COMMUNICATION IN COURNOT OLIGOPOLY

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Abstract. We study communication in a static Cournot duopoly model under the assumption that firms have unverifiable private information about their costs. We show that cheap talk between the firms cannot transmit any information. However, if the firms can communicate through a third party, communication can be informative even when it is not substantiated by any commitment or costly actions. We exhibit a simple mechanism that ensures informative communication and interim Pareto dominates the uninformative equilibrium for the firms.

1. Introduction

It is well recognized in both the theoretical literature and the antitrust law that information exchange between firms in an oligopolistic industry can have several effects (see, for example, Nalebuff and Zeckhauser (1986) and Kühn and Vives (1994)). On the one hand, more precise information about the market allows the firms to make more effective decisions. On the other hand, information exchange may facilitate collusion and increase barriers to entry, which reduce consumer surplus. Therefore, assessing the effects of communication on equilibrium prices and production is both interesting from the theoretical point of view and important for developing guidelines for competition policy. This paper contributes to the discussion by studying the possibility of informative communication in a Cournot oligopoly model where the firms have unverifiable private information about their costs.

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There is a large literature on information exchange in oligopoly with private information about costs. This literature considers the scenario where the firms participate in information exchange before playing a one-shot Cournot game. Information is assumed to be verifiable, i.e. a firm can conceal its private information but cannot misrepresent it. Examples include Fried (1984), Li (1985), Gal-Or (1986), Shapiro (1986), Okuno-Fujiwara, Postlewaite and Suzumura (1990), Raith (1996) and Amir, Jin and Troege (2010).[1] In addition, most of these papers assume that each firm decides whether to share its information or not before it observes the cost realization. (An exception is the paper by Okuno-Fujiwara, Postlewaite and Suzumura (1990), which assumes that each firm decides whether to reveal its cost realization after observing it). The conclusion from this literature is that in a Cournot oligopoly with linear demand, constant marginal cost and independently distributed cost shocks, each firm finds it profitable to commit to disclose its private information.

However, the assumption that private information is costlessly verifiable may be restrictive. Ziv (1993) notes that that information about a firm's cost function "is part of an internal accounting system that is not subject to external audit and not disclosed in the firm's financial statements," which makes it potentially costly or impossible to verify, and that even if the verification took place, punishment for misrepresenting the information is unavailable in a one-shot game, because contracts that prescribe such punishment may violate antitrust law. In some cases, external verification of information is impossible in principle, as when the communication between firms takes the form of planned production preannouncements (an empirical investigation of information exchange via production preannouncements can be found in Doyle and Snyder (1999)). Therefore, one may wish to examine whether the conclusions of the literature on information sharing in oligopoly are robust to the assumption that information is verifiable.

[1]A related strand of literature (Novshek and Sonnenschein (1982), Vives (1984), Gal-Or (1985), Kirby (1988)) studies information sharing between firms having private information about demand; Raith (1996) and Amir, Jin and Troege (2010) cover both cost uncertainty and demand uncertainty.

Ziv (1993) addresses this question in the framework of a Cournot oligopoly with linear demand and constant marginal costs. He assumes that the marginal costs are private information, and each firm can send a cheap-talk message to its competitors before choosing its output. He shows that if the information is unverifiable, the conclusion that each firm will be willing to share the information no longer holds. To understand this result, suppose that there exists an equilibrium where each firm announces its cost realization truthfully, the competitors take each announcement at face value, and the output of each type of each firm is positive. Then, regardless of the true cost realization, each firm would like to deviate and announce the lowest possible cost in order to appear more aggressive and thus make the competitors reduce their output.

Various mechanisms to make unverifiable cost announcements credible have been considered in the literature. For instance, different announcements can be accompanied by appropriate levels of 'money burning' (Ziv, 1993), the announcements can determine the amount of side payments in a collusive contract (Cramton and Palfrey, 1990), or the level of future 'market-share favors' from the competitors in repeated settings (Chakrabarti, 2010).

In this paper, we consider a Cournot duopoly model which generalizes the linear demand-constant marginal cost setting that is considered in almost all previous work. Each firm has unverifiable private information about the value of its marginal cost. We assume that the game is played only once, the firms cannot commit to information disclosure ex ante, and the communication between the firms cannot be substantiated by any costly actions.

We show that in this setting, unless some cost types are so unproductive that they prefer to shut down under all circumstances, then no information transmission is possible through one round of cheap talk (Theorem 1). This theorem generalizes the result of Ziv (1993) to a nonlinear setting where the techniques of that paper are no longer applicable. More generally, we prove that no cheap talk game that lasts for a pre-determined finite number of rounds has an informative equilibrium (Theorem 2).

However, we show that if the firms are allowed to use more complex communication protocols than one-shot cheap talk, informative communication is possible. In particular, we consider the scenario where the firms can communicate through a neutral and trustworthy third party (a mediator). The mediator can both receive costless and unverifiable reports from the firms about their cost realizations and send messages back to the firms. In this setting, we show that for a range of parameters there exist a simple communication protocol that makes information transmission possible in equilibrium and leaves every type of every firm better off than in the Bayesian-Nash equilibrium without communication (Theorem 3).[2] The reason for this is that the mediator can play the role of an information filter between the firms: a firm does not get to see the competitor's cost report directly, and the amount of information that it gets about the competitor's cost depends on its own report to the mediator.[3] Therefore, even though a higher cost report may lead to higher expected output by the competitor, it can cause the mediator to disclose more precise information about the competitor, which can make truthful reporting by the firms incentive compatible.[4]

Our paper belongs to the literature on mechanism design without enforcement, where, unlike in the standard mechanism design approach, the principal cannot commit to an outcome rule contingent on the agents' messages, but can only suggest actions

[2]Liu (1996) considers communication protocols which make use of a third party (correlated equilibria) in a Cournot oligopoly with complete information. He shows that the possibility of communication does not enlarge the set of possible outcomes: the only correlated equilibrium is the Nash equilibrium. We show that a similar result holds in our model too (Lemma 3). Therefore, for informative communication through a mediator to be possible, the mediator has to be able not only to send messages to the firms, but to receive cost reports from them as well.

[3]The idea that introducing noise into communication in sender-receiver games can improve information transmission has been introduced by Myerson (1991) and analyzed in detail by Blume, Board and Kawamura (2007).

[4]The idea that an informed party may be induced to reveal information by making the amount of information he gets about his competitor contingent on his own message appears in Baliga and Sjöström (2004), although the models and the results of that paper significantly differ from ours.

to the agents.[5] As a result, the firms are not doing as well as they could in a cartel with enforcement power.[6]

Our results have two implications for competition policy. First, they add a new aspect to the question of whether firms should be allowed to exchange disaggregated versus aggregate data. This issue is currently viewed mainly from the perspective of determining which of the regimes is more conducive to sustaining collusive equilibria when the firms interact repeatedly. From this point of view, the exchange of disaggregated data may be more harmful than the exchange of aggregate statistics, because, in case of a deviation from the collusive agreement, the former regime allows to establish the identity of the deviator (Kühn and Vives, 1994). For this reason, the competition policy views the exchange of aggregate statistics more favorably (for example, Kühn and Vives (1994) note that the European Commission "has no objection to the exchange of information on production or sales as long as the data does not go as far as to identify individual businesses."). What we show is that information aggregation can have another effect: it can relax the incentive compatibility constraint of the participants of the data exchange and thus lead to more information revelation.[7]

Second, our results contradict the notion that efficiency-enhancing exchange of unverifiable information is infeasible, and therefore the only possible purpose for the exchange of such information is to sustain a collusive agreement. For example, the 2010 OECD report on "Information Exchanges between Competitors under Competition Law" states:

[5] Myerson (1982) provides a revelation principle for mechanism design problems without enforcement. This approach has been used to study sealed-bid double auctions (Matthews and Postlewaite, 1989), battle of the sexes (Banks and Calvert, 1992), bargaining in the shadow of war (Hörner, Morelli and Squintani, 2011).

[6] See Cramton and Palfrey (1990) for the analysis of such cartels in a static setting. In the case of repeated interactions, cartel enforcement can be achieved by threats of future punishment (Chakrabarti, 2010).

[7] In their narrative analysis of the Sugar Institute, a cartel of sugar refiners that operated in the US in 1928-1936, Genesove and Mullin (1997) note that the confidentiality procedures adopted by the Institute in gathering and aggregating the data may have been adopted to insure incentive compatibility for participating firms. To our knowledge, this insight has never before been formalized within a theoretical oligopoly model.

> Empirical evidence shows that the positive effects for consumers of public announcements outweigh the possible collusive effects from the transparency they generate. Because of this, it can be very difficult in practice to distinguish whether public information exchanges have a procompetitive effect or simply facilitate collusion. One important factor that the literature points out is that communications between firms may have little value in facilitating coordination unless the information is verifiable. Information which is not verifiable can be dismissed as "cheap talk" and therefore disregarded. However, some have suggested that cheap talk can assist in a meeting of minds and allow firms to reach an understanding on acceptable collusive strategies. (p.34)

Similarly, Kühn (2001) notes that

> Since communication about future conduct is about something that is unobservable and unverifiable at the date of communication it cannot be used to transmit private information about market data, because firms would not have an incentive to reveal the truth. The problem of non-credibility arises because there is asymmetric information about the market environment. (pp. 183-184)

We show that this is not necessarily true, and that exchange of unverifiable information can be efficiency-enhancing.

The rest of the paper is organized as follows. In Section 2 we describe an example that illustrates the ideas behind our results. Section 3 contains a description of the model. In Section 4 we analyze unmediated public communication (cheap talk) and show that it cannot result in informative communication unless there exist unproductive types. In Section 5 we exhibit a simple mediated mechanism that ensures informative communication. Concluding comments are in Section 6. All proofs are relegated to the Appendix unless stated otherwise.

2. Example

Consider two symmetric firms producing a homogeneous good, the inverse demand for which is $P(Q) = 3 - Q$. Each firm has a linear cost function, the value of the marginal cost being its private information. Specifically, each firm can be either of type L, with the marginal cost of 0, or H, with the marginal cost of 2. The types are independently and identically distributed, and the probability of type L is $p \in (0,1)$. Regardless of the type realization, each firm has a capacity constraint of x units, where $x \in \left(\frac{1}{3}, 1\right)$.

Suppose that firm i's expectation of the opponent's output is Q_{-i}. Then firm i's optimal output maximizes its profit function $\pi_i(q_i, Q_{-i}, c_i) = (3 - q_i - Q_{-i} - c_i)q_i$, where c_i is the marginal cost of firm i. It is easy to check that for a firm of type L, the capacity constraint binds regardless of how much it expects the opponent to produce, and such a firm will find it optimal to produce x. On the other hand, the capacity constraint never binds for a firm of type H, and its optimal output is $q_i\left(Q_{-i}\right) = \frac{1-Q_{-i}}{2}$, which results in the profit of $\left(\frac{1-Q_{-i}}{2}\right)^2$.

To start, consider the Bayesian-Nash equilibrium of the Cournot game where the firms simultaneously choose their outputs. In this equilibrium, a firm of type L chooses x and a firm of type H chooses q_H that satisfies the equation

$$q_H = \frac{1 - (px + (1-p)q_H)}{2}$$

The solution to this equation is $q_H = \frac{1-px}{3-p}$.

Now suppose that the firms can commit to truthfully disclosing their cost realization to the competitor before making their production decisions. In this case, if the firms learn that both of them are of type H, both will produce $\frac{1}{3}$; if they learn that one of the firms is of type H and the other one of type L, the type-H firm will produce $\frac{1-x}{2}$. As before, a type-L firm will produce x regardless of what it knows about the opponent. It is straightforward to check that in this case, the ex ante expected profit of each firm is higher than in the case where the costs are private information.[8] Therefore, if the firms could participate in such an information-sharing agreement, they would have an incentive to do so.

Suppose, however, that such an information-sharing agreement is infeasible, and all a firm can do is make a public announcement about its marginal cost realization before choosing its output level. The announcements are made simultaneously, and are costless and unverifiable ("cheap talk"): a firm has no way to check whether its opponent has told the truth about its marginal cost. Let us show that in this case, the firms will not reveal their information truthfully in equilibrium.

