -1$, this implies that, for $x > x^*$,

$$\frac{T(x^*) - T(x)}{x - x^*} = \frac{x^* - T(x)}{x - x^*} < 1.$$

By the linearity of T, it follows that,

$$\frac{T(T(x)) - T(T(x^*))}{T(x^*) - T(x)} = \frac{T(x^*) - T(x)}{x - x^*} < 1$$

so that

$$T(T(x)) - T(T(x^*)) \leq x - x^*,$$

or

$$T(T(x)) \leq x.$$

The discussion above implies that, as long as

$$B + \delta V(0) = x \geq x^* = \frac{(1-p)B}{1+\delta\rho^*(1-p)},$$

the action profile is optimal. In other words, we need

$$V(0) \geq -\frac{1}{\delta}(\frac{p+\delta\rho^*(1-p)}{1+\delta\rho^*(1-p)})B.$$

Recalling from (4) that

$$V(0) = \frac{p(1-\frac{1}{\rho^*}+\delta(1-\rho^*))}{1-p\delta^2(1-\rho^*)}B.$$

So

$$\begin{aligned} & V(0) + \frac{1}{\delta}(\frac{p+\delta\rho^*(1-p)}{1+\delta\rho^*(1-p)})B \\ = & (\frac{p(1-\frac{1}{\rho^*}+\delta(1-\rho^*))}{1-p\delta^2(1-\rho^*)} + \frac{1}{\delta}(\frac{p+\delta\rho^*(1-p)}{1+\delta\rho^*(1-p)}))B \\ = & \frac{B}{\delta(1-p\delta^2(1-\rho^*))(1+\delta\rho^*(1-p))} \geq 0. \end{aligned}$$

This shows that the effort decisions are optimal.

Finally, to make sure that the agent always accepts the contract, we need to make sure (see the value function) that

$$B + \delta V(0) > 0.$$

But from the above, we see that

$$B + \delta V(0) \geq \frac{(1-p)B}{1+\delta\rho^*(1-p)} > 0,$$

and this shows that the proposed value function is truly the value function if the agent's action is truly optimal.

Following directly from (5),

$$\begin{aligned}\frac{c}{Bp} &= \frac{1}{\rho^*} - \frac{\rho^*}{p} + \delta\frac{(1-\rho^* - \delta(1-\rho^*)\rho^*)}{1-p\delta^2(1-\rho^*)} \\ &= \frac{\rho^* - p}{(1-p)\rho^*} + \delta\frac{(1-\rho^* - \delta(1-\rho^*)\rho^*)}{1-p\delta^2(1-\rho^*)},\end{aligned}$$

where we have used $(\rho^*)^2(1-p) = p(1-\rho^*)$ in simplifying $\frac{1}{\rho^*} - \frac{\rho^*}{p}$.

Now since

$$\begin{aligned}(1-p)\rho^{*2} + p\rho^* - p &= 0, \\ \rho^* &= \frac{-p + \sqrt{4p - 3p^2}}{2}.\end{aligned}$$

In other words, when p is small, ρ^* is roughly in the order of $\sqrt{p}$.

It is clear that as p goes to 0, both $\frac{\rho^* - p}{(1-p)\rho^*}$ and $\frac{(1-\rho^* - \delta(1-\rho^*)\rho^*)}{1-p\delta^2(1-\rho^*)}$ go to 1, so

$$\lim_{p\to 0} \frac{c}{Bp} = 1 + \delta.$$

This completes the proof. ■