Central Bank Digital Currency: Welfare and Policy Implications

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Abstract
A model of banking and means of payment is constructed to analyze the effects of the introduction of central bank digital currency (CBDC). That CBDC is interest-bearing is not an advantage, as replacement of physical currency with CBDC does not expand the attainable set of equilibrium allocations. CBDC can increase welfare by competing with private means of payment and shifting safe assets from the private banking sector to what is effectively a narrow banking facility. This uses the aggregate stock of safe collateral more efficiently, given incentive problems in private banking.

1 Introduction

As financial technology evolves, central banks need to re-evaluate their role, potentially introducing new central bank assets and liabilities, and altering their approach to monetary policy decision-making and implementation. Central banks, including those in Sweden, Canada, and the U.K., have shown an increasing interest in digital currencies (see Chapman and Wilkins 2019, Kumhof and Noone 2018, and Bech et al. 2018), typically referred to as CBDC (central bank digital currencies). Some of the interest in CBDC has been propelled by a flood, in recent years, of privately-issued cryptocurrencies. But, perhaps more importantly, there is a recognition that old-fashioned physical currency has serious drawbacks, reflected in the decline in use documented in means-of-payment...
surveys.\(^1\) This decline is driven by advances in electronic payments technologies and the migration of retailing to online platforms.

But if CBDC were issued by central banks, what form should it take, for example should it be traded in a decentralized fashion on a distributed ledger, like cryptocurrencies, or in a centralized fashion, as with conventional private bank deposit claims? Since no cryptocurrency has as yet achieved success as a means of payment, it appears fruitful to focus attention on the implications of the introduction of centralized CBDC, based in a system that involves wider access to central bank deposit accounts. Given such a system, should the central bank withdraw physical currency from circulation? Should the CBDC system compete with private banking? How will monetary policy work given a CBDC-issuing central bank? What would it take for CBDC issue to increase economic welfare?\(^2\)

The purpose of this paper is to construct a model of banking and means of payment, so that we can answer these questions. In the model, CBDC has potential advantages over physical currency in that it can bear interest, and it can be designed for use in transactions for which privacy is desired, and for use in digital transactions where transactors do not desire privacy. The model shows that what might seem an advantage for CBDC may not be, and also shows some advantages for CBDC issue that may not be obvious at the outset.

In monetary policy discussions, a frequently-cited benefit of CBDC issue is that it would permit the payment of interest on central bank liabilities used in retail payments, potentially decreasing the effective lower bound on the nominal interest rate, and allowing for more effective negative-interest-rate policy (see Bordo and Levin 2019). In our model, if the central bank were to replace physical currency with digital currency, this would not expand the feasible set of allocations relative to the existing policy regime. That is, the interest rate on CBDC is not an extra policy instrument, since what matters for conventional monetary policy is the margin between the nominal interest rate on central bank liabilities used in retail payments and the interest rate on government debt. That the nominal interest rate on physical currency is fixed at zero under current circumstances is irrelevant for how conventional monetary policy works.

An important issue for central banks in issuing CBDC is how these new central bank liabilities might compete with means of payment issued by private banks. Andolfatto (2020) shows how private financial intermediaries could be disciplined in a beneficial way by CBDC issue, when there is monopoly power in the banking system. As well, Keister and Sanches (2021) show how there can be a disintermediation effect of CBDC issue, whereby substitution of CBDC for private bank liabilities can result in a reduction in productive investment and welfare. In our model, private banks do not have monopoly power, being subject to free entry and earning zero profits in equilibrium, in present value terms. These banks also finance investment in private capital. A key element in the model is that the central bank competes with private sector financial intermediaries for safe assets – as is typical central banking practice, central bank

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\(^1\)See Henry et al. (2018), Foster et al. (2020), and Kim et al. (2020).

\(^2\)Some of these issues are also addressed in Davoodalhosseini (2018) and Chiu et al. (2019), for example.
liabilities are backed by government debt. This represents potential benefits from CBDC issue, and a potential constraint on CBDC.

Because of incentive problems in private banking – which are serious enough in practice that bank regulators constrain banks to mitigate such problems – the central bank can potentially make more efficient use of the stock of safe assets than can the private sector. Therefore, if consumers substitute CBDC for private bank liabilities as means of payment, then safe assets migrate from the private banking sector to the asset side of the central bank’s balance sheet, increasing the effective stock of collateral in the economy, and potentially increasing welfare. This also has the effective of reducing private investment, mitigating a capital over-accumulation problem familiar from Aiyagari (1994) or Diamond (1965), for example. Thus, there is a disintermediation effect, but this is beneficial. So, in the absence of monopoly, CBDC can discipline private banks, and disintermediation can be a good thing. The down side is that, under the assumption that the central bank is excluded from holding private assets, an increase in CBDC issue implies that government debt is more encumbered. This limits the potential for CBDC issue, if government debt is scarce.

Central banks are key suppliers of privacy in payments. Kahn et al. (2005) and Kahn (2018) examine how privacy is important in transactions. Use of physical currency as a means of payment contributes to privacy, sometimes in nefarious ways – for example, physical currency lowers the cost of illegal transactions and tax evasion (see Camera 2001). But payment with currency can also prevent the use of our personal information against us, and create useful finality in transactions. Privacy can thus be viewed as a social good, and there are arguments that the private sector provides too little privacy (Garratt and Van Oordt 2020). We take this as given. In our model, it is assumed that privacy in transactions can only be provided by the central bank, via currency or CBDC. We then focus on the different implications of having central bank liabilities that are an integral part of private deposit contracts vs. a world where there is public access to CBDC accounts with the central bank.

The basic structure in the model comes from Lagos and Wright (2005) and Rocheteau and Wright (2005). Consumers in the model value privacy in some transactions, and they can obtain privacy by choice of means of payment. Banks intermediate assets and play an insurance role (related to what exists in a Diamond-Dybvig 1983 model) in providing consumers with means of payment through bank deposit contracts subject to withdrawal on demand. As in Williamson (2016, 2019), and Gertler and Kiyotaki (2011), banks have limited commitment. Bank assets serve as collateral, backing deposit liabilities, and there can exist a shortage of safe collateral in equilibrium, which makes the real interest rate low.3 A low-real-interest-rate environment is an important feature shared by most countries in the world, particularly after the Global Financial Crisis.

In this context, how the issue of CBDC affects demand for the scarce stock

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3See also Ennis (2018) and Martin et al. (2016) for related mechanisms in the context of banking systems with large quantities of excess reserves.
of collateral is an important issue. The fundamental assets in the model are government debt, private capital, and central bank liabilities – physical currency and CBDC. Private banks hold government debt and private capital as assets, and write deposit contracts with consumers that permit withdrawal of central bank liabilities (physical currency or CBDC, depending on the regime considered) on demand, and use claims on the bank as means of payment. The central bank can issue liabilities only in exchange for government debt, and it is important that the central bank cannot purchase private capital. So, an advantage of private banks over central banks is that they can back their means-of-payment liabilities with a broader set of assets.

The first step in the analysis is to characterize a status quo benchmark. In this regime, designed to represent typical current central banking arrangements in the world, the central bank issues physical currency, and private bank deposits are subject to withdrawal on demand in currency. Conventional monetary policy works through open market operations that support a target for the nominal interest rate on government debt. A higher nominal interest rate implies substitution from transactions using currency to transactions using bank deposits, the real interest rate rises, as collateral constraints are relaxed, and the stock of private capital falls. There are two constraints on the central bank. First, the quantity of central bank liabilities outstanding cannot exceed the quantity of government debt in existence. Second, central bank independence requires that the central bank separate itself from government budgeting, so the central bank must generate sufficient revenue to pay its costs and make a nonnegative transfer to the fiscal authority each period. The latter constraint implies that the nominal interest rate on government debt must exceed zero in equilibrium. In this status quo regime, private banking is costly, due to incentive problems, and central banking is costly.

If CBDC is issued by the central bank, it will be important how CBDC is designed to perform specific payments functions, and what that implies for competition with private sector means of payment. Three possibilities are explored here. Under the first, physical currency is withdrawn from circulation and replaced by CBDC. It is assumed that CBDC is designed so that payments using CBDC can be private, and that CBDC can also be used in payments as a perfect substitute for bank deposits. That is, CBDC could exist as an account with the central bank, convertible into an object that can be exchanged anonymously. In this regime, the interest rate on CBDC is set so that it does not compete with private bank deposits as a means of payment in transactions not requiring privacy. With CBDC issue, proportional changes in the gross nominal interest rates on government debt and CBDC have no real effects, and only change the inflation rate in the same direction. Essentially, replacing physical currency with CBDC does not expand the set of equilibrium allocations that can be supported with alternative monetary policy settings.

The second CBDC regime considered is one where CBDC is issued as part of a narrow banking arrangement, in that all CBDC is backed by government debt in the central bank’s asset portfolio. Narrow banking proposals have played an important role in the history of thought in money and banking, and include
the Chicago Plan of 1933 (see Pennacchi 2012). As well, the 2018 Swiss referendum, Vollgeld (see Vollgeld Initiative 2018), and a proposal of Tobin (1987), are versions of narrow banking arrangements. In the narrow banking regime considered here, physical currency remains in circulation, with private banks writing deposit contracts allowing for withdrawal in physical currency on demand. Private banks then compete with the central bank’s narrow banking facility for depositors. Under this regime, an increase in the interest rate on CBDC causes substitution from private banking to the central bank’s narrow banking facility. Welfare rises, as the narrow bank uses safe assets more efficiently, given the incentive problems in private banking. As well, the increase in welfare coincides with a decline in investment and in the private capital stock. But the interest rate on CBDC cannot be too high, as the central bank needs to generate sufficient revenue to cover its costs, and the demand for CBDC cannot exceed the quantity of government debt available to back the narrow banking facility.