Indeed, suppose a truthful equilibrium exists. In such an equilibrium, if a firm truthfully announces type H, it will find it optimal to produce $\frac{1}{3}$ if the opponent announces H as well, and $\frac{1-x}{2}$ if the opponent announces L. A firm of type L that truthfully discloses its type will find it optimal to produce x no matter what the opponent announces. Suppose that a type-H firm discloses its type truthfully. Then with probability p it will learn from its opponent's announcement that the opponent will produce x, and with the remaining probability it will learn that the opponent will produce $\frac{1}{3}$. But suppose that a type-H firm deviates and announces that its type is L; then with probability p it will still learn that the opponent will produce x, but with the remaining probability it will learn that the opponent will produce $\frac{1-x}{2} < \frac{1}{3}$. Because the firm prefers the opponent to produce less, this deviation is profitable, and a truthful equilibrium does not exist. Therefore, even though the firms have an ex ante incentive to share their information, sharing it truthfully through cheap-talk messages is impossible: a high-cost firm will have an incentive to pretend that its cost is low in order to scare the opponent into producing less.[9]

To counteract this incentive, let us amend the information exchange scheme as follows. Suppose that, instead of announcing their types to each other, the firms report

[8]The difference in the profits between the complete information and the incomplete information case equals $\frac{p(1-p)^2(3x-1)(81x+5p-21-21px)}{36(3-p)^2}$, which is strictly positive for any $p \in (0,1)$ and $x \in \left(\frac{1}{3},1\right)$.

[9]In principle, the cheap-talk game could have a mixed-strategy equilibrium where the messages were partially informative about the types; however, in this example such equlilibria do not exist.

them privately to a neutral trustworthy third party (mediator). We still assume that the reports are costless and unverifiable. If both firms have reported that they are of type H, the mediator makes a public announcement to that effect; otherwise the mediator remains silent. We will show that in equilibrium, both firms will have an incentive to report truthfully, and their ex ante welfare will be higher than without communication.

Indeed, if both firms have truthfully announced that they are of type H, then they learn that this is the case, and each of them chooses to produce $\frac{1}{3}$. If a firm of type H has truthfully reported its type, but the mediator remains silent, then the firm learns that the opponent is of type L, and thus best responds with $\frac{1-x}{2}$. A firm of type L always finds it optimal to produce x. Therefore, conditional on any type profile, the equilibrium outputs are the same as in the case when the firms commit to disclosing their types truthfully, and therefore the ex ante profit is also the same. Let us now check that reporting truthfully is incentive compatible. Suppose a firm of type H reports truthfully. Then, as in the case of full revelation, with probability p it will learn that the opponent will produce x (and best respond with $\frac{1-x}{2}$), and with the remaining probability it will learn that the opponent will produce $\frac{1}{3}$ (and best respond with $\frac{1}{3}$). If a type-H firm deviates and reports L, its opponent's output will be equal to x with probability p and $\frac{1-x}{2}$ with probability $1-p$, just as in case of full revelation; but unlike that case, the firm will have to choose how much to produce without the benefit of knowing how much the opponent will produce. Its best response to the lottery over the opponent's output is to produce $\frac{1}{2}(1-\big(px+(1-p)\frac{1-x}{2}\big))$. The deviation is unprofitable if

$$p\left(\frac{1-x}{2}\right)^2+(1-p)\left(\frac{1}{3}\right)^2\geq\left(\frac{1-\big(px+(1-p)\frac{1-x}{2}\big)}{2}\right)^2$$

which is true if $p\geq\frac{3x+7}{9(3x-1)}$. It is also easy to check that a type-L firm will find it profitable to report truthfully for any values of $p\in(0,1)$ and $x\in\big(\frac{1}{3},1\big)$.

The intuition for why the mechanism above is incentive compatible is that, at the reporting stage, it makes the firms face a tradeoff between inducing the opponent to produce less in expectation (by sending message L) and learning exactly how much the opponent is going to produce (by sending message H). Different types of the firm resolve this tradeoff differently. A type-H firm values information about how much the opponent will produce; in contrast, a type-L firm always finds it optimal to choose the same output level and thus faces no need to coordinate with the opponent. This makes it possible for the firms to truthfully reveal their information and improve their expected profit relative to the no-communication case.[10]

3. The model and preliminaries

We consider a model of Cournot competition between two firms, A and B, with differentiated products. The inverse demand curve for firm i's product is given by $P(q_i, q_{-i}) = \max\{\rho(q_i) - \beta q_{-i}, 0\}$, where q_i is the output of firm i. We assume that $\rho(0) > 0$ and $-\rho'(q_i) \geq \beta > 0$ for every $q_i \geq 0$. The interpretation is that the products of the two firms are perfect or imperfect substitutes, and "own effect" on demand is greater than the "cross effect".[11] Firm i's cost function is $C(q_i, c_i)$ such that $C(0, c_i) = 0$, $\frac{\partial C(q_i,c_i)}{\partial q_i} \geq 0$ with strict inequality for $q_i > 0$, and $\frac{\partial^2 C(q_i,c_i)}{\partial q_i^2} \geq 0$. A higher value of the parameter c_i is associated with higher firm i's total cost and marginal cost: $\frac{\partial C(q_i,c_i)}{\partial c_i} \geq 0$ and $\frac{\partial^2 C(q_i,c_i)}{\partial c_i \partial q_i} \geq 0$. We assume that c_i is privately observed by firm i, and that c_A and c_B are independently distributed on $C = [0, \bar{c}]$ according to a continuous distribution function F with density $f > 0$.

In Lemma 4 in the Appendix we show that rational behavior by the firms always results in strictly positive prices, and thus we can take $P(q_i, q_{-i}) = \rho(q_i) - \beta q_{-i}$ from now on. The profit of firm i of type c_i when it produces q_i and its competitor produces

[10]Furthermore, it can be shown that for a range of parameters in this example, this mechanism is ex ante optimal in the class of all incentive compatible communication mechanisms.
[11]This is a standard assumption: see for example, Gal-Or (1986).

q_{-i} is

$$\pi_i(q_i, q_{-i}, c_i) = (\rho(q_i) - \beta q_{-i})\, q_i - C(q_i, c_i) \tag{1}$$

Let $q(q_{-i}, c_i)$ be the set of best responses of firm i of type c_i to the opponent's output q_{-i}:

$$q(q_{-i}, c_i) = \arg\max_{q_i \geq 0} \pi_i(q_i, q_{-i}, c_i) \tag{2}$$

We will impose the following conditions on the best response function q:

(A1)

$$q(q_{-i}, c_i) \text{ is single-valued, continuous everywhere, } C^1 \text{ on } \{(q_{-i}, c_i) : \; q(q_{-i}, c_i) > 0\}$$

(A2)

$$\text{If } q(q_{-i}, c_i) > 0, \text{ then } \frac{\partial q(q_{-i}, c_i)}{\partial c_i} \leq 0 \text{ and } \frac{\partial q(q_{-i}, c_i)}{\partial q_{-i}} \in (-1+\delta, 0) \text{ for some } \delta > 0$$

$$q(0,0) > 0, \;\; q(q(0,0), 0) > 0 \tag{A3}$$

To guarantee A1 and A2, it is enough to assume that the components of the profit are twice continuously differentiable and that ρ is "not too convex" (see Lemma 4 in the Appendix for the precise statement). In particular, the best response is nonincreasing in c_i and q_{-i} because of $\frac{\partial^2 C(q_i, c_i)}{\partial c_i \partial q_i} \geq 0$ and $\beta > 0$. Condition A3 simply requires that the most efficient type never chooses to shut down, even if facing the most efficient opponent who chooses the monopoly output.

For some results in the next section we will require that all types always produce strictly positive output:

$$q(q_{-i}, c_i) > 0 \text{ for every } q_{-i} \in [0, q(0,0)] \text{ and every } c_i \in C \tag{A4}$$

This can be guaranteed, for example, by assuming $\frac{\partial C(0, c_i)}{\partial q_i} = 0$ for every $c_i \in C$ (see Lemma 4 in the Appendix).

Let us illustrate these conditions with an example.

Example 1. *Let* $\rho(q_i) = A - q_i$, $C(q_i, c_i) = \frac{c_i}{\gamma} q_i^{\gamma}$ *such that* $A > 0$, $\gamma \geq 1$, *and* $\beta \in (0,1)$. *If* $\gamma > 1$, *then* $q_i(q_{-i}, c_i)$ *equals* 0 *if* $A - \beta q_{-i} \leq 0$, *and solves the first-order condition*

$$A - 2q_i - \beta q_{-i} - c_i q_i^{\gamma - 1} = 0$$

otherwise. It is easy to check that A1-A4 are satisfied. If $\gamma = 1$, *then* $q_i(q_{-i}, c_i) = \max\left\{0, \frac{1}{2}\left(A - \beta q_{-i} - c_i\right)\right\}$. *It is easy to check that A1-A3 are satisfied, while A4 is satisfied if* $\overline{c} < \frac{A}{2}$.

Substituting $q(q_{-i}, c_i)$ into the expression for the profit (1) we obtain the indirect profit function of firm i:

$$\Pi_i(q_{-i}, c_i) = \max_{q_i \geq 0} \; \pi_i(q_i, q_{-i}, c_i) = \pi_i(q(q_{-i}, c_i), q_{-i}, c_i) \tag{3}$$

4. Unmediated communication

In this section, we allow the firms to communicate directly with each other using costless and unverifiable messages before choosing their output levels. First, to provide a benchmark, we describe what happens in the game with no communication. After that we investigate the consequences of allowing one round of cheap talk communication. Finally, we look at games with any pre-determined finite number of rounds of cheap talk communication.

It is well-known that in the complete-information Cournot game with two firms, the unique intersection of the firms' best responses determines not only the unique Nash equilibrium strategy profile, but also the unique outcome of the iterated elimination of strictly dominated strategies.[12] In our setting, we have an analogous result for the game with no communication.[13]

[12] See for example Chapter 2 in Fudenberg and Tirole (1991).

[13] The proof of this Lemma follows from a more general result (Lemma 2).

Lemma 1. *Suppose that conditions A1-A3 hold. Then in the game with no communication the profile of strategies where each firm plays according to*

$$q^{NC}(c_i) = q(Q^{NC}, c_i) \text{ for every } c_i, \text{ where } Q^{NC} = \int q^{NC}(c_i)dF(c_i) \tag{4}$$

is the unique Bayesian-Nash equilibrium and the unique profile of strategies that survives iterated elimination of interim strictly dominated strategies.

Note that in the games with multiple equilibria, one possible role for preplay communication is to allow the players to coordinate among equilibria. Given Lemma 1, preplay communication in our setting cannot be used purely for coordination, but has to involve some information revelation.

We consider the following game where the firms can engage in cheap talk communication before making their output choices. Let M_A and M_B be the sets of possible messages for firms A and B. Each firm i sends a costless message $m_i \in M_i$, and the messages are publicly observed. Firm i's pure strategy is thus a pair of functions $(m_i(c_i), q_i(m_i, m_{-i}, c_i))$, where $m_i : C \to M_i$ is a message strategy and $q_i : M_i \times M_{-i} \times C \to \mathbb{R}_+$ is the output strategy in the continuation game following a pair of messages (m_i, m_{-i}) being observed.

Let us first consider the continuation game after a pair of messages (m_i, m_{-i}) is observed. Let $F_i(\cdot|m_i)$ be the c.d.f. of firm $-i$'s equilibrium beliefs about c_i after it has observed firm i's message m_i.[14] Similarly to Lemma 1, we can characterize what happens in such a continuation game.

Lemma 2. *Suppose that conditions A1-A3 hold. Then, in the game with one round of cheap-talk communication after a pair of messages (m_i, m_{-i}) is observed, the profile of*

[14] F_i does not depend on c_{-i}, because the types are independently distributed.

strategies given by

$$q_i(m_i, m_{-i}, c_i) = q\left(Q_{-i}(m_i, m_{-i}), c_i\right) \text{ for every } c_i,$$
$$\text{where } Q_i(m_i, m_{-i}) = \int q_i(Q_{-i}(m_i, m_{-i}), c_i) dF_i(c_i|m_i),\ i \in \{A, B\}$$

is the unique Bayesian-Nash equilibrium and the unique profile of strategies that survives iterated elimination of interim strictly dominated strategies.

Next we show that if some of the firms' cost types are so unproductive that they prefer to shut down under all circumstances, then the cheap-talk game can have informative equilibria.

Example 1 (continued) *Let* $C(q_i, c_i) = c_i q_i$ *(i.e.* $\gamma = 1$*), and thus* $q_i(q_{-i}, c_i) = \max\left\{0, \frac{1}{2}\left(A - \beta q_{-i} - c_i\right)\right\}$*, and let* $\bar{c} > A$*. Note that if* $c_i \geq A$*, then type* c_i *is so unproductive that it produces zero even if it is a monopolist:* $q_i(q_{-i}, c_i) = 0$ *for every* $q_{-i} \geq 0$*. There exists the following equilibrium with informative cheap talk: firm* A *sends one message when it is "productive"* $(c_A < A)$ *and another message otherwise; firm* B *always sends the same message regardless of its costs. To see that this is an equilibrium, first note that the "unproductive" types of firm* A *are indifferent between sending both messages, because their profit is always zero. The "productive" types prefer to tell the truth, because firm* B *behaves as a monopolist if it believes that firm* A *is "unproductive", and produces less if it believes that firm* A *is "productive".*[15]

The literature on oligopoly communication typically makes assumptions that rule out the possibility of such unproductive cost types. So for the rest of this section we investigate the possibility of informative cheap talk communication under the assumption that all types always choose positive outputs (Condition A4).

The question whether informative cheap talk between oligopolists is possible has been considered by Ziv (1993) in the context of a model with undifferentiated products, linear demand and constant marginal cost (which corresponds to Example 1 with $\beta = \gamma = 1$).

[15]Note that this equilibrium is not equivalent to the outcome under no communication. For example, the "productive" types of firm A can credibly reveal their productivity, and thus enjoy lower expected output of firm B than in the case of no communication.

Ziv's Proposition 3 shows that if the parameters are such that all cost types always find it optimal to produce, no informative equilibrium exists.[16] The logic behind this result is simple. First, every cost type of, say, firm A is strictly better off if firm B produces less. Second, firm B's equilibrium output choice depends on its expectation of firm A's cost: the higher this expectation, the more firm B will choose to produce, regardless of its cost type. Finally, if an informative cheap-talk equilibrium was possible, different messages by firm A would induce firm B to have a different expectations of firm A's cost. But then all types of firm A would have an incentive to deviate to the message that minimizes firm B's expectation of firm A's cost.

We find that this intuitive argument is not applicable to the case where the demand or the cost functions are nonlinear. In particular, the second step of the argument breaks down: it could be the case that one message corresponds to a higher expected level of the cost parameter than another, yet some types of the competitor choose to produce more after hearing the second message than the first one. This point is illustrated by the following numerical example.

Example 1 (continued) *Let* $\beta = 1$, $\gamma = \frac{3}{2}$ *and* $A = 10$. *To simplify the calculations, we will assume that the distribution of* c_i *is discrete: namely,* $c_i \in \{c_L, c_M, c_H\}$, *where* $c_L = 1, c_M = 2, c_H = 3$, *and* $P(c_i = c_L) = P(c_i = c_M) = 0.33$, $P(c_i = c_H) = 0.34$.

Suppose that each firm sends message m' *if its type is* c_M *and message* m *otherwise. Then, upon hearing the pair of messages* (m', m'), *it becomes common knowledge that each firm's type is* c_M. *A straightforward calculation establishes that each firm* i*'s optimal output is then* $q_i(m', m', c_M) \approx 2.318$. *Similarly, if firm* i *has sent message* m' *and firm* j *message* m, *firm* j *is sure that its opponent is of type* c_M, *and firm* i*'s posterior distribution over the opponent's type places probability* $\frac{0.34}{0.33+0.34} \approx 0.507$ *on* c_H, *and the complementary probability on* c_L. *The the optimal outputs*

[16]Formally, Proposition 3 states that a fully revealing equilibrium does not exist; however, what is in fact proved is that no information transmission is possible through cheap talk.

are $q_i(m', m, c_M) \approx 2.287, q_j(m, m', c_L) \approx 2.992, q_j(m, m', c_H) \approx 1.828$. *Therefore,* $q_i(m', m, c_M) < q_i(m', m', c_M)$*, despite the fact that* $E[c_j|m'] < E[c_j|m]$.

This example shows that in a nonlinear setting, the optimal output following a message profile depends not only on the expected value of the firm's posterior distribution over the opponent's type, but on the other characteristics of this distribution as well. However, using a different technique, we are still able to show that there are no informative equilibria in the game with one round of cheap talk.

Theorem 1. *Suppose that conditions A1, A2 and A4 hold. Then the game with one round of cheap talk communication has no informative equilibrium. That is, following any equilibrium message profile* (m_i, m_{-i}) *the expected output of each firm* i *satisfies* $Q_i(m_i, m_{-i}) = Q^{NC}$*, and firm* i *plays the same strategy as in the game without communication:* $q(Q_{-i}(m_i, m_{-i}), c_i) = q^{NC}(c_i)$*, for every* c_i*,* $i = A, B$*.*

The result of Theorem 1 extends to the setting where the firms can engage in finitely many rounds of cheap talk.[17] Specifically, suppose there are $T > 1$ possible communication stages, at each stage $t = 1, ..., T$ each firm simultaneously chooses a message, and their choices become commonly known at the end of the stage. After that, the firms choose outputs. We show that informative cheap talk is impossible in such a game with a pre-determined finite number of rounds.[18]

Theorem 2. *Suppose that conditions A1, A2 and A4 hold. Then the game with finitely many rounds of cheap talk communication has no informative equilibrium.*

The impossibility of informative cheap-talk communication in our model stands in contrast with a number of results on two-sided cheap talk with two-sided incomplete information. For example, informative cheap-talk equilibria have been shown to exist in the double auction game (Farrell and Gibbons (1989), Matthews and Postlewaite

[17]Games with multi-stage cheap talk have been studied both in the context of one-sided incomplete information (Aumann and Hart (2003), Krishna and Morgan (2001)), and two-sided incomplete information (Amitai, 1996).

[18]It remains an interesting open question whether cheap talk can be informative when there is no pre-determined bound on communication length.

(1989)), in the arms-race game (Baliga and Sjöström (2004)), and in the peace negotiations game (Hörner, Morelli and Squintani (2011)). However, in all these papers the underlying games have multiple equilibria, and the ability to have different continuation equilibria following different message profiles seems important for sustaining informative communication. In our setting, there is a unique continuation equilibrium for every posterior belief (Lemma 2), which makes it harder to sustain informative communication.

5. Mediated Communication

In this section, we assume that, before choosing how much to produce, the firms can communicate with a neutral and trustworthy third party (a mediator), which is initially ignorant of the firm's private information. Both firms, as well as the mediator, can send private or public messages according to a mediation rule, or mechanism, which specifies what messages the parties can send, in what sequence, and whether the messages are public or private. After the communication has ended, the firms simultaneously choose their outputs.

We assume that the mediator's role is limited to participating in communication between the firms and that it has no enforcement power over the firms' output choices. This distinguishes our setting from a standard mechanism design problem, where the mechanism designer can enforce the mechanism outcome, and makes it part of the literature on mechanism design without enforcement, which dates back to Myerson (1982). This literature suggests that in certain settings, mediated communication allows the players to strictly improve upon cheap talk.[19]

This is what we find in our model as well. Before exhibiting an informative mechanism, however, let us note that if the mediator is able only to send, but not to receive, messages from the firms, improving upon the uninformative Bayesian-Nash equilibrium

[19]See, for example, Banks and Calvert (1992), Goltsman, Hörner, Pavlov and Squintani (2009) and Hörner, Morelli and Squintani (2011). However, in finite games with a sufficiently large number of players, cheap talk can be as effective as mediated communication (see e.g. Forges (1990) and Ben-Porath (2003)).

outcome is impossible. More formally, suppose all the mediator can do is send the firms private messages m_A and m_B from some message sets M_A and M_B, generated according to a commonly known probability distribution $p \in \Delta(M_A \times M_B)$. (The Bayesian-Nash equilibria of communication games of this form are called the strategic form correlated equilibria of the game with no communication (Forges, 1993).) The following lemma is an immediate consequence of Lemma 1.

Lemma 3. *Under conditions A1-A3, all strategic form correlated equilibria are outcome equivalent to the Bayesian-Nash equilibrium of the game without communication.*

If the mediator can also receive messages from the firms, this result is no longer valid, as the example in Section 2 suggests. What we will do next is generalize the mechanism described in the example, and provide sufficient conditions for it to result in informative communication in our model.

Specifically, let $c^* \in (0, \overline{c})$, and consider the mechanism which works as follows. Each firm i sends a private message $\hat{c}_i \in [0, \overline{c}]$, which is interpreted as the firm's report about its cost, to the mediator. The mediator then publicly announces one message, m^0, if $\min\{\hat{c}_A, \hat{c}_B\} \leq c^*$ and another message, m^1, otherwise. After that, the firms choose their outputs. Let us call such a mechanism the **"min" mechanism with threshold** c^*.[20]

This mechanism induces a game between the firms, where a pure strategy for firm $i \in \{A, B\}$ consists of a reporting strategy $\hat{c}_i(c_i)$ and an output strategy $q_i(c_i, \hat{c}_i, m)$, where $m \in \{m^0, m^1\}$. We will say that the mechanism is **incentive compatible** if it has an equilibrium where the firms report their types truthfully: $\hat{c}_i(c_i) = c_i$, $\forall c_i \in [0, \overline{c}], i \in \{A, B\}$.

As in Section 2, the idea behind this mechanism is to give each firm a choice between having the competitor produce less in expectation and getting more information about how much the competitor will produce. Specifically, suppose that firm i reports $\hat{c}_i \leq c^*$.

[20] This mechanism is similar to the AND mechanism analyzed by Lehrer (1991), Gossner and Vieille (2001) and Vida (2005). Hugh-Jones and Reinstein (2011) suggest that a similar mechanism may improve welfare in a matching problem where players suffer disutility from being rejected.

Then, if firm j has reported $\hat{c}_j > c^*$, the mediator will announce message m^0, and firm j will learn that firm i has reported its cost to be low. This will make firm j produce less in expectation, which is favorable to firm i. However, firm i reporting $\hat{c}_i \leq c^*$ also deprives it of an opportunity to learn anything about firm j's report, because the mediator will announce m^0 regardless of firm j's report. Conversely, reporting $\hat{c}_i > c^*$ will result in firm j producing more in expectation, but will enable firm i to learn whether $\hat{c}_j$ is above or below c^*. The mechanism will be incentive compatible if different types of the firm resolve this tradeoff differently: types above c^* value additional information about the opponent more than the reduction in the opponent's expected output, while types below c^* exhibit the reverse preference.[21]

To guarantee the incentive compatibility of our mechanism, we will impose the following additional condition on the best response functions:

$$\text{(A5)} \qquad q(q_{-i}, c_i) \text{ is } C^2 \text{, and } \frac{\partial^2 \ln\left(q\left(q_{-i}, c_i\right)\right)}{\partial c_i \partial q_{-i}} < 0 \text{ on } \{(q_{-i}, c_i):\ q(q_{-i}, c_i) > 0\}$$

To interpret this condition, note that

$$\frac{\partial^2 \ln q(q_{-i}, c_i)}{\partial c_i \partial q_{-i}} = \frac{\partial}{\partial c_i}\left(\frac{\frac{\partial q(q_{-i}, c_i)}{\partial q_{-i}}}{q_i(q_{-i}, c_i)}\right) = -\frac{\partial}{\partial c_i}\left(\left|\frac{\frac{\partial^2 \Pi_i}{\partial q_{-i}^2}}{\frac{\partial \Pi_i}{\partial q_{-i}}}\right|\right)$$

The denominator of the latter expression measures how much the indirect profit of firm i changes with the expected output of the opponent, so it shows how much firm i values a reduction in the opponent's output. The numerator measures how convex the indirect profit function is, and thus how much the firm values information about the opponent's output. Condition A5 is a "single-crossing condition" on firms' preferences: it says that

[21] Similar logic lies behind the results of Seidmann (1990) and Watson (1996), who show that in a sender-receiver game with two-sided private information, an informative equilibrium can exist even if all the sender's types have the same preference ordering over the receiver's actions. This is because different types of the receiver respond differently to the sender's messages, and thus, from the sender's viewpoint, each message corresponds to a lottery over the receiver's actions. Informative communication is possible if different sender types have a different preference ranking over these lotteries. This effect has also been emphasized by Baliga and Sjöström (2004) in the context of an arms-race game. Unlike our model, however, these settings admit informative cheap talk.

the higher the firm's cost, the more it values information about the opponent relative to reduction in opponent's expected output.

In addition, we will impose a condition that guarantees that each firm's output sufficiently varies with respect to its type:

$$\text{(A6)} \qquad \lim_{c_i \to \infty} q(q_{-i}, c_i) = 0 \qquad \text{for every } q_{-i} \geq 0$$

Example 1 (continued) *In this example,* $\frac{\partial^2 \ln q(q_{-i}, c_i)}{\partial c_i \partial q_{-i}} = \frac{2\beta(2-\gamma) q_i^{\gamma-1}}{\left(-2q_i - c_i(\gamma-1) q_i^{\gamma-1}\right)^3}$. *Therefore,* A5 *holds if* $\gamma < 2$, *and* A6 *is always satisfied.*

Condition A5 implies that to ensure that the "min" mechanism is incentive compatible, it is enough to choose threshold c^* to be the type of the firm that is indifferent between reporting $\hat{c} \leq c^*$ and $\hat{c} > c^*$: if type c^* is indifferent, then any type above c^* will strictly prefer reporting $\hat{c} > c^*$, and any type below c^* will strictly prefer reporting $\hat{c} \leq c^*$. The following theorem shows that when the support of the cost distribution is large enough, such c^* can be found.

Theorem 3. *Suppose that conditions A1-A3, A5 and A6 hold, and that* $\overline{c}$ *is large enough. Then there exists* $c^* \in (0, \overline{c})$ *such that the "min" mechanism with threshold* c^* *is incentive compatible.*

Next, we show that whenever a "min" mechanism is incentive compatible, it interim Pareto dominates the Bayesian-Nash equilibrium without communication for the firms.