The third regime with CBDC issue examined is one where private banks issue deposits subject to withdrawal in physical currency, and the central bank replicates the essential properties of the private deposit contract, by paying appropriate interest rates on CBDC used for private transactions and CBDC used for transactions where privacy is foregone. This regime has similar properties to the narrow banking arrangement, in that CBDC issue can improve welfare. But this setup permits an analysis of the case where incentive problems in banking are confined to issues related to the intermediation of private capital. This analysis shows that, under these conditions, CBDC issue is irrelevant for the equilibrium allocation, illustrating the importance of private bank incentive problems for the efficacy of CBDC issue.

The paper is organized as follows. In the second section, the model is constructed, then in Section 3 an equilibrium is defined and characterized, for the general case. In Section 4 a status quo equilibrium with physical currency is analyzed. In Sections 5, 6, and 7, respectively, there is an analysis of CBDC issued through private banks, CBDC issued through the central bank’s narrow banking facility, and CBDC issued through a narrow banking facility that replicates private deposit contracts. In Section 8, some related literature is discussed, and Section 9 is a conclusion.

2 Model

The model builds on a basic Rocheteau and Wright (2005) framework, with additional structure added to address the particulars of this problem. Periods are indexed by \( t = 0, 1, 2, \ldots \), and in each period there are two sub-periods, the centralized market (CM), followed by the decentralized market (DM). There is a continuum of buyers, with unit mass, each of whom is infinite-lived with preferences

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ -H_t + u(x_t) \right],
\]
where \(0 < \beta < 1\), \(H_t\) denotes labor supply in the CM, and \(x_t\) denotes consumption in the DM. Assume that \(u(\cdot)\) is strictly increasing, strictly concave, twice continuously differentiable, and has the properties \(u(0) = 0\), \(u'(0) = \infty\), and \(u'(\hat{x}) - \hat{x} = 0\) for some \(\hat{x} > 0\). Define \(x^*\) as the solution to \(u'(x^*) = 1\), and assume
\[
0 < -x u''(x) u'(x) < 1,
\]
for \(x \geq 0\), which implies, roughly, that asset demand in the model increases with the asset’s own rate of return, that is substitution effects dominate income effects.

There also exists a continuum of bankers with unit mass, each of whom has preferences
\[
E_0 \sum_{t=0}^{\infty} [\mathclap{-H_t + X_t}],
\]
where \(H_t\) and \(X_t\) are, respectively, labor supply and consumption for the banker in the CM. In addition, there is a continuum of sellers with unit mass, each with preferences
\[
E_0 \sum_{t=0}^{\infty} \beta^t [X_t - h_t],
\]
where \(X_t\) is consumption in the CM, and \(h_t\) is labor supply in the DM. In the CM and the DM, one unit of labor supply produces one unit of the perishable consumption good. Buyers cannot produce in the DM, and sellers cannot produce in the CM (except using private capital, as specified in what follows).

In the CM, all agents are together in one location. At the beginning of the CM, debts acquired in the previous period are settled, then production, consumption, and exchange take place, buyers write contracts with bankers, and assets are traded. Finally, at the end of the period, buyers’ idiosyncratic shocks are realized, and each buyer can make contact with his or her bank.

In the DM, each buyer is matched at random with a seller. But sometimes the buyer wants privacy in a DM transaction, and sometimes not. We will take a broad view of privacy, in line with Kahn et al. (2005). That is, if an economic agent gives up privacy, this opens up the possibility that some other economic agent could take something from him or her. In the model, we will assume that each buyer receives a privacy shock each period, realized at the end of the CM, after consumption and production take place. With probability \(\rho\), the buyer requires his or her transaction in the DM to be private, and with probability \(1 - \rho\), privacy in the DM is of no value. Privacy in the DM will depend on what the buyer uses as a means of payment. In the model, there are potentially three means of payment used in equilibrium, though we will examine alternative regimes in which one or another means of payment is not offered. These three potential means of payment are bank deposits, physical currency, and CBDC. By assumption, a transaction in the DM using physical currency is private, but a transaction using bank deposits – a claim on a bank – is not private. As well, assume that CBDC is designed so that CBDC transactions in the DM...
can be private. For example, CBDC could be a central bank liability that can be used in transactions in the same way as bank deposits, through electronic debits and credits on a centralized ledger. Those transactions would not be private. But, assume that CBDC is also convertible into digital entries that can be transferred anonymously, retaining privacy. Further, for convenience, assume that physical currency is inferior to CBDC and bank deposits, in that physical currency cannot be used in transactions where buyers choose to forego privacy. For example, physical currency cannot be used in online transactions.

The model includes government debt and private capital. We will assume there is no technology that permits the trade of government debt or private capital in the DM, although banks can hold these assets and issue tradeable bank deposits as liabilities. There is limited commitment, in that no economic agent can be forced to work, and there is no recordkeeping, so buyers cannot trade personal IOUs in the DM.

The basic assets (before transformation by financial intermediaries) in this economy are physical currency, CBDC, one-period nominal government debt, and private capital. Physical currency bears a nominal interest rate of zero. One unit of CBDC held at the beginning of the CM of period \( t \) yields the digital currency holder \( R_{m_{t-1}} \) units of digital currency, paid by the central bank. The gross nominal interest rate on government debt is \( R_b \). Private capital is perfectly divisible, and can be produced, one-for-one, from labor supplied by either a buyer or a bank, in the CM. Capital produced in the CM of period \( t \) does not become productive until the CM of period \( t + 1 \). Each seller has a production technology that produces \( f(k) \) units of consumption goods from \( k \) units of productive capital in the CM. Assume that \( f(\cdot) \) is twice continuously differentiable, with \( f(0) = 0 \), \( f'(0) = \infty \), \( f''(k) < 0 \), and \( f'\infty) = 0 \). Capital produced in period \( t \) depreciates by 100\% at the end of the CM in period \( t + 1 \).

2.1 Government

Confine attention to policies that are constant for all \( t \), and to stationary equilibria. Assume that there is no consolidated government debt outstanding at the beginning of period \( t = 0 \), so the period 0 consolidated government budget constraint is given by

\[
\bar{c} + \bar{m} + \bar{b} = \tau_0,
\]

where \( \bar{c} \), \( \bar{m} \), and \( \bar{b} \) denote the quantities of physical currency, CBDC, and one-period government debt issued in period 0 (and in each succeeding period), all in units of the CM good in period 0. As well, \( \tau_0 \) denotes the lump sum transfer to each buyer at \( t = 0 \). Then, in each subsequent period, again confining attention to stationary policies and stationary equilibria,

\[
\bar{c} + \bar{m} + \bar{b} = \frac{\bar{c}}{\pi} + \frac{R_m\bar{m}}{\pi} + \frac{R_b\bar{b}}{\pi} + \tau
\]

where \( \pi \) is the price level.
Here, $\pi$ denotes the gross inflation rate, and $\tau$ is the lump-sum transfer\textsuperscript{4} that goes to each buyer in the CM. This transfer is constant for $t = 1, 2, 3, \ldots$. The left-hand side of (3) is the sum of total consolidated government liabilities outstanding after new liabilities are issued, while the right-hand side is the sum of the total redemption value of consolidated government liabilities from the previous period, plus the transfer to buyers.

It is important for the analysis how we specify fiscal policy, as this will help determine the aggregate supply of collateral, which plays an important role in the analysis. We will assume, as in Williamson (2016, 2019) that the fiscal authority sets $\tau_0$ and $\tau$ in response to monetary policy so that the real value of the consolidated government debt is a constant, $v$, forever. That is,

$$v = \bar{c} + \bar{m} + \bar{b}$$

(4)

Given this fiscal policy rule, the fiscal authority determines the total value of the consolidated government debt, while the central bank determines its composition.\textsuperscript{5}

3 Equilibrium

In this section, we will provide a general setup that applies across the different policy regimes we want to study (except for the last). Special cases will then be considered in subsequent sections. Here, we first look at the behavior of private banks, and model the provision of CBDC by the central bank. Then, we define an equilibrium for the general case, and characterize the equilibrium.

3.1 Private Banks

Buyers write deposit contracts with private banks in the CM before learning their privacy shocks. As in Williamson (2012, 2016, 2019), the deposit contract will provide insurance, in that the privacy shock outcome will determine what means of payment the depositor will use. We first want to solve a private bank’s problem, for the general case in which all means of payment are potentially used, then we will look at special cases in subsequent sections, using our solution here.

A private bank can insure depositors by providing each with an option either to withdraw a means of payment that supplies privacy – physical currency or CBDC – at the end of the CM, or else trade a claim on the bank. The bank offers

\textsuperscript{4}Note that, due to the absence of wealth effects, the distribution of transfers across agents in the CM is irrelevant.

\textsuperscript{5}The fiscal policy rule is important, as it will constrain the total real supply of public safe assets in equilibrium. This fiscal rule provides a nice separation of fiscal and monetary policy – fiscal policy determines the total quantity of consolidated government debt outstanding, and monetary policy determines the composition of that debt – and is arguably a realistic representation of government debt management policy. The form of the fiscal policy rule matters for the scarcity of government debt, and for the effects of monetary policy, which are critical to our analysis. But, there exists a wide class of fiscal policy rules that would not change the qualitative results.
a deposit contract \((a, c, m, d)\), where \(a\) is the quantity of \(CM\) goods deposited with the bank by the depositor at the beginning of the \(CM\), \(c\) and \(m\) are the quantities of physical currency and CBDC, respectively, that the depositor can choose to withdraw (in real terms) at the end of the \(CM\), and \(d\) is the quantity of claims to \(CM\) goods in the next period that the depositor can trade in the \(DM\) if he or she does not withdraw currency and/or CBDC at the end of the \(CM\). As well, in the \(CM\) the bank acquires a portfolio consisting of \(b\) government bonds and \(k\) private capital. Then, in equilibrium, the bank solves

\[
\max_{a, c, m, d, b, k} \left[ -a + \rho u \left( \frac{\beta(c + R^m m)}{\pi} \right) + (1 - \rho)u(\beta d) \right]
\]

subject to

\[
a - b - k - \rho(c + m) + \beta \left[ \frac{R^b b}{\pi} + rk - (1 - \rho)d \right] \geq 0,
\]

and

\[
\left( \frac{R^b b}{\pi} + rk \right) (1 - \gamma) - \gamma \rho \left( \frac{c + R^m m}{\pi} \right) - (1 - \rho)d \geq 0.
\]
This is essentially “sweat equity,” i.e. internally generated bank capital. In the next \( CM \), the bank pays off on its outstanding deposit liabilities, receives the payoffs on its assets, and consumes whatever is left.