Theorem 4. *If an incentive compatible "min" mechanism exists, then every type of every firm is better off under this mechanism than in the Bayesian-Nash equilibrium without communication. If, in addition, condition A4 holds, then every type of every firm is strictly better off.*

The intuition behind this theorem is that, when a "min" mechanism is in place, reporting $\hat{c} \leq c^*$ results in higher expected profit for every type than the Bayesian-Nash equilibrium without communication. This is because in both cases, the firm gets no information, but reporting $\hat{c} \leq c^*$ results in lower expected output by the opponent than the uninformative equilibrium. Since reporting $\hat{c} \leq c^*$ is possible for every type and the mechanism is incentive compatible, in equilibrium every type's expected profit must be at least as high as the one guaranteed by this action.

6. Discussion

Our model can be extended to accommodate the case of more than two firms. Specifically, suppose that the inverse demand for firm i's product is $\max\{\rho(q_i) - \beta q_{-i}, 0\}$, where $q_{-i} = \sum_{j \neq i} q_j$ is the aggregate output of all firms other than i, and, as before, let $q(q_{-i}, c_i)$ be the best response function of each firm. The proofs of Lemma 1, Lemma 2 and Theorem 1 go through once we replace the second part of Condition A2 by a stronger assumption $\frac{\partial q(q_{-i}, c_i)}{\partial q_{-i}} \in (-\frac{1-\delta}{n-1}, 0)$.[22]

The proof of Theorem 2 also extends to the case of more than two firms, if Condition A2 is modified as above. However, this theorem covers only the case where all the communication between the firms is public. With two firms, this is clearly without loss of generality, but with three or more firms, one can also consider communication protocols whereby each firm can also send private messages to a subset of other firms. There are reasons to expect that the result of Theorem 2 will no longer hold once private communication is allowed: indeed, Ben-Porath (2003) proves that in a finite game, any communication equilibrium that assigns only rational probabilities to outcomes can be replicated by a sequential equilibrium of some unmediated communication protocol, if the number of players is at least three. Despite the fact that Ben-Porath's result is not directly applicable in this case because of the finiteness assumption, it might be possible to extend it to cover at least some simple communication equilibria (such as the "min"

[22] To see how the proof of Theorem 1 should be modified, fix any firm i, and let (m_i, m_{-i}) be a message profile. Let $BR_{-i}(q_i|m_{-i}) = \sum_{j \neq i} q_j$, where $(q_j)_{j \neq i}$ are a solution to the system of equations $q_j = BR_j(q_{-j}|m_j)$, $j \in \{1, \ldots, n\} \setminus \{i\}$ (this solution, and therefore the function BR_{-i}, depends on m_i and q_i). Then define $\left(\underline{q}_i, \overline{q}_i, \underline{q}_{-i}, \overline{q}_{-i}\right)$ analogously to $\left(\underline{q}_A, \overline{q}_A, \underline{q}_B, \overline{q}_B\right)$. As in Theorem 1, we get $(1-\delta)\left(\overline{q}_{-i} - \underline{q}_{-i}\right) \geq \overline{q}_i - \underline{q}_i$. The definition implies that $\sum_{j \neq i}\left(\overline{q}_j - \underline{q}_i\right) \geq \overline{q}_{-i} - \underline{q}_{-i}$. Combining these inequalities and summing up over i results in $(1-\delta)(n-1)\left(\sum_{i=1}^{n}\left(\overline{q}_i - \underline{q}_i\right)\right) \geq \sum_{i=1}^{n}\left(\overline{q}_i - \underline{q}_i\right)$, which is impossible unless $\overline{q}_i = \underline{q}_i$ for every i.

mechanism) in our model. Finally, if we extend the definition the “min” mechanism as the mechanism that informs the firms whether the minimum of the reported costs is above or below a certain threshold, then we expect the proof of Theorem 3 to go through.

Next, suppose that, instead of cost shocks, the firms face private demand shocks. In particular, suppose θ_i is a private (iid) demand shock that affects firm i as follows: $P(q_i, q_{-i}, \theta_i) = \max\{\rho(q_i, \theta_i) - \beta q_{-i}, 0\}$ with $\rho_\theta < 0$. Then we can define the best response function $q(q_{-i}, \theta_i)$, make the same assumptions A1-A6 with θ_i in place of c_i, and replicate all the analysis.

The question of whether any of the results would extend to the case where cost or demand shocks are correlated is more difficult. To see why, suppose that each firm receives a signal about a common cost parameter. Now each firm might prefer to be perceived as having a high cost signal rather than a low cost signal, because if the opponent believes the report about the high cost signal, then it may decide to produce less. We leave this question for future research.

Finally, one may also ask whether the results of the paper apply to a Bertrand model with differentiated products. Because prices are strategic complements, each firm will have an incentive to overstate its type, opposite to what happens in the Cournot model. Nevertheless, we believe that, when the assumptions are adjusted to reflect this change, the results of the paper will go through with the “max” mechanism (the mediator announcing whether the maximum of the cost reports exceeds a certain threshold) replacing the “min” mechanism in Theorem 3.

7. Appendix

7.1. **Proof of Lemmas 2 and 4.**

Lemma 4. (i) $\rho(q_i) - \beta q_{-i} \geq 0$ *for every pair* (q_i, q_{-i}) *that is rationalizable for some for some* (c_i, c_{-i}).

(ii) *Suppose* $C(q_i, c_i)$ *is* C^2 *in* q_i, $\frac{\partial C_i(q_i,c_i)}{\partial q_i}$ *is* C^1 *in* c_i, ρ *is* C^2, *and, for some* $\varepsilon > 0$, $\rho''(q_i)q_i + (1-\varepsilon)\rho'(q_i) < 0$ *for every* q_i. *Then* $q(q_{-i}, c_i)$ *is single-valued, continuous at every* (q_{-i}, c_i), C^1 *on* $\{(q_{-i}, c_i) : q(q_{-i}, c_i) > 0\}$. *If* $q(q_{-i}, c_i) > 0$, *then* $\frac{\partial q(q_{-i},c_i)}{\partial c_i} \leq 0$ *and* $\frac{\partial q(q_{-i},c_i)}{\partial q_{-i}} \in (-\frac{1}{1+\varepsilon}, 0)$.

(iii) *Suppose A1 and A2 hold, and* $\frac{\partial C(0,c_i)}{\partial q_i} = 0$ *for every* $c_i \in C$. *Then* $q(q_{-i}, c_i) > 0$ *for every* $q_{-i} \in [0, q(0,0)]$ *and every* $c_i \in C$.

Proof. (i) Let $\overline{q}$ be the revenue maximizing quantity when $q_{-i} = 0$, i.e. $\overline{q} = \underset{q_i \geq 0}{\arg\max} P(q_i, 0)\, q_i$. Since $|\rho'(q_i)| \geq \beta$, $\overline{q}$ cannot be greater than $\frac{\rho(0)}{\beta}$. This, together with the fact that the revenue is continuous in q_i, implies that $\overline{q}$ exists. Since the revenue is zero at $q_i = 0$ and $q_i = \frac{\rho(0)}{\beta}$, the solution is interior and satisfies the first-order condition: $\rho'(\overline{q})\,\overline{q} + \rho(\overline{q}) = 0$.

Note that no type $c_i \in C$ will find it optimal to choose quantities higher than $\overline{q}$ regardless of the conjecture about the opponent's play. This is because such quantities result in (weakly) lower revenue than $\overline{q}$ (not just when $q_{-i} = 0$, but for every $q_{-i} \geq 0$), and strictly higher cost (because $\frac{\partial C(q_i,c_i)}{\partial q_i} > 0$ when $q_i > 0$). Hence

$$\rho(q_i) - \beta q_{-i} \geq \rho(\overline{q}) - \beta\overline{q} = (-\rho'(\overline{q}) - \beta)\,\overline{q} \geq 0$$

where the first inequality is because $\rho' < 0$ and $\beta > 0$, the equality is by definition of $\overline{q}$, the second inequality is due to $|\rho'(q)| \geq \beta$.

(ii) Note that

$$(5)\quad \frac{\partial^2 \pi_i(q_i, q_{-i}, c_i)}{\partial q_i^2} = \rho''(q_i)q_i + 2\rho'(q_i) - \frac{\partial^2 C(q_i, c_i)}{\partial q_i^2} < (1+\varepsilon)\,\rho'(q_i) \leq -(1+\varepsilon)\,\beta < 0$$

for every $q_i \geq 0$. Thus π_i is strictly concave in q_i, and q is single valued. By the Theorem of the Maximum, q is continuous in (q_{-i}, c_i). Note that q equals 0 if $\rho(0) - \beta q_{-i} - \frac{\partial C_i(0,c_i)}{\partial q_i} \leq 0$, and solves the first-order condition

$$\rho'(q_i)q_i + \rho(q_i) - \beta q_{-i} - \frac{\partial C_i(q_i, c_i)}{\partial q_i} = 0$$

otherwise. By the Implicit Function Theorem q is continuously differentiable in (q_{-i}, c_i) whenever $q(q_{-i}, c_i) > 0$, i.e. $\rho(0) - \beta q_{-i} - \frac{\partial C_i(0,c_i)}{\partial q_i} > 0$, with

$$\frac{\partial q(q_{-i}, c_i)}{\partial c_i} = \frac{\frac{\partial^2 C_i(q_i,c_i)}{\partial q_i \partial c_i}}{\frac{\partial^2 \pi_i(q_i,q_{-i},c_i)}{\partial q_i^2}} \leq 0, \quad \frac{\partial q(q_{-i}, c_i)}{\partial q_{-i}} = \frac{\beta}{\frac{\partial^2 \pi_i(q_i,q_{-i},c_i)}{\partial q_i^2}}.$$

Using (5) we get $\frac{\partial q(q_{-i},c_i)}{\partial q_{-i}} \in \left(-\frac{1}{1+\varepsilon}, 0\right)$.

(iii) Let $\overline{q}$ be as defined in part (i). Then

$$\begin{aligned} \frac{\partial \pi(0, q_{-i}, c_i)}{\partial q_i} &= \rho(0) - \beta q_{-i} - \frac{\partial C(0, c_i)}{\partial q_i} \\ &\geq \rho(0) - (-\rho'(\overline{q}))\,\overline{q} \geq \rho(0) - \rho(\overline{q}) > 0 \end{aligned}$$

where the first inequality uses the facts that that $\beta \leq -\rho'(\overline{q})$, $q_{-i} \leq \overline{q}$, and $\frac{\partial C(0,c_i)}{\partial q_i} = 0$; the second inequality uses the first-order condition for $\overline{q}$. Thus $q(q_{-i}, c_i) > 0$ for every $q_{-i} \in [0, q(0,0)] \subseteq [0, \overline{q}]$. ■

Proof of Lemma 2. Let

$$BR_i(q_{-i} \mid m_i) = \int q(q_{-i}, c_i)\, dF_i(c_i \mid m_i) \qquad \text{for } i \in \{A, B\}$$

Let $M_A = M$ and $M_B = N$ be the sets of equilibrium messages for firms A and B, respectively, and (m, n) be a representative element of $M \times N$. Then the expected outputs in a Bayesian-Nash equilibrium following messages (m, n) satisfy

$$Q_A(m,n) = BR_A(Q_B(m,n) \mid m),\ Q_B(m,n) = BR_B(Q_A(m,n) \mid n) \tag{6}$$

Let $H(q_A, q_B) = (BR_A(q_B|m), BR_B(q_A|n))$. By A2 and the fact that $q(q_{-i}, c_i)$ is decreasing in c_i, H maps the interval $[0, q(0,0)]^2$ into itself. A2 implies that for every c_i and $Q_{-i} \neq Q'_{-i}$:

$$\left|q(Q_{-i}, c_i) - q\left(Q'_{-i}, c_i\right)\right| < (1-\delta)\left|Q_{-i} - Q'_{-i}\right| \tag{7}$$

This in turn implies that H is a contraction mapping in the sup norm.

Consider the sequence $\left\{Q_A^k, Q_B^k\right\}_{k=0}^{\infty}$ defined by

$$Q_A^0 = Q_B^0 = 0;$$
$$(Q_A^k, Q_B^k) = H(Q_A^{k-1}, Q_B^{k-1}),\ k \geq 1$$

and for $k \geq 1$, let

$$I_i^k = \left[\min\left\{Q_i^{k-1}, Q_i^k\right\}, \max\left\{Q_i^{k-1}, Q_i^k\right\}\right]$$

Because H is a contraction mapping on $[0, q(0,0)]^2$, the sequence $\left\{Q_A^k, Q_B^k\right\}_{k=0}^{\infty}$ converges. By continuity of $BR_i(\cdot|m_i)$, its limit satisfies (6) and thus defines the expected outputs in a Bayesian-Nash equilibrium.