Banks, just like the other individuals in this economy, are subject to limited commitment, and (7) is a collateral constraint, which states that the bank weakly prefers to pay out physical currency and CBDC to those requesting withdrawal in the current \( CM \), and to pay off on its deposit liabilities in the subsequent \( CM \), rather than absconding. The bank’s assets – government bonds and private capital – are effectively posted by the bank as collateral, but the bank can abscond with fraction \( \gamma \) of this collateral, should it default. When the bank defaults, off equilibrium, it will not fulfill requests for withdrawal, and will abscond in the next \( CM \) with fraction \( \gamma \) of the physical currency and CBDC acquired in the current \( CM \). Collateral constraints similar to (7) appear in Gertler and Kiyotaki (2011) and Williamson (2016, 2019).  

The incentive problem captured by the collateral constraint (7) plays an important role in generating potential benefits from CBDC issue in the model. Since it may not be clear to the reader how the incentive problem modeled here maps into real world issues in private banking, we should elaborate. Discussion of incentive problems in banking often focuses on moral hazard – that is, deposit insurance and the prospect of bailouts for large banks (too-big-to-fail) encourages excessive risk-taking. And moral hazard problems have motivated bank regulation, particularly risk-based capital requirements. But what about limited commitment problems in banking? As evidence that this is a significant practical problem, O’Keefe and Yom (2017) find that 37% of failed banks in the United States had issues of insider abuse and internal fraud, over the period 1989-2015. Further, under Basel III banking regulations, implemented in the United States as a supplementary leverage ratio (SLR) requirement, all bank assets – and not just risk-weighted assets – appear in the denominator when calculating the SLR. Thus, it is not far-fetched to model the bank in our model as having the option of absconding with liquid assets – cash and government debt. And we will show, in what follows, that the constraint (7) can be interpreted as a regulatory constraint.  

Let \( x^c \), \( x^m \) and \( x^d \) denote the quantities of consumption in the \( DM \) purchased, respectively, with currency, CBDC, and deposits claims, so

\[
x^c = \frac{\beta c}{\pi}, \quad (8)
\]

\[
x^m = \frac{\beta R^m m}{\pi}, \quad (9)
\]

and

\[
x^d = \beta d. \quad (10)
\]

6Though note that in Williamson (2016, 2019) it is not possible for the bank to abscond with currency, so this setup is more general.

7We could add to the model by including an entry cost for banks, so that active banks earn a strictly positive net profit, and would have something additional to lose from off-equilibrium default, as in Williamson (2021). This would provide extra discipline for banks, but would still admit binding incentive constraints, and the results would go through.
3.2 The Central Bank’s CBDC Facility

CBDC issue potentially permits a private banking arrangement that works much like banking with deposit accounts subject to withdrawal of currency on demand. The only difference is that, instead of withdrawing currency, the depositor can convert their deposit account into CBDC, and then conduct transactions using CBDC as a means of payment. But, the central bank could also offer CBDC through a type of narrow banking facility, in line with a proposal by Tobin (1987).

In the model, it is assumed that all central bank liabilities are backed one-for-one with government debt. Assuming that CBDC allows privacy, and can also be used in all the transactions for which buyers might otherwise use bank deposits, depositors might potentially choose to defect entirely from private banking, and use CBDC in all transactions. Depositors who choose to do this would be engaging in narrow banking, as they would be using means of payment backed only by safe assets – government debt. The narrow banking facility set up by the central bank then has the advantage of not being subject to the same costs as private banking. That is, private banking is costly because banks need to be provided with incentives not to abscond. Assuming central bankers can be trusted, this problem does not exist for the narrow bank. But, we assume that central banking is potentially costly, in two ways. First, to maintain its independence from the fiscal authority, the central bank needs to generate sufficient profits to pay its costs. Second, the narrow banking facility requires a sufficient supply of government debt to back the CBDC it issues. We will further discuss these problems in the next subsection.

If a buyer participates in the CBDC narrow banking facility, he or she chooses, in a stationary equilibrium, a quantity of CBDC, \( m_N \), to acquire in the CM, and then uses those CBDC balances in all transactions in the subsequent DM. The buyer then solves

\[
\max_{m_N} \left[ -m_N + u\left( \frac{\beta R m_N}{\pi} \right) \right].
\]

Define the DM consumption quantity for the participant in the central bank’s narrow bank by

\[
x m_N = \frac{\beta R m_N}{\pi}.
\]

3.3 Market-Clearing, Constraints on the Central Bank, and Definition of Equilibrium

Let \( \alpha \) denote the fraction of buyers who choose to deposit in banks in the CM, and \( 1 - \alpha \) the fraction of buyers who choose to use CBDC supplied by the narrow banking facility of the central bank. In equilibrium, all asset markets clear, so the demands for currency, CBDC, and government debt, respectively, are equal to the supplies, that is

\[
\alpha \rho c = \bar{c}; \alpha \rho m + (1 - \alpha)m_N = \bar{m}; \alpha b = \bar{b}.
\]
As well, optimization by sellers, who own the technology that produces output using capital as an input, gives

\[ f'(k) = r, \] (14)

which defines the supply function for capital.

From the private bank’s problem, (5) subject to (6) and (7), optimized utility for a bank depositor can be written as

\[ U^P = \rho[u(x_c + x_m) - u'(x_c + x_m)(x_c + x_m)] + (1 - \rho)[u(x_d) - u'(x_d)x_d], \] (15)

and note in particular that \( U^P \) is strictly increasing in \( x_c + x_m \), and in \( x_d \), for \( 0 \leq x_c + x_m < x^* \), and for \( 0 \leq x_d < x^* \). As well, from (11) and (12), we can write optimized utility for a participant in the central bank’s narrow banking facility as

\[ U^N = u(x_{mN}) - u'(x_{mN})x_{mN}, \] (16)

and similar to the case for private banks, \( U^N \) is strictly increasing in \( x_{mN} \) for \( 0 \leq x_{mN} < x^* \). Then, in equilibrium, \( \alpha \) is determined by

\[ \alpha = \arg \max_{\alpha'} [\alpha'U^P + (1 - \alpha')U^N], \] (17)

that is, depositors choose the banking arrangement that maximizes expected utility.

A key element in the structure of a typical modern central bank is a monopoly on the issue of physical currency. This monopoly enhances the central bank’s ability to earn a profit, based on the difference between the interest rate on government debt (the rate of return on the central bank’s assets), and zero-interest currency. Basically, the inflation tax yields revenue for the central bank, the central bank pays its costs, and then the central bank typically transfers the remainder to the central government. If the central bank cannot earn sufficient revenue to pay its costs, it is dependent on capital infusions from the central government, and this can become a threat to its independence. That is, central bank independence depends on the ability of the central bank to keep its distance from the central government’s budgetary process.

The flow of transfers from the central bank to the fiscal authority appears to have been mostly ignored in the United States until the balance sheet expansion conducted by the Federal Reserve System under its post-financial crisis large-scale asset purchase programs. The chief concern in that case related to a significant lengthening in the average maturity of assets in the Fed’s portfolio, along with the payment of interest on a large stock of reserve liabilities. Under scenarios whereby the Fed would be in a position in which it had to raise short term interest rates significantly or sell assets at a loss, Carpenter et al. (2013) and Greenlaw et al. (2013) show that there could have been situations where the Fed booked deferred assets and was forced to reduce transfers to the U.S. Treasury to zero. Thus, there was concern with a potentially temporary cessation in central bank transfers to the fiscal authority. But, large and permanent changes in the structure of the assets and liabilities of the central bank

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– such as the introduction of CBDC – potentially imply even more concerning implications for the relationship of the central bank with the fiscal authority.

To capture this in the model, we need to separate the activities of the fiscal authority and the central bank. In period 0, in a stationary equilibrium, the fiscal authority issues $v$ units of government bonds, in units of $CM$ goods, and the central bank issues $\bar{c} + \bar{m}$ units of central bank liabilities (currency plus CBDC) to purchase $\bar{c} + \bar{m}$ units of government debt. In a stationary equilibrium, in each succeeding period the central bank maintains $\bar{c} + \bar{m}$ units of central bank liabilities outstanding, purchases $\bar{c} + \bar{m}$ units of government debt, pays the interest and principal on the central bank liabilities carried over from the previous period, receives interest on the government debt held over from the previous period, and makes a transfer $\tau_{CB}$ to the fiscal authority. Assume that total central bank profits are transferred to the fiscal authority period-by-period.

As well, suppose there is a fixed cost $F$ per period of running the central bank. Assume that $F$ is paid from the revenue generated from issuing consolidated government liabilities in period 0. But, in each succeeding period, central bank independence (in the sense of independence from the fiscal authority) requires $\tau_{CB} \geq F$, or

$$\bar{c}\left(\frac{R^b - 1}{\pi}\right) + \bar{m}\left(\frac{R^b - R^m}{\pi}\right) \geq F.$$  

(18)

So, on the right-hand side of (18), the revenue generated by the central bank is determined by the stocks of physical currency ($\bar{c}$) and CBDC ($\bar{m}$) outstanding, and by the effective tax rates on those liabilities. In each case, the effective tax rate is the difference between the real interest rate on government debt and the real interest rate on the respective central bank liability. In general, the usual factors will come into play in determining central bank revenue. That is, monetary policy can increase the tax rates on central bank liabilities, which tends to increase central bank revenue, but increases in tax rates also tend to reduce the tax base, which in this case is the stock of central bank liabilities.

We will assume that the central bank operates under rules that prevent it from purchasing private capital. Such rules can be justified in practice by arguing that central banks are inefficient in intermediating private assets, and that doing so would enter into the realm of fiscal policy. We recognize that purchases of private assets have become more common for central banks in recent years, for example for the Bank of Japan, the Swiss National Bank, the European Central Bank, and the Federal Reserve System. But, it seems useful to explore whether or not CBDC issue could matter for a central bank constrained to purchase only government debt. In this model, if the central bank cannot purchase private capital, then

$$\bar{m} + \bar{c} \leq v,$$  

(19)

which implies, given the fiscal policy rule (4), that the stock of government debt held by the public is nonnegative.

An equilibrium can then be defined as follows.
Definition 1: A stationary equilibrium consists of asset quantities \((a, c, m, d, b, k, m_N, \bar{m}, \bar{c}, \bar{b})\), gross inflation rate \(\pi\), capital rental rate \(r\), taxes \(\tau_0\) and \(\tau\), DM consumption quantities \((x^c, x^m, x^d, x^{mN})\), maximized utilities \(U^P\) and \(U^N\), and a fraction of private banking participants \(\alpha\), satisfying the government’s budget constraints (2) and (3), the fiscal policy rule (4), the private bank’s problem, (5) subject to (6) and (7), the problem of a participant in the narrow bank (11), and (15)-(17), (18), and (19), given exogenous monetary policy \((R^b, R^m)\) and fiscal policy \(v\).

Note in the definition that exogenous monetary policy is defined by the nominal interest rates on government debt and CBDC, and exogenous fiscal policy is defined by the real quantity of consolidated government debt. The central bank determines market interest rates by setting the interest rate on CBDC administratively, and then purchases sufficient government debt and issues sufficient currency and CBDC to achieve its target for the nominal interest rate on government debt. That is, the central bank’s balance sheet policy supports its nominal interest rate target, and the nominal interest rate chosen for CBDC. The fiscal authority sets taxes so as to satisfy the government’s budget constraints and its fiscal policy rule, given monetary policy.

3.4 Characterization of Equilibrium

We first solve the bank’s problem, (5) subject to (6) and (7). Note that (6) holds with equality at the optimum, i.e. each bank will earn zero profits, in present value terms. Then, from the bank’s optimization problem, optimal choice of \(d\) implies

\[
\lambda = \beta \left[ u'(x^d) - 1 \right],
\]

where \(\lambda\) denotes the multiplier associated with the bank’s collateral constraint (7). Recall that \(x^*\) denotes the surplus-maximizing quantity of goods exchanged in a DM transaction, where surplus when \(x\) goods are consumed by the buyer is given by \(u(x) - x\). So in (20), the greater is the inefficiency in DM exchange involving bank deposits, the tighter is the bank’s collateral constraint (7). Then, optimal choice of \(c\) and \(m\) in the bank’s problem, using (8)-(10) and (20), gives asset pricing relationships for physical currency and digital currency,

\[
-1 + \frac{\beta}{\pi} \left[ u'(x^c + x^m) - \gamma u'(x^d) + \gamma \right] \leq 0,
\]

and

\[
-1 + \frac{\beta R^m}{\pi} \left[ u'(x^c + x^m) - \gamma u'(x^d) + \gamma \right] \leq 0.
\]

As well, from the bank’s optimization problem, the following asset pricing relationships for government debt and private capital hold in equilibrium:

\[
\frac{R^b}{\pi} = r = \frac{1}{\beta \left[ \gamma + (1 - \gamma) u'(x^d) \right]},
\]
So, the greater is the inefficiency in $DM$ exchange involving bank deposits, that is the larger is $u'(x^d)$, the lower are the real rates of return on government debt and private capital. That is, binding collateral constraints for banks impart liquidity premia to government debt and private capital, which back bank deposits, which in turn are used in exchange.

If the bank’s collateral constraint (7) binds, then since (6) holds with equality, the quantity of bank capital – labor supplied by the bank in the current $CM$ – is

$$b + k + \rho(c + m) - a = \beta \gamma \left[ \frac{R^d b}{\pi} + \frac{\rho c}{\pi} + \frac{\rho R^m m}{\pi} \right]$$

so a binding collateral constraint is reflected in positive bank capital. That is, for the bank to demonstrate that it has the incentive to pay off its debts in the future, it has to acquire bank capital equal to the present value of the asset portfolio it could potentially abscond with in the future $CM$. An alternative interpretation of the parameter $\gamma$ is that this captures a regulatory capital constraint or leverage constraint.

Next, from the problem for a participant in the central bank’s narrow bank, (11), the first order condition for an optimum can be written, given (12),

$$-1 + \frac{\beta R^m}{\pi} u'(x^{m,N}) = 0,$$

which is another asset pricing relationship for CBDC that must hold.

Then, confine attention to an equilibrium in which banks’ collateral constraints (7) bind. If collateral constraints do not bind, this is more straightforward, less interesting, and an easy extension. So, given (7) holds with equality, substitute in (7) using (4), (8)-(10), (21), (22), (13), (14), and (24) to obtain

$$v + k = \alpha \rho \left( x^c + x^m \right) \left[ \frac{\gamma}{1 - \gamma} + u'(x^c + x^m) \right] + \alpha(1 - \rho) x^d \left[ \frac{\gamma}{1 - \gamma} + u'(x^d) \right] + (1 - \alpha)x^{m,N} u'(x^{m,N}).$$

In (25), the left-hand side is the sum of the supplies of public and private collateral, $v$ and $k$ respectively. The right-hand side is the sum of demands for collateral, derived from the quantities of $DM$ consumption using each means of payment – currency and CBDC withdrawn from private banks, private bank deposits, and CBDC issued by the central bank’s narrow bank, respectively. Then, from (14) and (23), the marginal product of capital is equal to the real rate of interest on government debt:

$$f'(k) = \frac{1}{\beta [\gamma + (1 - \gamma)u'(x^d)].}$$

If $x^c > 0$, then from (21) and (23), the nominal rate of interest depends on the margin between the inefficiency in exchange using currency, and inefficiency in
exchange using bank deposits,

$$R^b = \frac{u'(x^c + x^m) - \gamma[u'(x^d) - 1]}{u'(x^d) - \gamma[u'(x^d) - 1]}.$$  \hspace{1cm} (27)

And, if $x^m > 0$, then from (22) and (23),

$$R = \frac{u'(x^c + x^m) - \gamma[u'(x^d) - 1]}{u'(x^d) - \gamma[u'(x^d) - 1]},$$  \hspace{1cm} (28)

where

$$R \equiv \frac{R^b}{R^m}.$$  \hspace{1cm} (29)

So, note that (28) is identical to (27), except that $R$ appears on the left-hand side of (28), as in this case the relative inefficiencies relate to exchange using CBDC vs. exchange using bank deposits. As well, from (24) and (22), if $x^m > 0$ we get

$$u'(x^m) - \gamma[u'(x^d) - 1] = u'(x^{mN}),$$  \hspace{1cm} (30)

Using (13), (8)-(10), (12), (21), (22), (23), and (24), we can write inequality (18) as

$$\left\{ \frac{[\alpha \rho(x^c + x^m)] [u'(x^c + x^m) - u'(x^d)]}{+(1 - \alpha)x^{mN} [u'(x^{mN}) - \gamma - (1 - \gamma)u'(x^d)]} \right\} \geq F$$  \hspace{1cm} (31)

and we can write inequality (19) as

$$\alpha \rho(x^c + x^m) \left\{ u'(x^c + x^m) - \gamma [u'(x^d) - 1] \right\} + (1 - \alpha)x^{mN} u'(x^{mN}) \leq v.$$  \hspace{1cm} (32)

In (31) and (32), respectively, we have expressed the constraint that central bank profits be sufficient to fund the central bank, and that total central bank liabilities cannot exceed the stock of government debt, in terms of the quantities of goods exchanged in each type of DM transaction.

In each special case we consider, we can use (25)-(28), (30)-(32), and (15)-(17) to solve for the DM consumption allocation given policy $(R^b, R, v)$.

4 Typical Status Quo: Equilibrium With Physical Currency

We will start with a representation of the current typical central banking arrangement. Under this system, the central bank is restricted to buying only the debt issued by the fiscal authority, and the government grants the central bank a monopoly on a specific means of payment, physical currency. There is no CBDC, and private banks also provide a means of payment, but as part of deposit contracts that permit withdrawal of physical currency on demand. Any central bank profits are transferred to the fiscal authority.\footnote{Note that we abstract from central bank lending and the central bank’s lender-of-last-resort function. These elements could be important for some aspects of CBDC issue (financial stability implications for example), but not for what we address here.}
As no CBDC is issued in this regime, \( x^m = x^{mN} = 0 \) and \( \alpha = 1 \). Then, from (25),
\[
v + k = \rho x^c \left[ \frac{\gamma}{1 - \gamma} + u'(x^c) \right] + (1 - \rho)x^d \left[ \frac{\gamma}{1 - \gamma} + u'(x^d) \right],
\]
and from (27), (31) and (32), respectively,
\[
R^b = \frac{u'(x^c) - \gamma[u'(x^d) - 1]}{u'(x^d) - \gamma[u'(x^d) - 1]},
\]
\[
\rho x^c \left[ u'(x^c) - u'(x^d) \right] \geq F,
\]
and
\[
\rho x^c \left\{ u'(x^c) - \gamma [u'(x^d) - 1] \right\} \leq v
\]
From (21), we can solve for the inflation rate,
\[
\pi = \beta \left[ u'(x^c) - \gamma u'(x^d) + \gamma \right]
\]
So, in this case policy \((R^b, v)\) determines the DM consumption allocation \((x^c, x^d)\) and \(k\), as the solution to (33), (34), and (55). Then, (37) solves for the inflation rate, given \(x^c\) and \(x^d\). The policy \((R^b, v)\) is feasible if and only if \((x^c, x^d)\) satisfies (35) and (36), so that transfers from the central bank to the fiscal authority are nonnegative, and so that there is sufficient government debt to back the currency demanded by buyers at market prices.