Next, let us prove that any strategy $q_i(m_i, m_{-i}, c_i)$ of firm i that survives k rounds of elimination of interim strictly dominated strategies has to satisfy $\int q_i(m_i, m_{-i}, c_i)dF(c_i|m_i) \in I_i^k$. Indeed, the statement holds for $k = 1$: for every i, $\int q_{-i}(m_{-i}, m_i, c_{-i})dF(c_{-i}|m_{-i}) \geq 0$ implies that any strategy $q_i(m_i, m_{-i}, c_i)$ such that $q_i(m_i, m_{-i}, c_i) > q(0, c_i)$ is interim strictly dominated for type c_i. Thus the first round of elimination leaves only strategies such that $\int q_i(m_i, m_{-i}, c_i)dF(c_i|m_i) \in [0, BR_i(0|m_i)] = I_i^1$. Suppose that the statement holds for $k \geq 1$, i.e. k rounds of elimination result in strategies for firm $-i$ such that $\int q_{-i}(m_{-i}, m_i, c_{-i})dF(c_{-i}|m_{-i}) \in I_{-i}^k$. Conditional on firm $-i$ using such strategies, any strategy $q_i(m_i, m_{-i}, c_i)$ of firm i such that

$$\begin{aligned} q_i(m_i, m_{-i}, c_i) &\notin \left[q(\max\left\{Q_{-i}^{k-1}, Q_{-i}^k\right\}, c_i), q(\min\left\{Q_{-i}^{k-1}, Q_{-i}^k\right\}, c_i)\right] \\ &= \left[\min\left\{q(Q_{-i}^{k-1}, c_i), q(Q_{-i}^k, c_i)\right\}, \max\left\{q(Q_{-i}^{k-1}, c_i), q(Q_{-i}^k, c_i)\right\}\right] \end{aligned}$$

is interim strictly dominated for type c_i. Therefore, firm i's strategies surviving $k+1$ rounds of elimination satisfy

$$\int q_i(m_i, m_{-i}, c_i)dF(c_i|m_i) \in \left[\min\left\{BR_i(Q_{-i}^{k-1}|m_i), BR_i(Q_{-i}^k|m_i)\right\},\right.$$
$$\left.\max\left\{BR_i(Q_{-i}^{k-1}|m_i), BR_i(Q_{-i}^k|m_i)\right\}\right] = \left[\min\left\{Q_i^k, Q_i^{k+1}\right\}, \max\left\{Q_i^k, Q_i^{k+1}\right\}\right] = I_i^{k+1}$$

Let $(Q_A(m,n), Q_B(m,n)) = \lim_{k\to\infty}(Q_A^k, Q_B^k)$ be the equilibrium expected output following messages (m,n). Then $Q_i(m,n) = \lim_{k\to\infty} \min\left\{Q_i^{k-1}, Q_i^k\right\} = \lim_{k\to\infty} \max\left\{Q_i^{k-1}, Q_i^k\right\}$ for $i = A, B$. Therefore, any strategy profile that survives iterated elimination of interim strictly dominated strategies has to satisfy $\int q_i(m_i, m_{-i}, c_i) dF(c_i|m_i) = Q_i(m,n)$, and the only strategy profile that survives the elimination is the one satisfying $q_i(m_i, m_{-i}, c_i) = q(Q_i(m,n), c_i)$, which is the condition for the Bayesian-Nash equilibrium. ■

7.2. **Proof of Theorems 1 and 2.** Suppose there exists an informative cheap talk equilibrium. The fact that the equilibrium is informative implies that $\max\{|M|, |N|\} \geq 2$. We will assume, without loss of generality, that every message induces a different distribution over the opponent's output. To state this assumption formally, let $\sigma_i(\cdot|c_i)$ be a probability distribution over M_i defining the message strategy of firm i, and let $G_{-i}(x|m_i) = Pr\left(Q_{-i}(m_i, m_{-i}) \leq x | m_i\right) = \int\int 1_{\{Q_{-i}(m_i,m_{-i})\leq x\}} d\sigma_{-i}(m_{-i}|c_{-i}) dF(c_{-i})$ be the distribution function of firm $-i$'s expected output conditional on firm i sending message m_i. Then we will assume that $G_{-i}(x|m_i) \neq G_{-i}(x|m_i')$, $\forall m_i, m_i' \in M_i$, $i \in \{A, B\}$.

Lemma 5. *Suppose A1-A4 hold. For every $m, m' \in M$ such that $m \neq m'$, there exist $n, n' \in N$ such that $Q_B(m,n) > Q_B(m',n)$ and $Q_B(m,n') < Q_B(m',n')$. Symmetrically, for every $n, n' \in N$ such that $n \neq n'$, there exist $m, m' \in M$ such that $Q_A(m',n) > Q_A(m',n')$ and $Q_A(m',n) < Q_A(m',n')$.*

Proof. Suppose the conclusion of the lemma does not hold for $m, m' \in M$; e.g. $\forall n \in N$, $Q_B(m,n) \geq Q_B(m',n)$. This implies that $\forall x \geq 0$, $G(x|m) \leq G(x|m')$. Then the difference in expected profit of type c_A from sending message m as opposed to m' is

$$\begin{aligned} & \int \Pi_A(q_B, c_A)\, dG(q_B \mid m) - \int \Pi_A(q_B, c_A)\, dG(q_B \mid m') \\ = & \int \frac{d\Pi_A(q_B, c_A)}{dq_B}(1 - G(q_B \mid m))\, dq_B - \int \frac{d\Pi_A(q_B, c_A)}{dq_B}(1 - G(q_B \mid m'))\, dq_B \\ = & -\beta \int q(q_B, c_A)(G(q_B \mid m') - G(q_B \mid m))\, dq_B \leq 0 \end{aligned}$$

where the first equality is obtained through integration by parts (the validity of integration by parts is guaranteed by Theorem II.6.11 of Shiryaev (2000), which applies because the support of q_B is bounded and Π_A is decreasing in q_B), and the second equality is by the envelope theorem. Moreover, A4 implies that $q\left(q_B, c_A\right) > 0$ for every (q_B, c_A), so, because $G(x|m) \neq G(x|m')$, the inequality is strict. Hence every type c_A strictly prefers sending message m' to message m, which is a contradiction. ■

Lemma 6. *Suppose A1-A4 hold. For every $n, n' \in N$ such that $n \neq n'$, $\exists q^*(n, n') = (q_A^*(n, n'), q_B^*(n, n'))$ such that $q_B^*(n, n') = BR_B(q_A^*(n, n')|n) = BR_B(q_A^*(n, n')|n')$. Moreover, $\exists m, m' \in M$ s.t. $q_A^*(n, n')$ is strictly between $Q_A(m, n)$ and $Q_A(m', n)$. A symmetric statement holds for any $m, m' \in M$ such that $m \neq m'$.*

Proof. By Lemma 5, there must exist $m, m' \in M$ such that $Q_A\left(m, n\right) > Q_A\left(m, n'\right)$ and $Q_A\left(m', n\right) < Q_A\left(m', n'\right)$.

Let

$$\psi\left(q_A\right) := BR_B\left(q_A \mid n'\right) - BR_B\left(q_A \mid n\right)$$

and

$$\phi\left(q_A; \widetilde{m}, \tilde{n}\right) := BR_B\left(q_A \mid \tilde{n}\right) - BR_A^{-1}\left(q_A \mid \widetilde{m}\right)$$

Function ϕ is increasing in q_A, since BR_A^{-1} is steeper than BR_B. By (6), $\phi\left(Q_A(\widetilde{m}, \tilde{n}); \widetilde{m}, \tilde{n}\right) = 0$ for every $(\widetilde{m}, \tilde{n})$.

Note that

$$(8) \qquad \psi\left(Q_A\left(m, n\right)\right) = \phi\left(Q_A(m, n); m, n'\right) > \phi\left(Q_A(m, n'); m, n'\right) = 0$$

where the equalities use (6); the inequality holds because $Q_A\left(m, n\right) > Q_A\left(m, n'\right)$ and because ϕ is increasing. Similarly,

$$(9) \qquad \psi\left(Q_A\left(m', n\right)\right) = \phi\left(Q_A(m', n); m', n'\right) < \phi\left(Q_A\left(m', n'\right); m', n'\right) = 0$$

Since the best responses, and thus ψ, are continuous, from (8) and (9) it follows that there exists $q^*(n,n')$ at which $BR_B(\cdot \mid n)$ and $BR_B(\cdot \mid n')$ intersect, and $q_A^*(n,n')$ is strictly between $Q_A(m,n)$ and $Q_A(m',n)$ by construction. ■

Let $\underline{q}_A = \inf_{(m,n)\in M\times N} Q_A(m,n)$; that is, $\forall(m,n) \in M \times N$, $Q_A(m,n) \geq \underline{q}_A$, and $\forall\varepsilon > 0$, $\exists(m,n) \in M \times N$: $Q_A(m,n) \leq \underline{q}_A + \varepsilon$. For $\varepsilon > 0$, let $\overline{q}_B(\varepsilon) = \sup_{(m,n)\in M\times N: Q_A(m,n)\leq \underline{q}_A+\varepsilon} Q_B(m,n)$ and let $\overline{q}_B = \lim_{\varepsilon\to 0}\overline{q}_B(\varepsilon)$. Similarly, let $\overline{q}_A = \sup_{(m,n)\in M\times N} Q_A(m,n)$; for $\varepsilon > 0$, let $\underline{q}_B(\varepsilon) = \inf_{(m,n)\in M\times N: Q_A(m,n)\geq \overline{q}_A-\varepsilon} Q_B(m,n)$ and let $\underline{q}_B = \lim_{\varepsilon\to 0}\underline{q}_B(\varepsilon)$. Both $\overline{q}_A$ and $\overline{q}_B$ are finite, because $Q_i(m,n) \leq q_i(0,0) < \infty$. By definition, $\underline{q}_A \leq \overline{q}_A$; the fact that the equilibrium is informative implies that $\underline{q}_A < \overline{q}_A$ (indeed,if $\underline{q}_A = \overline{q}_A = q$, then $Q_A(m,n) = q$, $\forall(m,n) \in M \times N$; therefore, $Q_B(m,n)$ is also constant with respect to (m,n), and the equilibrium is uninformative).

Lemma 7. *For all* $(m,n) \in M \times N$, $\underline{q}_i \leq Q_i(m,n) \leq \overline{q}_i$, $i \in \{A,B\}$.

Proof. The fact that $\underline{q}_A \leq Q_A(m,n) \leq \overline{q}_A$ follows immediately from the definitions of $\underline{q}_A$ and $\overline{q}_A$. Let us prove that $\forall(m,n) \in M \times N$, $Q_B(m,n) \leq \overline{q}_B$ (the proof that $\underline{q}_B \leq Q_B(m,n)$ is analogous). Suppose $\exists(\tilde{m},\tilde{n}) \in M \times N$: $Q_B(\tilde{m},\tilde{n}) > \overline{q}_B$. Since $\overline{q}_B(\varepsilon)$ weakly increases in ε, this implies that $\exists\overline{\varepsilon} > 0$: $\forall\varepsilon \in (0,\overline{\varepsilon})$, $Q_B(\tilde{m},\tilde{n}) > \overline{q}_B(\varepsilon)$. Therefore, if $Q_A(m,n) < \underline{q}_A + \overline{\varepsilon}$, then $Q_B(\tilde{m},\tilde{n}) > Q_B(m,n)$. This implies that $Q_A(\tilde{m},\tilde{n}) \geq \underline{q}_A + \overline{\varepsilon}$.

Let us prove that there exists $(m,n) \in M \times N$ such that

$$\begin{cases} Q_A(m,n) \leq Q_A(\tilde{m},\tilde{n}); \\ Q_A(m,n) \leq Q_A(m,\tilde{n}); \\ Q_B(m,n) < Q_B(\tilde{m},\tilde{n}) \end{cases} \tag{10}$$

Indeed, pick any (m',n') such that $Q_A(m',n') < \underline{q}_A + \overline{\varepsilon}$ and let $(m,n) = (m',n')$ if $Q_A(m',n') \leq Q_A(m',\tilde{n})$ and $(m,n) = (m',\tilde{n})$ otherwise. Then, by definition, $Q_A(m,n) \leq Q_A(m,\tilde{n})$. Since $Q_A(m,n) < \underline{q}_A + \overline{\varepsilon}$, $Q_B(\tilde{m},\tilde{n}) > Q_B(m,n)$ and, as noted above, $Q_A(m,n) \leq Q_A(\tilde{m},\tilde{n})$.

Let

$$\phi(q_A) := BR_B(q_A \mid \tilde{n}) - BR_A^{-1}(q_A \mid m)$$

Note that ϕ is increasing, and $\phi(Q_A(m, \tilde{n})) = 0$.

Suppose (m, n) satisfies condition (10). Then

(11)
$$\phi(Q_A(m,n)) = BR_B(Q_A(m,n) \mid \tilde{n}) - Q_B(m,n) \geq BR_B(Q_A(\tilde{m},\tilde{n}) \mid \tilde{n}) - Q_B(m,n)$$
$$= Q_B(\tilde{m},\tilde{n}) - Q_B(m,n) > 0 = \phi(Q_A(m,\tilde{n}))$$

where the first inequality is because by condition (10), $Q_A(m,n) \leq Q_A(\tilde{m},\tilde{n})$, and because BR_B is downward sloping; the second inequality is again by (10).

Thus $\phi(Q_A(m,n)) > \phi(Q_A(m,\tilde{n}))$, and, since ϕ is increasing, $Q_A(m,n) > Q_A(m,\tilde{n})$. But this contradicts (10). ■

Lemma 7 implies that $\underline{q}_B \leq \overline{q}_B$. The fact that the equilibrium is informative implies that the inequality is strict: if $\underline{q}_B = \overline{q}_B = q$, then, by Lemma 7, $Q_B(m,n) = q$, $\forall(m,n) \in M \times N$, and the equilibrium cannot be informative.

Proof of Theorem 1. Suppose an informative equilibrium exists. Let us first prove that

$$(1-\delta)\left(\overline{q}_A - \underline{q}_A\right) \geq \overline{q}_B - \underline{q}_B \tag{12}$$

Fix $\varepsilon > 0$. By definition of $\underline{q}_A$ and $\overline{q}_B$, there exists $(m,n) \in M \times N$ such that $Q_A(m,n) \in \left[\underline{q}_A, \underline{q}_A + \varepsilon\right)$ and $Q_B(m,n) \in (\overline{q}_B - \varepsilon, \overline{q}_B]$. Similarly, there exists $(m',n') \in M \times N$ such that $Q_A(m',n') \in (\overline{q}_A - \varepsilon, \overline{q}_A]$ and $Q_B(m',n') \in \left[\underline{q}_B, \underline{q}_B + \varepsilon\right)$. Since $\underline{q}_A < \overline{q}_A$ and $\underline{q}_B < \overline{q}_B$, $Q_A(m,n) < Q_A(m',n')$, $Q_B(m,n) > Q_B(m',n')$ if ε is small enough.

If $n = n'$, both $Q(m,n)$ and $Q(m',n')$ satisfy the equation $q_B = BR_B(q_A|n)$. Then by $A2$, and since $Q_A(m,n) < Q_A(m',n')$, we have

$$(1-\delta)(Q_A(m',n') - Q_A(m,n)) > Q_B(m,n) - Q_B(m',n') \tag{13}$$

Since $\overline{q}_A - \underline{q}_A \geq Q_A(m',n') - Q_A(m,n)$, and $Q_B(m,n) - Q_B(m',n') > \overline{q}_B - \underline{q}_B - 2\varepsilon$, we get

$$(1-\delta)\left(\overline{q}_A - \underline{q}_A\right) > \overline{q}_B - \underline{q}_B - 2\varepsilon \tag{14}$$

Since the equation above has to hold for any $\varepsilon > 0$, however small, we get (12).

If $n \neq n'$, by Lemma 6 there exists $q^*(n,n') = (q_A^*(n,n'), q_B^*(n,n'))$ such that $q_B^*(n,n') = BR_B(q_A^*(n,n')|n) = BR_B(q_A^*(n,n')|n')$, and $q_A^*(n,n') \in (Q_A(\hat{m},n), Q_A(\tilde{m},n))$ for some $\hat{m}, \tilde{m} \in M$. There are three cases to consider.

Case 1: $Q_A(m,n) < q_A^*(n,n') < Q_A(m',n')$.

The first inequality, together with the fact that both $Q(m,n)$ and $q^*(n,n')$ satisfy the equation $q_B = BR_B(q_A|n)$, implies

$$(1-\delta)\left(q_A^*(n,n') - Q_A(m,n)\right) > Q_B(m,n) - q_B^*(n,n') \tag{15}$$

Similarly, the second inequality implies

$$(1-\delta)\left(Q_A(m',n') - q_A^*(n,n')\right) > q_B^*(n,n') - Q_B(m',n') \tag{16}$$

Summing up (15) and (16) gives (13), which as when $n = n'$, implies (12).

Case 2: $q_A^*(n,n') \leq Q_A(m,n) < Q_A(m',n')$.

Like in Case 1, $q_A^*(n,n') < Q_A(m',n')$ implies (16). Since $\underline{q}_A \leq Q_A(\hat{m},n) < q_A^*(n,n')$, we have $\overline{q}_A - \underline{q}_A \geq Q_A(m',n') - q_A^*(n,n')$. Since $q^*(n,n')$ and $Q(m,n)$ lie on the curve $q_B = BR_B(q_A|n)$, which is downward sloping, $q_B^*(n,n') \geq Q_B(m,n) > \overline{q}_B - \varepsilon$. Hence, $q_B^*(n,n') - Q_B(m',n') > \overline{q}_B - \underline{q}_B - 2\varepsilon$. Thus we get (14), which as when $n = n'$, implies (12).

Case 3: $Q_A(m,n) < Q_A(m',n') \leq q_A^*(n,n')$.

Like in Case 1, $Q_A(m,n) < q_A^*(n,n')$ implies (15). Since $q_A^*(n,n') < Q_A(\tilde{m},n) \leq \overline{q}_A$, we have $\overline{q}_A - \underline{q}_A \geq q_A^*(n,n') - Q_A(m,n)$. Since $q^*(n,n')$ and $Q(m',n')$ lie on the curve $q_B = BR_B(q_A|n')$, which is downward sloping, $q_B^*(n,n') \leq Q_B(m',n') < \underline{q}_B + \varepsilon$. Hence,

$Q_B(m,n) - q_B^*(n,n') > \overline{q}_B - \underline{q}_B - 2\varepsilon$. Thus we get (14), which as when $n = n'$, implies (12).

Symmetrically, we can show

$$(1-\delta)\left(\overline{q}_B - \underline{q}_B\right) \geq \overline{q}_A - \underline{q}_A$$

which is in contradiction with (12) and the fact that $\delta \in (0,1)$. ■

Proof of Theorem 2. Note that nowhere in the proof of Theorem 1 did we use the fact that each firm's cost types are distributed according to the same distribution F. In fact Theorem 1 holds even if we assume that the cost types of firms A and B are distributed according to distributions F_A and F_B, respectively, independently of each other.

Specifically, first note that in the case of different prior distributions, F_A and F_B, in the game without cheap talk communication by Lemma 2 there is a unique Bayesian Nash equilibrium, which is also a unique outcome of the iterated dominance procedure. This strategy profile is given by $q_i^{NC}(c_i) = q\left(Q_{-i}^{NC}, c_i\right)$, where $Q_i^{NC} = \int q_i(Q_{-i}^{NC}, c_i) dF_i(c_i)$ for $i = A, B$. Next, following the steps of the proof of Theorem 1, one can verify that in the game with one round of cheap talk, following any message profile (m_i, m_{-i}) the expected quantity of firm i satisfies $Q_i(m_i, m_{-i}) = Q_i^{NC}$, for $i = A, B$. Following any message profile firm i plays the same strategy as in the Bayesian Nash equilibrium: $q(Q_i(m_i, m_{-i}), c_i) = q\left(Q_i^{NC}, c_i\right)$, for every c_i, $i = A, B$.

Next, suppose there exist no informative T-round cheap talk equilibrium. We will show that then every $T+1$-round cheap talk equilibrium is uninformative as well. Suppose the message profile in the first round is (m_A, m_B), and the posterior beliefs are $(F_A(\cdot \mid m_A), F_B(\cdot \mid m_B))$. The continuation game starting from period 2 has no informative cheap talk equilibrium. That is, the expected quantities are always the same as in the game without communication, $\left(Q_A^{NC}, Q_B^{NC}\right)$ calculated for beliefs

$(F_A(\cdot \mid m_A), F_B(\cdot \mid m_B))$:

$$Q_A^{NC} = BR_A\left(Q_B^{NC} \mid m_A\right), Q_B^{NC} = BR_B\left(Q_A^{NC} \mid m_B\right)$$

Thus if in $T+1$-round cheap talk game there exists an informative equilibrium, then there exists an outcome equivalent informative equilibrium where the firms use the same first-period communication strategies, and use babbling strategies in the remaining periods. However this implies that in one-round cheap talk game there exists an outcome equivalent informative equilibrium where the firms use the same first-period communication strategies as above, which is a contradiction with Theorem 1. ■

7.3. **Proof of Theorems 3 and 4.** Consider a "min" mechanism with threshold $c^* \in (0, \bar{c})$. After m^1 is announced, the expected output of firm $-i$ is $Q^{H2}(c^*)$ that solves

$$\Phi(Q_{-i}, c^*) = Q_{-i} - \frac{1}{1-F(c^*)}\int_{c^*}^{\infty} q(Q_{-i}, c_i)\, dF(c_i) = 0 \tag{17}$$

Lemma 8. *For every c^*, there exists a unique $Q^{H2}(c^*)$ that solves (17), and thus there exists a unique continuation equilibrium following message m^1, which is symmetric. The function $Q^{H2}(c^*)$ is continuous and decreasing in c^*, $Q^{H2}(0) = Q^{NC}$, $\lim_{c^* \to \infty} Q^{H2}(c^*) = 0$.*

Proof. Note that Φ is continuous in all variables by A1 and the continuity of F; $\Phi(0, c^*) = -\frac{1}{1-F(c^*)}\int_{c^*}^{\infty} q(0, c_i)\, dF(c_i) < 0$ by A3. Let $Q'_{-i} > Q_{-i}$; then

$$\Phi(Q'_{-i}, c_i) - \Phi(Q_{-i}, c_i) = Q'_{-i} - Q_{-i} - \frac{1}{1-F(c^*)}\int_{c^*}^{\infty}\left(q\left(Q'_{-i}, c_i\right) - q\left(Q_{-i}, c_i\right)\right) dF(c_i)$$

$$\geq Q'_{-i} - Q_{-i}$$

where the inequality is by A2. Therefore equation (17) has a unique solution, which we will call $Q^{H2}(c^*)$. The function $Q^{H2}(c^*)$ is continuous by Theorem 2.1 in Jittorntrum (1978). Let us prove that $Q^{H2}(c^*)$ is decreasing in c^*. First, note that for any Q_{-i}, the function $\frac{1}{1-F(c^*)}\int_{c^*}^{\infty} q\left(Q_{-i},c_i\right)dF\left(c_i\right)$ decreases in c^*. Indeed, if $\tilde{c}^* < c^*$, then

(18)

$$\begin{aligned}
&\frac{1}{1-F\left(\tilde{c}^*\right)}\int_{\tilde{c}^*}^{\infty} q\left(Q_{-i},c_i\right)dF\left(c_i\right)-\frac{1}{1-F\left(c^*\right)}\int_{c^*}^{\infty} q\left(Q_{-i},c_i\right)dF\left(c_i\right)\\
&=\frac{1}{1-F\left(\tilde{c}^*\right)}\int_{\tilde{c}^*}^{c^*} q\left(Q_{-i},c_i\right)dF\left(c_i\right)-\frac{F(c^*)-F\left(\tilde{c}^*\right)}{(1-F\left(\tilde{c}^*\right))(1-F\left(c^*\right))}\int_{c^*}^{\infty} q\left(Q_{-i},c_i\right)dF\left(c_i\right)\\
&\geq\frac{1}{1-F\left(\tilde{c}^*\right)}\int_{\tilde{c}^*}^{c^*} q\left(Q_{-i},c_i\right)dF\left(c_i\right)-\frac{F(c^*)-F\left(\tilde{c}^*\right)}{1-F\left(\tilde{c}^*\right)}q(Q_{-i},c^*)\\
&=\frac{F(c^*)-F\left(\tilde{c}^*\right)}{1-F\left(\tilde{c}^*\right)}\left(\frac{1}{F(c^*)-F\left(\tilde{c}^*\right)}\int_{\tilde{c}^*}^{c^*} q\left(Q_{-i},c_i\right)dF\left(c_i\right)-q(Q_{-i},c^*)\right)\geq 0
\end{aligned}$$

where both inequalities follow from A2. Therefore, if $\tilde{c}^* < c^*$, and $Q^{H2}\left(\tilde{c}^*\right) < Q^{H2}\left(c^*\right)$, then

$$\begin{aligned}
&Q^{H2}\left(c^*\right)-Q^{H2}\left(\tilde{c}^*\right)\\
&=\frac{1}{1-F\left(c^*\right)}\int_{c^*}^{\infty} q\left(Q^{H2}\left(c^*\right),c_i\right)dF\left(c_i\right)-\frac{1}{1-F\left(\tilde{c}^*\right)}\int_{\tilde{c}^*}^{\infty} q\left(Q^{H2}\left(\tilde{c}^*\right),c_i\right)dF\left(c_i\right)\\
&\leq\frac{1}{1-F\left(c^*\right)}\int_{c^*}^{\infty} q\left(Q^{H2}\left(\tilde{c}^*\right),c_i\right)dF\left(c_i\right)-\frac{1}{1-F\left(\tilde{c}^*\right)}\int_{\tilde{c}^*}^{\infty} q\left(Q^{H2}\left(\tilde{c}^*\right),c_i\right)dF\left(c_i\right)\leq 0
\end{aligned}$$

which contradicts the assumption $Q^{H2}\left(\tilde{c}^*\right) < Q^{H2}\left(c^*\right)$ (the first inequality above follows from A2 and $Q^{H2}\left(\tilde{c}^*\right) < Q^{H2}\left(c^*\right)$, and the second from (18)). By definition,

$$Q^{H2}(0)=\int_{0}^{\infty} q\left(Q^{H2}(0),c_i\right)dF\left(c_i\right)$$

and therefore $Q^{H2}(0)=Q^{NC}$. Finally, $\lim_{c^*\to\infty} Q^{H2}\left(c^*\right)=0$ by A6. ■