If we measure welfare by summing lifetime utility across agents in this equilibrium, then welfare is proportional to
\[
W = -k + \beta f(k) + \rho[u(x^c) - x^c] + (1 - \rho)[u(x^d) - x^d].
\]
That is, as is usual in Lagos-Wright (2005) settings, the disutility of production nets out with the utility from consuming goods, when production and consumption of the good occur during the same CM. For private capital, the net contribution to aggregate welfare is minus the disutility of producing the capital in the current CM, plus the discounted value of the consumption goods produced from the capital in the subsequent CM. This accounts for the first two terms in (38), and the next two terms are the total surplus exchanged in the DM.

**Proposition 1:** If we ignore (35), then efficiency is attainable if and only if
\[
v \geq \max \left( x^* \frac{x^*}{1 - \gamma} - k^*, \rho x^* \right),
\]
and \( R^b = 1 \), where \( k^* \) is defined by
\[
f'(k^*) = \frac{1}{\beta}.
\]
Proof: Online appendix.

Proposition 1 establishes conditions under which efficiency is achieved in equilibrium, i.e. the equilibrium allocation is identical to what could be achieved by a social planner choosing $k$, $x^c$ and $x^d$ to maximize $W$ in (38). But in an efficient equilibrium, $\lambda = 0$ from (20), so banks’ collateral constraints do not bind. Therefore, given that we are interested in equilibria with binding collateral constraints, we will assume that

$$v < \frac{x^*}{1 - \gamma} - k^*, \quad (41)$$

that is collateral is sufficiently scarce in the aggregate, and

$$v > \rho x^*, \quad (42)$$

which will guarantee that (36) can hold in this equilibrium. Note that a necessary condition for (41) and (42) to be satisfied is

$$x^* > \frac{k^*(1 - \gamma)}{1 - \rho(1 - \gamma)} \quad (43)$$

This then establishes a benchmark for the other regimes we consider.

Proposition 2: If (41) and (42) hold, and $R^b$ is sufficiently close to 1, then, ignoring (35), an equilibrium with binding incentive constraints exists, it is unique, and an increase in $R^b$ results in an increase in $x^d$, a decrease in $x^c$, an increase in the real interest rate on government debt, a decrease in $k$, and an increase in $\pi$.

Proof: Online appendix.

In Proposition 2, the effects of conventional monetary policy under binding collateral constraints are in line with the properties of the model in Williamson (2016), for this special case (though Williamson 2016 has no private capital, and lacks some of the other details incorporated in this model). An increase in the nominal interest rate on government debt is supported by permanently lower purchases of government debt by the central bank. With less physical currency outstanding, in real terms, the quantity of transactions supported by physical currency, $x^c$, must fall. However, since the central bank releases more government debt, this increases the stock of safe collateral backing bank deposits, so the quantity of transactions supported by bank deposits, $x^d$, rises. As well, the relaxation of banks’ collateral constraints acts to increase the real rate of interest on government debt. The rental rate on capital is equal to the gross real rate of interest on government debt, so the rental rate rises, the marginal product of private capital rises, and the quantity of private capital falls. There is a Fisher effect, in that the inflation rate increases, but less than in proportion to the increase in the nominal interest rate, as the real interest rate rises.

We are also interested in the effects of a change in $R^b$ on welfare, and on central bank profits.
Proposition 3: At the margin, an increase in $R^b$, from $R^b = 1$, reduces welfare, and increases central bank profits.

Proof: Online appendix.

From (34) and (35), $R^b = 1$ implies that $x^c = x^d$ in equilibrium, and so the central bank collects no revenue when the nominal interest rate is zero. But given the binding collateral constraint there is DM inefficiency when $R^b = 1$, in that $x^c = x^d < x^*$, and the economy is over-capitalized, that is $f'(k) < 1/\beta$. From Proposition 2, raising $R^b$ from $R^b = 1$ reduces inefficiency in DM transactions using bank deposits and in capital accumulation, as $x^d$ rises and $k$ falls. But there is greater inefficiency in DM transactions using physical currency, as $x^c$ falls. On balance, the reduction in efficiency in cash transactions outweighs the increases in efficiency from other sources and, from Proposition 3, welfare falls.

But, from (34) and (35), a positive nominal interest rate ($R^b > 1$) is necessary for (35) to be satisfied. Proposition 3 implies that, if $R^b$ is chosen by the central bank to maximize welfare, given fiscal policy $v$, and if $F$ is sufficiently small, then $R^b > 1$ at the optimum and (35) holds with equality at the optimum. Given (42), therefore (36) holds for $R^b = 1$, so by continuity (36) holds for $R^b$ sufficiently close to 1.

This case will be our baseline for evaluating alternative approaches to delivering CBDC. As long as the costs of running the central bank are small, at the optimum the central bank sets the nominal interest rate so that just enough central bank revenue is generated to cover the central bank’s costs. The latter property is interesting, as it gives a rationale for a deviation from zero nominal interest rates.

5 CBDC Use Integrated With Private Banks

The first CBDC regime we will consider is one where physical currency is withdrawn from circulation by the central bank, and replaced by CBDC. In this case, we assume that CBDC takes the form of an account with the central bank that can be accessed and used in transactions in such a way as to preserve privacy. But, because CBDC is digital, it can be used in the same transactions as bank deposits. In this regime, though, the central bank sets $R^m$ sufficiently low that buyers will not use CBDC in a narrow banking arrangement. That is, in equilibrium all buyers will participate in private banking arrangements, where the deposit contracts permit “withdrawal” of CBDC (that is, conversion of bank deposits to CBDC accounts) at the end of the CM. In equilibrium, CBDC will be used in DM transactions for which buyers want privacy, and bank deposits will be used in all other transactions.

In this case, $x^c = 0$ and $\alpha = 1$ in equilibrium, so from (25), (28), (31), and (32), we get, respectively,

$$v + k = \rho x^m \left[ \frac{\gamma}{1-\gamma} + u'(x^m) \right] + (1-\rho)x^d \left[ \frac{\gamma}{1-\gamma} + u'(x^d) \right],$$

(44)
\[ R = \frac{u'(x^m) - \gamma [u'(x^d) - 1]}{u'(x^d) - \gamma [u'(x^d) - 1]}, \quad (45) \]

\[ \frac{\rho x^m [u'(x^m) - u'(x^d)]}{\beta \{\gamma + (1 - \gamma) u'(x^d)} \geq F, \quad (46) \]

and

\[ \rho x^m \{u'(x^m) - \gamma [u'(x^d) - 1]\} \leq v. \quad (47) \]

To solve for inflation, from (22) we get

\[ \pi = \beta R^m [u'(x^m) - \gamma u'(x^d) + \gamma]. \quad (48) \]

As well, (55) and (30) hold, and from (15)-(17), we have

\[ \rho [u(x^m) - u'(x^m)x^m] + (1 - \rho)[u(x^d) - u'(x^d)x^d] \geq u(x^{mN}) - u'(x^{mN})x^{mN} \quad (49) \]

Here, (49) states that buyers must weakly prefer, in the CM, a deposit contract with a private bank to using the narrow bank for all transactions.

So, in this regime, (44), (45), and (55) solve for \( x^m, x^d \), and \( k \), given policy \( (R, v) \). Then (30) solves for \( x^{mN} \), (48) solves for \( \pi \), and the solution must satisfy (46)-(49) for the policy to be feasible. Here, note that \( x^{mN} \) is what a buyer would consume in the DM, out of equilibrium, given equilibrium prices, were he or she to participate in the narrow bank.

Then, if we compare the equilibrium solution in the typical status quo case to what we get here, by replacing \( R^b \) with \( R \) and \( x^c \) with \( x^m \) in (33), (34), and (55), it is easy to see that we get (44), (45), and (55). So given \( R^b = R \), the equilibrium allocation and welfare are identical to the status quo case. The difference in this digital currency world is that what matters, rather than the difference between the interest rate on government debt and zero-interest currency as in the status quo, is the difference between the interest rate on government debt and the interest rate on CBDC. The level of nominal interest rates matters in this regime only for inflation, from (48). That is, if \( R^b \) and \( R^m \) increase in proportion, then \( R \) is unaffected, and there are no consequences for real variables or welfare. But the gross inflation rate \( \pi \) will increase in proportion to \( R^m \), from (48) – there is a pure Fisher effect.

An important difference from the status quo is that we add the constraint (49). So, either that constraint does not bind, and optimal policy yields the same equilibrium allocation in this regime as under the status quo, or the constraint binds and welfare must be lower.

**Proposition 4:** Inequality (49) is a binding constraint for some parameter values.

Proof: Online appendix.

In this regime, CBDC is introduced as a substitute for physical currency, in such a way that there is no net welfare benefit. Monetary policy is conducted so that CBDC will not compete with private bank deposits, i.e. so that a
narrow banking arrangement involving CBDC will not be preferred by buyers to conducting business with a private bank. This means that \( R \) must be sufficiently high to make CBDC is sufficiently unattractive. But, just as under the status quo, \( R \) must also be sufficiently high to generate enough revenue to keep the central bank running. How high \( R \) needs to be to make CBDC sufficiently unattractive depends on the costs to conducting private banking, represented by \( \gamma \), which determines the magnitude of the incentive problem in private banking. So CBDC, in this regime, tends to produce an inferior outcome to the status quo when the incentive costs of private banking are high. This is related to concerns that CBDC could generate financial instability (e.g. Kumhof and Noone 2018). The typical concern in this respect has more to do with flight to safety during financial crises, and how CBDC might encourage that (addressed in Williamson 2021, for example). But here, if \( \gamma \) is high, then \( R \) must be high to prevent private bank depositors from fleeing to narrow banking services, so the issues are related.

The conclusion is that, if the central bank goes out of its way not to make CBDC attractive enough to compete with private bank deposits, then at best economic welfare is unchanged. But, the introduction of CBDC may actually reduce economic welfare, if private banking is sufficiently inefficient.

6 CBDC Used Through the Central Bank’s Narrow Banking Facility

In this regime, the central bank takes a more liberal approach to CBDC issue. The central bank retains physical currency issue, so private banks operate as under the status quo, with deposit contracts permitting withdrawal of physical currency on demand. But the central bank issues CBDC through a narrow banking arrangement, and buyers who opt to conduct transactions with CBDC are able to use CBDC when they want privacy, and when they do not. Narrow banking then competes with private banks, and monetary policy will matter for the allocation of deposit liabilities between narrow banking and private banking.