Let $Q^{L}\left(c^*\right)$ be the expected output of firm $-i$ if m^0 was announced and firm i reported $\hat{c}_i < c^*$, and let $Q^{H1}\left(c^*\right)$ be the expected output of firm $-i$ if m^0 was announced and firm i reported $\hat{c}_i > c^*$. Then $Q^{L}\left(c^*\right)$ and $Q^{H1}\left(c^*\right)$ solve

(19)

$$\begin{cases}
\Psi\left(Q_{-i}^{L},Q_{-i}^{H1},c^*\right)=Q_{-i}^{L}-\int_{0}^{c^*} q\left(Q_{-i}^{L},c_i\right)dF\left(c_i\right)-\int_{c^*}^{\infty} q\left(Q_{-i}^{H1},c_i\right)dF\left(c_i\right)=0\\
\Omega\left(Q_{-i}^{L},Q_{-i}^{H1},c^*\right)=Q_{-i}^{H1}-\frac{1}{F(c^*)}\int_{0}^{c^*} q\left(Q_{-i}^{L},c_i\right)dF\left(c_i\right)=0
\end{cases}$$

Lemma 9. *For every c^* there exist unique $Q^L(c^*)$ and $Q^{H1}(c^*)$ that solve equations (19), and thus there exists a unique continuation equilibrium after public message m^0, which is symmetric. Both $Q^L(c^*)$ and $Q^{H1}(c^*)$ are continuous; $Q^L(c^*)$ is increasing and $Q^{H1}(c^*)$ is decreasing in c^*; $Q^L(c^*) \leq Q^{H1}(c^*)$; $Q^L(0) > 0$; $\lim_{c^* \to \infty} Q^L(c^*) = \lim_{c^* \to \infty} Q^{H1}(c^*) = Q^{NC}$.*

Proof. Denote

$$\overline{\Psi}\left(Q_{-i}^L, c^*\right) = Q_{-i}^L - \int_0^{c^*} q\left(Q_{-i}^L, c_i\right) dF(c_i) - \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)}\int_0^{c^*} q\left(Q_{-i}^L, \widehat{c}\right) dF(\widehat{c}), c_i\right) dF(c_i)$$

Note that $Q_{-i}^L(c^*)$ is defined by $\overline{\Psi}\left(Q_{-i}^L(c^*), c^*\right) = 0$.

By A1 and the continuity of F, $\overline{\Psi}$ is continuous. By A3,

$$\overline{\Psi}(0, c^*) = -\int_0^{c^*} q(0, c_i) dF(c_i) - \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)}\int_0^{c^*} q_i(0, \widehat{c}) dF(\widehat{c}), c_i\right) dF(c_i) < 0$$

By A2 and the fact that $q(q_{-i}, c_i)$ is decreasing in c_i,

$$\begin{aligned}\overline{\Psi}(q(0,0), c^*) &= q(0,0) - \int_0^{c^*} q(q(0,0), c_i) dF(c_i) \\ &- \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)}\int_0^{c^*} q(q(0,0), \widehat{c}) dF(\widehat{c}), c_i\right) dF(c_i) > 0\end{aligned}$$

If $Q'_{-i} > Q_{-i}$, then

$$\begin{aligned}&\overline{\Psi}(Q'_{-i}, c^*) - \overline{\Psi}(Q_{-i}, c^*) = Q'_{-i} - Q_{-i} - \int_0^{c^*} \left(q\left(Q'_{-i}, c_i\right) - q\left(Q_{-i}, c_i\right)\right) dF(c_i) \\ &- \int_{c^*}^{\infty} \left(q\left(\frac{1}{F(c^*)}\int_0^{c^*} q\left(Q'_{-i}, \widehat{c}\right) dF(\widehat{c}), c_i\right) - q\left(\frac{1}{F(c^*)}\int_0^{c^*} q\left(Q_{-i}, \widehat{c}\right) dF(\widehat{c}), c_i\right)\right) dF(c_i) \\ &\geq Q'_{-i} - Q_{-i} - (1-\delta)\int_{c^*}^{\infty}\left(\frac{1}{F(c^*)}\int_0^{c^*}\left(q\left(Q_{-i}, \widehat{c}\right) - q\left(Q'_{-i}, \widehat{c}\right)\right) dF(\widehat{c})\right) dF(c_i) \\ &= Q'_{-i} - Q_{-i} - (1-\delta)\frac{1-F(c^*)}{F(c^*)}\int_0^{c^*}\left(q\left(Q_{-i}, \widehat{c}\right) - q\left(Q'_{-i}, \widehat{c}\right)\right) dF(\widehat{c}) \\ &\geq Q'_{-i} - Q_{-i} - (1-\delta)^2(1-F(c^*))(Q'_{-i} - Q_{-i}) \\ &= (Q'_{-i} - Q_{-i})(1 - (1-\delta)^2(1-F(c^*))) > 0\end{aligned}$$

where the inequalities follow from A2. Therefore for every c^* there exists a unique $Q^L(c^*) \in (0, q_i(0,0))$ such that $\overline{\Psi}\left(Q^L(c^*), c^*\right) = 0$, and a unique $Q^{H1}(c)$ defined by $\Omega\left(Q^L(c^*), Q^{H1}(c^*), c^*\right) = 0$. The functions $Q^L(c^*)$ and $Q^{H1}(c)$ are continuous by Theorem 2.1 in Jittorntrum (1978).

Next we show that $Q^L(c^*) \leq Q^{H1}(c^*)$. If $Q^L(c^*) > Q^{H1}(c^*)$, then

$$Q^L(c^*) - Q^{H1}(c^*) = \int_{c^*}^{\infty} q\left(Q^{H1}(c^*), c_i\right) dF(c_i) - \frac{1 - F(c^*)}{F(c^*)} \int_0^{c^*} q\left(Q^L(c^*), c_i\right) dF(c_i)$$

$$\leq (1 - F(c^*))\left(q\left(Q^{H1}(c^*), c^*\right) - q\left(Q^L(c^*), c^*\right)\right) < (1 - F(c^*))\left(Q^L(c^*) - Q^{H1}(c^*)\right)$$

which is a contradiction (the first inequality is by q_i decreasing in c_i, the second by A2).

Next, note that the function $\frac{1}{F(c)} \int_0^c q\left(Q^L, c_i\right) dF(c_i)$ decreases in c for every Q^L. Indeed, if $\tilde{c}^* < c^*$, then

$$\begin{aligned}
&\frac{1}{F(c^*)} \int_0^{c^*} q\left(Q^L, c_i\right) dF(c_i) - \frac{1}{F(\tilde{c}^*)} \int_0^{\tilde{c}^*} q\left(Q^L, c_i\right) dF(c_i) \\
&= \frac{1}{F(c^*)} \int_{\tilde{c}^*}^{c^*} q\left(Q^L, c_i\right) dF(c_i) - \frac{F(c^*) - F(\tilde{c}^*)}{F(c^*) F(\tilde{c}^*)} \int_0^{\tilde{c}^*} q\left(Q^L, c_i\right) dF(c_i) \\
&\leq \frac{1}{F(c^*)} \int_{\tilde{c}^*}^{c^*} q\left(Q^L, c_i\right) dF(c_i) - \frac{F(c^*) - F(\tilde{c}^*)}{F(c^*)} q\left(Q^L, \tilde{c}^*\right) \\
&= \frac{F(c^*) - F(\tilde{c}^*)}{F(c^*)} \left(\frac{1}{F(c^*) - F(\tilde{c}^*)} \int_{\tilde{c}^*}^{c^*} q\left(Q^L, c_i\right) dF(c_i) - q\left(Q^L, \tilde{c}^*\right)\right) \leq 0
\end{aligned} \tag{20}$$

where the inequalities follow from A2.

Let us now show that $Q^L(c^*)$ is increasing in c^*. Suppose that $\tilde{c}^* < c^*$ and $Q^L(\tilde{c}^*) > Q^L(c^*)$. Then $\overline{\Psi}(Q^L(\tilde{c}^*), c^*) > \overline{\Psi}(Q^L(c^*), c^*)$, because $\overline{\Psi}$ is strictly increasing in Q^L.

Since $\overline{\Psi}(Q^L(c^*), c^*) = 0$ and $\overline{\Psi}(Q^L(\tilde{c}^*), \tilde{c}^*) = 0$, we get

$$
\begin{aligned}
(21)\quad & 0 < \overline{\Psi}(Q^L(\tilde{c}^*), c^*) - \overline{\Psi}(Q^L(\tilde{c}^*), \tilde{c}^*) \\
= \; & \int_0^{\tilde{c}^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF\left(c_i\right) + \int_{\tilde{c}^*}^{\infty} q\left(Q^{H1}\left(\tilde{c}^*\right)\right) dF\left(c_i\right) \\
& - \int_0^{c^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF\left(c_i\right) - \int_{c^*}^{\infty} q\left(\frac{1}{F\left(c^*\right)} \int_0^{c^*} q\left(Q^L(\tilde{c}^*), \widehat{c}\right) dF\left(\widehat{c}\right), c_i\right) dF\left(c_i\right) \\
\leq \; & -\int_{\tilde{c}^*}^{c^*} \left(q\left(Q^L(\tilde{c}^*), c_i\right) - q\left(Q^{H1}\left(\tilde{c}^*\right)\right)\right) dF\left(c_i\right) \leq 0
\end{aligned}
$$

where the second inequality follows from A2, (20), and definition of Q^{H1}; the third inequality follows from equality from $\tilde{c}^* < c^*$, $Q^L(\tilde{c}^*) \leq Q^{H1}(\tilde{c}^*)$ and A2. Hence we get a contradiction. Therefore, $Q^L(\tilde{c}^*) \leq Q^L(c^*)$, and

$$
\begin{aligned}
Q^{H1}(c^*) - Q^{H1}(\tilde{c}^*) &= \frac{1}{F\left(c^*\right)} \int_0^{c^*} q\left(Q^L(c^*), c_i\right) dF\left(c_i\right) - \frac{1}{F\left(\tilde{c}^*\right)} \int_0^{\tilde{c}^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF\left(c_i\right) \\
&\leq \frac{1}{F\left(\tilde{c}^*\right)} \int_0^{\tilde{c}^*} q\left(Q^L(c^*), c_i\right) dF\left(c_i\right) - \frac{1}{F\left(\tilde{c}^*\right)} \int_0^{\tilde{c}^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF\left(c_i\right) \leq 0
\end{aligned}
$$

where the first inequality follows from (20) and the second from $Q^L(\tilde{c}^*) \leq Q^L(c^*)$ and A2. This proves that $Q^{H1}(c^*)$ is decreasing in c^*.