To simplify the analysis, confine attention to a policy with \( R_b = 1 \), that is a zero nominal interest rate on government debt, so monetary policy is determined by \( R \), which in this instance is in turn determined by the nominal interest rate on CBDC, that is \( R = \frac{1}{1 + \pi} \). Our results would not change substantively if we permitted \( R_b > 1 \). A policy with \( R_b = 1 \) has the virtue of eliminating the relative distortion in transactions using physical currency and bank deposits, respectively. Also, in this equilibrium any central bank profits will be generated by negative interest on CBDC, which seems a novel idea, and monetary policy works through changes in \( R^m \), as we will show. Finally, this setup allows for a convenient comparison to the status quo.

For this case we have \( x^m = 0, x^c > 0, 0 < \alpha < 1 \), and, from (27) and \( R_b = 1 \), \( x^c = x^d = x \). Then, from (15)-(17), we have \( x^{mN} = x \). So, in this equilibrium,
all buyers consume $x$ in the $DM$. Then, from (25) and (28), we get, respectively,

$$v + k = xu'(x) + \frac{\alpha x \gamma}{1 - \gamma},$$

and

$$R = \frac{u'(x)}{\gamma + (1 - \gamma)u'(x)}.$$  

(50)  

(51)

From (21),

$$\pi = \beta [\gamma + (1 - \gamma)u'(x)]$$

(52)

Then, from (31) and (32) we get, respectively,

$$\frac{(1 - \alpha)x \gamma [u'(x) - 1]}{\beta \{\gamma + (1 - \gamma)u'(x)x\}} \geq F,$$

and

$$\alpha \rho x \gamma + xu'(x) [\alpha \rho (1 - \gamma) + 1 - \alpha] \leq v.$$  

(53)  

(54)

So, in this equilibrium, $x$ is determined by (51), given $R$. Then, (55) and (50) solve for $\alpha$, the fraction of buyers who participate in private banking, and $k$, the stock of private capital. Inflation is determined from (52), given $x$. Finally, $x$ must satisfy the constraints (53) and (54).

In the above characterization of the narrow banking regime, we have assumed that private banks’ collateral constraints bind. What about the possibility that the introduction of CBDC through a narrow banking facility could relax collateral constraints to the point where these constraints do not bind? Proposition 5 addresses this.

**Proposition 5:** In the narrow banking regime, an equilibrium in which banks’ collateral constraints do not bind is not feasible, in that (53) does not hold.

Proof: Online appendix.

Proposition 5 allows us to restrict attention to the equilibrium with binding bank collateral constraints, which is characterized above. Next, we want to determine the response of the equilibrium allocation to a change in monetary policy in the narrow banking regime.

**Proposition 6:** In the narrow banking regime, a decrease in $R$ increases $x$, decreases $\alpha$, decreases $k$, increases the real interest rate on government debt, and decreases the inflation rate. Welfare increases, and constraint (54) tightens.

Proof: Online appendix.

**Proposition 7:** In the narrow banking regime, a decrease in $R$, when $\alpha = 1$, increases central bank revenue at the margin.
Proof: Online appendix.

Monetary policy works somewhat differently in this regime. With the nominal interest rate on government debt pegged at zero, monetary policy affects real quantities, inflation, and welfare through administered changes in $R^m$, the gross nominal interest rate on CBDC. An increase in $R^m$ – a reduction in $R$ – makes CBDC more attractive and results in substitution from private banking to narrow banking, that is $\alpha$ falls. This acts to free up scarce collateral, as private banking is more demanding of the stock of collateral, due to incentive problems in private banking. So, more transactions are accommodated, and $x$ increases, that is consumption rises in all $DM$ exchange. Further, since the collateral constraints of banks have been relaxed, the real interest rate rises. The increase in $R^m$ causes inflation to fall, since the nominal interest rate on government debt is fixed at zero and the real interest rate has risen.

Is a narrow banking regime feasible given some setting for $R$, that is do policies exist such that (53) and (54) are satisfied in equilibrium? If $\alpha = 1$, then the left-hand side of (53) is equal to zero, and from Proposition 6 a decrease in $R$ will increase the left-hand side of (53). So, by continuity, for $F$ sufficiently small, (53) will hold for $\alpha$ sufficiently close to 1. Further, under assumption (42), (54) is a strict inequality when $\alpha = 1$. So, for $F$ sufficiently small, narrow banking is feasible.

**Proposition 8:** Welfare can be higher in the narrow banking regime than with the status quo.

Proof: Online appendix.

If $R^m$ is set low enough in the narrow banking regime, then $\alpha = 1$, and the allocation is identical to what we would get with the status quo, since there is no narrow banking. But that allocation, with $R^b = 1$, has higher welfare than at the optimum with the status quo, since a higher nominal interest rate is required to generate sufficient central bank revenue. So, with no narrow banking, that is $\alpha = 1$, the narrow banking regime is preferable to what we get with optimal policy under the status quo. But when $\alpha = 1$, (53) is not satisfied, that is the central bank generates no revenue. But if the central bank raises $R^m$ to the point where $\alpha$ starts to fall, then welfare increases, as does central bank revenue. So, the central bank can do better than the status quo, provided that it can generate enough revenue to satisfy (53) without violating (54). But note that there is a tension here, as increasing $R^m$ also increases the left-hand side of (54). More narrow banking relieves the scarcity of collateral in general, but it puts higher demands on publicly-supplied collateral.

What is the optimal monetary policy, i.e. the optimal setting for $R$ (or equivalently $R^m$) under a narrow banking regime? A policy that implies $\alpha = 1$ in the narrow banking regime implies that the left-hand side of (53) is equal to zero, so, given $F > 0$, (53) does not hold. Further, Proposition 9 tells us that a narrow banking equilibrium with $\alpha = 0$ is not feasible either.

**Proposition 9:** A narrow banking regime with $\alpha = 0$ in equilibrium is not feasible.
Proof: Online Appendix.

The proof of Proposition 9 shows that if $\alpha = 0$ in this regime then (54) is violated. That is, narrow banking cannot drive out private banking, as there is necessarily too little government debt to back the required quantity of CBDC.

For a policy to be optimal, it must be feasible, in that (53) and (54) hold. Further, one of (53), or (54) must bind, as otherwise, from Proposition 5, $R$ could be reduced, with a resulting increase in welfare. So, in the narrow banking regime, at the optimum, monetary policy is either constrained by the need to earn sufficient central bank profits, or constrained by the quantity of available government debt. Further, we have shown that, at the optimum, CBDC and private banking coexist, and banks’ collateral constraints bind, so there is inefficiency in exchange in all trades that occur in the DM.

7 A Narrow Bank That Replicates Private Deposit Contracts

This regime will be related to the one in the previous section, in that CBDC is offered only by a narrow bank operated by the central bank. The difference here is that we differentiate explicitly between means of payment offered by the narrow bank that can be used when privacy is needed, vs. means of payment issued by the narrow bank that do not protect privacy. That is, in a stationary equilibrium, the narrow bank offers a form of CBDC that protects privacy (privacy-CBDC), and bears gross nominal interest rate $R_p$, and offers CBDC that does not protect privacy (non-privacy-CBDC) and bears gross nominal interest rate $R_n$. These payments instruments are offered through an insurance arrangement that is essentially identical to private deposit contracts, with participants either converting their deposit to privacy-CBDC at the end of the CM, or trading non-privacy-CBDC in the subsequent DM. For CBDC issued in period $t$, holders of privacy-CBDC receive a nominal interest payment $R_p - 1$ in the DM of period $t + 1$, while non-privacy CBDC holders receive $R_n - 1$.

So, if $x^p$ and $x^n$ denote, respectively, the quantities of DM consumption for a buyer who uses privacy-CBDC, and a buyer who uses non-privacy-CBDC, then an optimal CBDC arrangement, given $R_p$ and $R_n$, satisfies

$$1 = \frac{\beta R_p}{\pi} u'(x^p),$$

and

$$1 = \frac{\beta R_n}{\pi} u'(x^n).$$

Then, assume that the central bank sets $R_p$ and $R_n$ so that buyers consume the same quantities in DM transactions as they would if they held deposits in private banks, that is $x^p = x^c$ and $x^n = x^d$. This implies that buyers are indifferent between depositing in a private bank and depositing in the narrow bank at the beginning of the CM.
As well, in this regime we will permit differences in a private bank’s ability to abscond across assets. That is, let \( \gamma^c \), \( \gamma^b \), and \( \gamma^k \) denote, respectively, the fraction of a bank’s currency, bond, or private asset holdings that the bank can abscond with, should it choose (off equilibrium) to default on its liabilities. Then, given this setup, and given \( x^p = x^c \) and \( x^n = x^d \), along with (55) and (56), we obtain, analogous to (55)-(32),

\[
\frac{k(1 - \gamma^k)}{(1 - \gamma^b)[\gamma^k + (1 - \gamma^k)u'(x^d)]} = \frac{\alpha \rho x^c \gamma^c}{1 - \gamma^b} + \frac{\alpha(1 - \rho)x^d \gamma^c}{1 - \gamma^b} + \rho x^c u'(x^c) + (1 - \rho)x^d u'(x^d),
\]

(57)

\[
f'(k) = \frac{1}{\beta [\gamma^k + (1 - \gamma^k)u'(x^d)]},
\]

(58)

\[
R^b = \frac{u'(x^c) - \gamma^c [u'(x^d) - 1]}{u'(x^d) - \gamma^b [u'(x^d) - 1]},
\]

(59)

\[
R^p = \frac{u'(x^c) - \gamma^c [u'(x^d) - 1]}{u'(x^c)}.
\]

(60)

\[
R^n = \frac{u'(x^c) - \gamma^c [u'(x^d) - 1]}{u'(x^d)},
\]

(61)

\[
\rho x^c u'(x^c) - \alpha \rho x^c \gamma^c [u'(x^d) - 1] + (1 - \alpha)(1 - \rho)x^d u'(x^d) \leq v,
\]

(62)

\[
\alpha \rho x^c (R^b - 1) + (1 - \alpha) \rho x^c \left( \frac{R^b}{R^p} - 1 \right) + (1 - \alpha)(1 - \rho)x^d \left( \frac{R^b}{R^n} - 1 \right) \geq \beta F
\]

(63)

Equations (59) and (61) solve for \( x^c \) and \( x^d \) given monetary policy \((R^b, R^n)\), and then (60) solves for \( R^p \) given \( x^c \) and \( x^d \). That is, \( R^p \) needs to be set so that depositors are indifferent between private banking contracts and CBDC. Finally, equations (57) and (58) solve for \( \alpha \) and \( k \) given \( x^c \) and \( x^d \).

**Proposition 10:** If \( \gamma^k > \gamma^b > 0 \), then, holding \( R^b \) constant, an increase in \( R^n \) causes increases in \( x^c \) and \( x^d \), a decrease in \( k \), an increase in \( R^p \), and a decrease in \( \alpha \).

Proof: Online appendix.

Proposition 10 states that, if private banks can abscond with private capital and government debt, and government debt has greater pledgeability than private capital, then a change in monetary policy that makes CBDC more attractive will increase the quantity of exchange in all DM meetings, and will increase welfare. But we need to check whether such a welfare-improving policy is feasible, that is (62) and (63) must be satisfied.

**Proposition 11:** If \( \gamma^k > \gamma^b > 0 \), and there exists an optimal policy \((\bar{R}^b, \bar{R}^n)\) with \( \alpha = 1 \) imposed, then there exists a policy \((\hat{R}^b, \hat{R}^n)\) implying \( 0 < \alpha < 1 \) that strictly dominates \((\bar{R}^b, \bar{R}^n)\) in welfare terms.
Proof: Online appendix.

Proposition 11 states that, if there is an optimal monetary policy in this regime, under which CBDC is not issued, then there exists another monetary policy under which CBDC is issued, and the equilibrium allocation under this alternative dominates the equilibrium allocation with no CBDC issued. So, a policy that encourages the use of CBDC is optimal. Given (42), under the optimal monetary policy with no CBDC issued, (62) is slack. As well, an optimal policy must be feasible, so (63) holds. Then, Proposition 10 tells us that given any policy, if we hold $R^b$ constant and increase $R^n$, then welfare increases, so that must be true in moving from the optimal policy with no CBDC by increasing $R^n$. But we need to check that the policy under which $R^n$ increases is feasible, at least at the margin. Since (62) is slack under the initial monetary policy, therefore (62) must hold, and we show in the proof of Proposition 11 that, at the margin, increasing $R^n$ when $\alpha = 1$ increases the left-hand side of (63). Therefore the proposed monetary policy is feasible.

Alternatively, suppose that $\gamma^k > 0$ and $\gamma^b = \gamma^c = 0$, so that private banks can abscond with private capital, but not with government debt or physical currency. Then, (59), (60), and (61) imply that $R^b = R^n$, $R^p = 1$, and $\alpha$ is indeterminate. Further, the left-hand side of (63) does not depend on $\alpha$, but an increase in $\alpha$ relaxes (62). So, if incentive problems in private banking are confined to private assets, then if CBDC is designed as a perfect substitute for private bank instruments, its introduction is irrelevant, except that CBDC uses government debt more intensively than do private sector banks, so government debt can limit the extent of CBDC issue.

So, this regime makes clear how the incentive problems in private banking matter for the effects of the introduction of CBDC in the model. In this regime, CBDC does not have any advantages over private sector bank deposits subject to withdrawal in currency, in terms of its properties as a means of payment. But, if private banks need to be disciplined to prevent them from absconding with government debt, monetary policy that encourages buyers to substitute from private banking to CBDC can improve welfare. The potential cost of more CBDC issue is that it could result in a decrease in central bank profits. As well, CBDC issue could be constrained by the available supply of government debt. But, at the margin, these concerns are not an issue. However, if the incentive problems in private banking are restricted to private capital, then the introduction of CBDC is irrelevant for economic activity and welfare, and serves only to encumber a larger quantity of government debt.

If a key benefit from CBDC issue is the mitigation of safe asset scarcity, would it not be more straightforward for the fiscal authority to solve the scarcity problem by issuing more government debt? First, a fiscal solution would of course require that the fiscal authority recognize the problem and take the necessary steps. There is no guarantee of either in practice. Second, in practice there is a limit on the quantity of the outstanding government debt, determined by the ability of the government, under alternative scenarios, to service the government debt. There may be circumstances under which relieving a safe asset scarcity is not feasible for the government, as this would increase the probability
of sovereign default beyond the acceptable threshold. In such circumstances, increasing efficiency in the usage of safe assets through CBDC issue could prove very useful.

Why should CBDC be offered directly by the central bank? It has been suggested (e.g. Bordo and Levin 2019, Adrian 2019), that it may be preferable to have “synthetic CBDC,” i.e. accounts offered by private banks, but backed by central bank liabilities – conventional narrow banks, essentially. The arrangement considered in Section 6 is one such setup, and in our model that framework is not without incentive problems. As a result, it does no better than the Section 5 monetary institution with physical currency. Narrow private banking in the environment of this section could consist of private banks that hold only government debt on their balance sheets, but that would be irrelevant, as private banks would just allocate assets in such a way that the narrow banks hold only government debt and other banks hold private capital as well, with no implications for the equilibrium allocation. 9

8 Relationship to Some Other Research on CBDC

Work by Keister and Sanches (2021) (KS) uses a related model to address the efficacy of CBDC issue. The approach in KS is to consider alternative properties for CBDC, and then evaluate the welfare consequences, in a framework where CBDC potentially competes with private banking. KS emphasizes a potential tradeoff between CBDC and private banking arrangements, in that the crowding out of private bank deposits by CBDC could have negative consequences. The approach in our paper is different, in assuming particular properties for CBDC, and then studying how monetary policy and CBDC implementation affect CBDC adoption and welfare. Our approach focuses on how assets are intermediated, the central bank’s asset portfolio, and the use of collateral in backing CBDC and private bank liabilities.

Brunnermeier and Niepelt (2019) establish conditions under which private and public money are equivalent, in the sense that swaps of public for private money are irrelevant. How do their results relate to what is going on in our model? The introduction of CBDC matters here because there are different costs associated with public and private means of payment, and because these means of payment potentially play different roles. Section 8 shows an example in which private banks cannot abscond with government debt or physical currency, which implies that the introduction of CBDC is irrelevant for equilibrium quantities and prices, simply displacing private bank deposits one-for-one. This has the flavor of Brunnermeier and Niepelt’s (2019) result, as a special case.

Other related work has been done by Fernandez-Villaverde et al. (2020a), which is an extension of the Diamond-Dybvig (1983) model. Their paper shows

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9 One could make a case that it would be inefficient for central banks to set up the payments infrastructure needed to support the clearing and settlement of a very large number of CBDC transactions. But that would take us beyond the scope of this paper.
that, under specific conditions, CBDC issue by the central bank can substitute perfectly for private bank deposits, but only if the central bank engages in maturity transformation. This contrasts with our results, where CBDC issue can matter, and improve welfare, but with CBDC intermediation activity restricted to narrow banking. As well, Fernandez-Villaverde et al. (2020a) shows how CBDC issue can promote financial stability. This is different from Williamson (2021), for example, which shows how CBDC can promote instability.

9 Conclusion

There are many issues left to explore regarding potential CBDC issue. For example, direct CBDC issue by the central bank would require a different platform for clearing and settlement from what exists for interbank payments – the Fedwire system in the United States for example. Clearing and settlement of CBDC payments would involve clearing and settling a very high volume of small payments daily, as opposed to the relatively low volume of large-value transactions that occurs currently on the central bank’s books. Presumably CBDC payments will require fast clearing and settlement, on a 24/7 basis. As well, in this paper we have not addressed issues of financial instability and flight to safety associated with the introduction of CBDC. Those issues are covered in Williamson (2021).

Since the provision of privacy arises as such an important issue for CBDC issue, it would be useful to explore in future research how central banks may, or may not, have an advantage in supplying privacy in transactions. Perhaps the ideas in Garratt and Van Oordt (2020) will be useful in such research.

10 References


11 Online Appendix

**Proof of Proposition 1:** A social planner free to choose $k$, $x^c$, and $x^d$ to maximize the right-hand side of (38) will choose $k = k^*$, where $k^*$ satisfies (40), and $x^c = x^d = x^*$. To support this as an equilibrium allocation requires, first, that $R^b$ satisfy (35) given the equilibrium allocation, which implies that $R^b = 1$. From (20), banks’ collateral constraints do not bind in equilibrium, but we need to check that the demand for collateral does not exceed the supply in equilibrium. In equation (33), which holds when the collateral constraint binds, the left-hand and right-hand sides are, respectively the supply of collateral and the demand, even if collateral constraints do not bind. That the left-hand side exceeds the right-hand side in equilibrium gives

$$v \geq \frac{x^*}{1-\gamma} - k^*. \quad (64)$$

As well, (36) must hold in equilibrium, so

$$v \geq \rho x^*. \quad (65)$$

This gives us (39). But note that (35) does not hold in this efficient equilibrium, since a zero nominal interest rate on government debt implies that the central bank collects no revenue. This is why we neglect (35) in the Proposition.

**Proof of Proposition 2:** First, from (55), in equilibrium $k$ is a decreasing function of $x^d$, so using (55) and (33) we can write

$$v = \psi(x^c, x^d), \quad (66)$$

where $\psi(\cdot, \cdot)$ is strictly increasing in both arguments for $0 \leq x^c < x^*$ and $0 \leq x^d < x^*$. So, equation (66) defines a downward-sloping locus in $(x^c, x^d)$ space, which is continuous. Then, letting $R^b = 1$, equation (34) becomes

$$x^c = x^d, \quad (67)$$

and equation (67) defines an upward sloping locus in $(x^c, x^d)$ space which is continuous. If (66) and (67) solve for $(x^c, x^d)$, then that is an equilibrium solution. If $\frac{(1-\rho)x^*}{1-\gamma} - k^* \leq v$, then given (41) there exists some $x^{**}$ with $0 \leq x^{**} < x^*$ such that (66) is satisfied with $x^c = x^{**}$ and $x^d = x^*$. Alternatively, if $\frac{(1-\rho)x^*}{1-\gamma} - k^* > v$, then (66) is satisfied for $x^c = 0$ and $x^d = x^{***}$, with $0 < x^{***} < x^*$. Then, by virtue of (41) and (42), there exists a solution to (66) with $x^c = x^*$ and $x^d = \hat{x}$, with $0 < \hat{x} < x^*$. So, by continuity and monotonicity, there exists a unique solution to (66) and (67) with $0 < x^c = x^d < x^*$. Then, by continuity, there exists a unique solution to (66) and (34) for $R^b > 1$, but sufficiently close to 1. For the comparative statics, since (66)
defines a downward-sloping locus, and (34) an upward-sloping locus, it is immediate that an increase in \( R_b \) increases \( x^d \), and reduces \( x^c \). Therefore, from (55) \( k \) must fall, and the real rate of interest on government debt rises. Further, from (37), \( \pi \) rises.

**Proof of Proposition 3:** In equilibrium, equations (55), (33), and (34) solve for \( k, x^c, \) and \( x^d \). Then, totally differentiate (55), (33), and (34) and evaluate the derivatives for \( R_b = 1 \), which implies, from (34), that \( x^c = x^d = x \). Then, evaluate the derivative of total welfare from (38), obtaining

\[
dW \over dR_b = \left[ f'(k)(1 - \gamma)^2 x u''(x) \left( \Gamma f''(k) \frac{\gamma + (1 - \gamma)u'(x)}{\gamma + (1 - \gamma)u'(x)} \right)^2 \right] < 0,
\]

where

\[
\Gamma = -u''(x) \frac{\gamma + u'(x) + xu''(x)}{\gamma + (1 - \gamma)u'(x)} - \frac{f'(k)(1 - \gamma)^2 [u''(x)]^2}{f''(k) [\gamma + (1 - \gamma)u'(x)]^2} > 0
\]

**Proof of Proposition 4:** Suppose that \( F = 0 \), and consider a policy \( R = 1 \). Then, from (45) \( x^m = x^d = x \), where from (44) and (55), \( x \) and \( k \) satisfy

\[
v + k = x \left[ \frac{\gamma}{1 - \gamma} + u'(x) \right]
\]

and

\[
f'(k) = \frac{1}{\beta [\gamma + (1 - \gamma)u'(x)]}.
\]

Assuming (41) and (42), \( 0 < x < x^* \), and (46) and (47) are satisfied. But from (30), \( x^m > x \), so (49) does not hold. Therefore, since the policy \( R = 1 \) is optimal ignoring constraint (49), but (49) does not hold for \( R = 1 \), and all other constraints are satisfied, therefore (49) must bind if policy is conducted optimally. Then, if we increase \( F \) by a small amount, by continuity (49) must bind at the optimum in that case as well.

**Proof of Proposition 5:** Suppose that the bank’s collateral constraint (7) does not bind in the narrow banking regime. Then, from the bank’s problem, (5) subject to (6), we get \( x^d = x^* \), \( u'(x^c) = \frac{x^*}{\beta} \), and \( R^b = \frac{\gamma}{\pi} \). In the narrow banking regime, \( R^b = 1 \), so \( \pi = \beta \) and \( x^c = x^* \). Then, since buyers are indifferent between holding CBDC and private bank deposits in equilibrium in the narrow banking regime, we have \( x = x^m > x^d = x^c = x^* \). Since \( u'(x^*) = 1 \), therefore the left-hand side of (53) equals zero, which implies that, given \( F > 0 \), (53) does not hold. So, if the bank’s collateral constraint does not bind, then the equilibrium allocation is not feasible.
Proof of Proposition 6: In equation (51), the right-hand side is strictly decreasing in \( x \), for \( 0 < x < x^* \), so an increase in \( R \) will increase \( x \), in equilibrium. Then, from (55), \( k \) decreases. So, from (50), since the left-hand side is strictly increasing in \( k \), and the right-hand side is strictly increasing in \( x \) and in \( \alpha \), therefore, since \( x \) rises and \( k \) falls, \( \alpha \) must decrease in equilibrium. As well since \( f'(k) \) rises, the real interest rate on government debt rises. From (52), since \( x \) increases, \( \pi \) decreases. Then, since \( k > k^* \) and \( x^c = x^d = x < x^* \), therefore an increase in \( x \) must increase welfare. The demand for government-supplied collateral is the left-hand side of (54), which is strictly increasing in \( x \) and strictly decreasing in \( \alpha \). So, if \( x \) increases and \( \alpha \) decreases, therefore the demand for government-supplied collateral increases.

Proof of Proposition 7: Let \( \phi(x, \alpha) \) denote that left-hand side of (53), so we can re-state (53) as
\[
\phi(x, \alpha) \geq F. \tag{72}
\]
Then, from (53),
\[
\frac{\partial \phi(x, 1)}{\partial R} = \frac{-x \gamma [u'(x) - 1]}{\beta [\gamma + (1 - \gamma)u'(x)]} \frac{\partial \alpha}{\partial R} > 0. \tag{73}
\]
Proof of Proposition 8: Let \( \bar{x} \) denote the value of DM consumption in the narrow banking equilibrium when \( \alpha = 1 \) in equilibrium. Then, from (55) and (50), \((\bar{x}, \bar{k})\) is the unique solution to
\[
f'(\bar{k}) = \frac{1}{\beta [\gamma + (1 - \gamma)u'(\bar{x})]}, \tag{74}
\]
and
\[
v + \bar{k} = \bar{x} \left[ u'(\bar{x}) + \frac{\gamma}{1 - \gamma} \right]. \tag{75}
\]
Then, the critical gross nominal interest rate on reserves, that supports this equilibrium is, from (51),
\[
\bar{R}^m = \frac{\gamma}{u'(\bar{x})} + 1 - \gamma. \tag{76}
\]
Then \( x^c = x^d = \bar{x} \) and \( k = \bar{k} \) satisfies all the conditions for an equilibrium allocation under the status quo if \( R^b = 1 \), except for (35), if we assume (42). From Proposition 3, we know that, under the status quo, and if \( F \) is sufficiently small, satisfying (35) requires \( R^b > 1 \) and lower welfare than with the \((\bar{x}, \bar{k})\) allocation. But, in the narrow banking equilibrium, to satisfy (53) requires \( R^m > \bar{R}^m \), as this will reduce \( \alpha \), increase \( x \), and increase the left-hand side of (53) above zero. So, as long as \( F \) is sufficiently small, there is an equilibrium in the narrow banking regime that satisfies the constraints (53) and (54), delivers higher welfare than the \((\bar{x}, \bar{k})\) allocation, and therefore higher welfare than under the status quo.
Proof of Proposition 9: Suppose that $\alpha = 0$ in narrow banking equilibrium. Then, from (50),
\[ v + k = xu'(x), \]  
(77)
and from (54),
\[ xu'(x) \leq v. \]  
(78)
But, since $k > 0$ in equilibrium, (77) and (78) cannot both hold. That is, an equilibrium with $\alpha = 0$ is not feasible.

Proof of Proposition 10: From (59) and (61), we obtain
\[ \frac{R^b}{R^n} = \frac{u'(x^d)}{u'(x^d)(1 - \gamma^b) + \gamma^b}, \]  
(79)
which implies that, with $R^b$ fixed, an increase in $R^n$ implies an increase in $x^d$, since the right-hand side of (79) is strictly decreasing in $x^d$ for $0 < x^d < x^\ast$. Therefore, from (59), with $R^b$ fixed, $x^c$ must increase as well. Since $x^d$ increases, equation (58) implies that $k$ decreases. In equation (57), the left-hand side is increasing in $k$, and if $\gamma^k > \gamma^b$, the left-hand side is strictly decreasing in $x^d$. As well, the right-hand side of (57) is strictly increasing in $x^c$ and $x^d$, and strictly increasing in $\alpha$. So, since $k$ falls, $x^c$ increases, and $x^d$ increases, equation (57) implies that $\alpha$ falls. In this equilibrium, welfare is given by (38) and, since $x^c$ and $x^d$ increase and $k$ decreases, and since $x^c < x^\ast$, $x^d < x^\ast$, and $k > k^\ast$, therefore welfare increases.

Proof of Proposition 11: Construct an equilibrium with $\alpha = 1$, such that buyers are just indifferent between the deposit contract offered by private banks, and the contract offered by the central bank’s narrow banking facility. That is, $\alpha = 1, k, x^c, x^d,$ and $R^p$ solve (57)-(61) and satisfy (62) and (63). Then, choose $R^b$ and $R^n$ so as to maximize welfare, given by (38), in this equilibrium, and let $(\bar{R}^b, \bar{R}^n)$ denote the optimal policy. Given (41), (42), and (1), constraint (62) is slack at the optimum. So, if we hold $R^b$ constant at $\bar{R}^b$ and increase $R^n$ from $\bar{R}^n$, then Proposition 10 states that, at the margin, welfare increases, $x^d$ increases, $x^c$ increases, and $\alpha$ falls. Since with the previous optimal policy (62) was slack, therefore (62) holds for a marginal change in the policy, as specified. Further, if we let $\phi$ denote the left-hand side of (63), differentiate, and evaluate at the previous optimal policy, we get
\[ \frac{\partial \phi}{\partial R^n} = \rho \left( R^b - 1 \right) \frac{\partial x^c}{\partial R^n} - [\rho x^c \left( \frac{R^b}{R^p} - 1 \right) + (1 - \rho) x^d \left( \frac{R^b}{R^n} - 1 \right)] \frac{\partial \alpha}{\partial R^n} > 0, \]  
(80)
so this marginal change in policy relaxes constraint (63), so the change in policy is feasible. And since the change in policy increases welfare, some CBDC issue is optimal.