Next,

$$
Q^{H1}\left(0\right) = q\left(Q^L\left(0\right), 0\right) \leq q\left(0, 0\right)
$$

by A2, and therefore

$$
q\left(Q^{H1}\left(0\right), 0\right) \geq q\left(q\left(0, 0\right), 0\right) > 0
$$

where the first inequality is by A2 and the second by A3. Therefore, by A1 and the fact that $f > 0$,

$$
Q^L\left(0\right) = \int_0^{\infty} q\left(Q^{H1}\left(0\right), c_i\right) dF\left(c_i\right) > 0
$$

Finally, $\lim_{c^* \to \infty} Q^L\left(c^*\right) = \lim_{c^* \to \infty} Q^{H1}\left(c^*\right) = Q^{NC}$ by (19) and the definition of Q^{NC}. ■

For firm i of type c_i, let $\Delta\Pi\left(c_i;c^*\right)$ be the gain from reporting $\hat{c}_i < c^*$ compared to reporting $\hat{c}_i > c^*$ when the "min" mechanism with threshold c^* is in place:

$$\Delta\Pi\left(c_i;c^*\right) = \Pi_i\left(Q^L\left(c^*\right),c_i\right) - F\left(c^*\right)\Pi_i\left(Q^{H1}\left(c^*\right),c_i\right) - \left(1-F\left(c^*\right)\right)\Pi_i\left(Q^{H2}\left(c^*\right),c_i\right)$$

$$= \Pi_i\left(Q^L\left(c^*\right),c_i\right) - \Pi_i\left(Q^{H1}\left(c^*\right),c_i\right) - \left(1-F\left(c^*\right)\right)\left(\Pi_i\left(Q^{H2}\left(c^*\right),c_i\right) - \Pi_i\left(Q^{H1}\left(c^*,c_i\right)\right)\right)$$

By the envelope theorem:

$$\Delta\Pi\left(c_i;c^*\right) = \beta\left(F\left(c^*\right)\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c_i\right)dq_{-i} - \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q\left(q_{-i},c_i\right)dq_{-i}\right)$$

$$= \beta\left(\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c_i\right)dq_{-i} - \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c_i\right)dq_{-i}\right)$$

Lemma 10. *If* $\Delta\Pi\left(c;c^*\right)=0$*, then either* $\Delta\Pi\left(c';c^*\right)=0$, $\forall c' \geq c$*; or* $\frac{\partial\Delta\Pi(c;c^*)}{\partial c} < 0$*.*

Proof. Suppose first that

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c\right)dq_{-i} = \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c\right)dq_{-i} = 0$$

Then $\forall c' \geq c$, $\forall q_{-i} > \min\left\{Q^L\left(c^*\right),Q^{H2}\left(c^*\right)\right\}$, $q\left(q_{-i},c'\right)=0$. Hence $\Delta\Pi\left(c';c^*\right)=0$, $\forall c' \geq c$.

Suppose next that

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c\right)dq_{-i} = \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c\right)dq_{-i} \neq 0$$

Since $Q^{H1}\left(c^*\right) \geq Q^L\left(c^*\right)$ (Lemma 9), we have

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c\right)dq_{-i} = \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q\left(q_{-i},c\right)dq_{-i} > 0$$

This in turn implies $Q^L(c^*) < Q^{H1}(c^*)$, $q\left(Q^L\left(c^*\right),c\right) > 0$ and (since $q\left(q_{-i},c\right) \geq 0$) $Q^L\left(c^*\right) > Q^{H2}\left(c^*\right)$.

Let $Q(c) = \min\left\{q_{-i} \geq 0 : q(q_{-i},c)=0\right\}$. The value of $Q(c)$ is determined by the first-order condition: $Q(c) = \frac{1}{\beta}\left(\rho(0) - \frac{\partial C(0,c)}{\partial q_i}\right)$. The function $Q(c)$ is differentiable and decreasing in c. The fact that $q\left(Q^L\left(c^*\right), c\right) > 0$ implies that $Q^L\left(c^*\right) < Q(c)$. Finally, by the definition of $Q(c)$, $\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c) dq_{-i} = \int_{Q^L(c^*)}^{\min\left\{Q(c), Q^{H1}(c^*)\right\}} q(q_{-i}, c) dq_{-i}$.

Condition A5 implies that for $q_{-i} \in \left(Q^L\left(c^*\right), Q(c)\right)$,

$$\frac{\partial q\left(q_{-i}, c\right)}{\partial c} < \frac{\frac{\partial q\left(Q^L\left(c^*\right), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} q\left(q_{-i}, c\right) \tag{22}$$

Equation (22) implies

$$\int_{Q^L(c^*)}^{\min\left\{Q(c), Q^{H1}(c^*)\right\}} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} < \frac{\frac{\partial q\left(Q^L\left(c^*\right), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \int_{Q^L(c^*)}^{\min\left\{Q(c), Q^{H1}(c^*)\right\}} q\left(q_{-i}, c\right) dq_{-i} \tag{23}$$

Since $q\left(Q^L\left(c^*\right), c\right) > 0$ and $q(q_{-i}, c)$ is decreasing in q_{-i}, we have $q\left(q_{-i}, c\right) > 0$, $\forall q_{-i} \in \left[Q^{H2}\left(c^*\right), Q^L\left(c^*\right)\right)$. Therefore, by A5, $\frac{\partial q(q_{-i}, c)}{\partial c} > \frac{\frac{\partial q\left(Q^L\left(c^*\right), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} q\left(q_{-i}, c\right)$ for every $q_{-i} \in \left[Q^{H2}\left(c^*\right), Q^L\left(c^*\right)\right)$, and thus

$$\int_{Q^{H2}(c^*)}^{Q^L(c^*)} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} > \frac{\frac{\partial q\left(Q^L\left(c^*\right), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \int_{Q^{H2}(c^*)}^{Q^L(c^*)} q\left(q_{-i}, c\right) dq_{-i} \tag{24}$$

Suppose first that $Q(c) < Q^{H1}(c^*)$. Then equations (23) and (24) and the fact that $q(Q(c), c) = 0$ imply

$$\begin{aligned}
&\frac{\partial \Delta\Pi\left(c; c^*\right)}{\partial c} = \beta F\left(c^*\right) \int_{Q^L(c^*)}^{Q(c)} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} + \beta F(c^*) \frac{dQ(c)}{dc} q(Q(c), c) \\
&- \beta\left(1 - F\left(c^*\right)\right) \int_{Q^{H2}(c^*)}^{Q^L(c^*)} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} \\
&< \beta \frac{\frac{\partial q\left(Q^L\left(c^*\right), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \left(F\left(c^*\right) \int_{Q^L(c^*)}^{Q(c)} q\left(q_{-i}, c\right) dq_{-i} - \left(1 - F\left(c^*\right)\right) \int_{Q^{H2}(c^*)}^{Q^L(c^*)} q\left(q_{-i}, c\right) dq_{-i}\right) \\
&= \frac{\frac{\partial q\left(Q^L\left(c^*\right), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \Delta\Pi\left(c; c^*\right) = 0
\end{aligned} \tag{25}$$

Now suppose that $Q(c) > Q^{H1}(c^*)$. Then equations (23) and (24) imply

(26)

$$\begin{aligned}
\frac{\partial \Delta\Pi\left(c;c^*\right)}{\partial c} &= \beta\left(F\left(c^*\right)\int_{Q^L(c^*)}^{Q^{H1}(c^*)}\frac{\partial q\left(q_{-i},c\right)}{\partial c}dq_{-i} - \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^L(c^*)}\frac{\partial q\left(q_{-i},c\right)}{\partial c}dq_{-i}\right)\\
&< \beta\frac{\frac{\partial q\left(Q^L(c^*),c\right)}{\partial c}}{q\left(Q^L\left(c^*\right),c\right)}\left(F\left(c^*\right)\int_{Q^L(c^*)}^{Q^{H1}(c^*)}q\left(q_{-i},c\right)dq_{-i} - \left(1-F\left(c^*\right)\right)\int_{Q^{H2}(c^*)}^{Q^L(c^*)}q\left(q_{-i},c\right)dq_{-i}\right)\\
&= \frac{\frac{\partial q\left(Q^L(c^*),c\right)}{\partial c}}{q\left(Q^L\left(c^*\right),c\right)}\Delta\Pi\left(c;c^*\right) = 0
\end{aligned}$$

Finally, suppose that $Q(c) = Q^{H1}(c^*)$. Then $\frac{\partial\Delta\Pi(c_+;c^*)}{\partial c}$ is given by the first line in (25), and $\frac{\partial\Delta\Pi(c_-;c^*)}{\partial c}$ is given by the first line in (26). Since $q(Q(c),c) = 0$ and $Q(c) = Q^{H1}(c^*)$, we have $\frac{\partial\Delta\Pi(c_+;c^*)}{\partial c} = \frac{\partial\Delta\Pi(c_-;c^*)}{\partial c} < 0$. ■

Lemma 11. *There exists $\eta > 0$ such that for every $c_i \in [0,\overline{c}]$ and every $q_{-i} \leq q(0,0)$*

(27) $$q\left(q'_{-i},c_i\right) \geq q\left(q_{-i},c_i\right) + \eta q\left(q_{-i},c_i\right)\left(q_{-i}-q'_{-i}\right) \qquad \forall q'_{-i}\in\left(0,q_{-i}\right).$$

Proof. Let $\eta = \inf\left\{-\frac{\frac{\partial q(\widetilde{q}_{-i},0)}{\partial q_{-i}}}{q(\widetilde{q}_{-i},0)} \mid \widetilde{q}_{-i}\in[0,q(0,0)]\right\}$. It is well defined since, by A3, $q\left(\widetilde{q}_{-i},0\right) > 0$ for every $\widetilde{q}_{-i} \in [0,q(0,0)]$, and $\frac{\partial q(\widetilde{q}_{-i},0)}{\partial q_{-i}}$ is continuous by A1. By A2, $\eta > 0$.

If $q\left(q_{-i},c_i\right) = 0$, then (27) clearly holds. If $q\left(q_{-i},c_i\right) > 0$, then, by A2, $q\left(\widetilde{q}_{-i},c_i\right) > 0$ for every $\widetilde{q}_{-i} \in [0,q_{-i}]$. By A5:

$$\frac{\frac{\partial q(\widetilde{q}_{-i},c_i)}{\partial q_{-i}}}{q\left(\widetilde{q}_{-i},c_i\right)} < \frac{\frac{\partial q(\widetilde{q}_{-i},0)}{\partial q_{-i}}}{q\left(\widetilde{q}_{-i},0\right)} \leq -\eta$$

Thus for every $q'_{-i} \in (0,q_{-i})$,

$$q\left(q_{-i},c_i\right) - q\left(q'_{-i},c_i\right) = \int_{q'_{-i}}^{q_{-i}}\frac{\partial q\left(\widetilde{q}_{-i},c_i\right)}{\partial q_{-i}}d\widetilde{q}_{-i} \leq -\eta q\left(q_{-i},c_i\right)\left(q_{-i}-q'_{-i}\right)$$

■

Lemma 12. *If* $Q^L(c^*) \geq Q^{H2}(c^*)$*, then*

$$\Delta\Pi(c^*; c^*) \leq \beta q\left(Q^L(c^*), c^*\right)(1 - F(c^*))\left(Q^{H2}(c^*) - \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right) \tag{28}$$

where $\eta > 0$ *satisfies* (27).

Proof. By Lemma 11, there exists $\eta > 0$ such that (27) holds for every c^* and every $q_{-i} \leq q(0,0)$. In particular, since $Q^L(c^*) \leq q(0,0)$, we have that for every $q_{-i} \in \left[Q^{H2}(c^*), Q^L(c^*)\right]$,

$$q(q_{-i}, c^*) \geq q\left(Q^L(c^*), c^*\right) + \eta q\left(Q^L(c^*), c^*\right)\left(Q^L(c^*) - q_{-i}\right)$$

Therefore

$$\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q(q_{-i}, c^*)\, dq_{-i} \geq q\left(Q^L(c^*), c^*\right)\int_{Q^{H2}(c^*)}^{Q^L(c^*)}\left(1 + \eta\left(Q^L(c^*) - q_{-i}\right)\right) dq_{-i}$$

$$= q\left(Q^L(c^*), c^*\right)\left(\left(Q^L(c^*) - Q^{H2}(c^*)\right) + \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right) \tag{29}$$

For every $q_{-i} \in \left[Q^L(c^*), Q^{H1}(c^*)\right]$, $q(q_{-i}, c^*) \leq q\left(Q^L(c^*), c^*\right)$, and thus

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c^*)\, dq_{-i} \leq q\left(Q^L(c^*), c^*\right)\left(Q^{H1}(c^*) - Q^L(c^*)\right) \tag{30}$$

Equations (29) and (30) imply

$$\Delta\Pi(c^*; c^*) = \beta\left(F(c^*)\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c^*)\, dq_{-i} - (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q(q_{-i}, c^*)\, dq_{-i}\right)$$

$$\leq \beta\left(\begin{array}{c} F(c^*)\, q\left(Q^L(c^*), c^*\right)\left(Q^{H1}(c^*) - Q^L(c^*)\right) \\ -(1 - F(c^*))q\left(Q^L(c^*), c^*\right)\left(\left(Q^L(c^*) - Q^{H2}(c^*)\right) + \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right) \end{array}\right)$$

$$= \beta q\left(Q^L(c^*), c^*\right)\left(\begin{array}{c} \left(Q^{H1}(c^*) - Q^L(c^*)\right) \\ -(1 - F(c^*))\left(\left(Q^{H1}(c^*) - Q^{H2}(c^*)\right) + \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right) \end{array}\right)$$

Note that by definition of $Q^{H1}(c^*)$ and $Q^L(c^*)$,

$$Q^{H1}(c^*) - Q^L(c^*) = (1-F(c^*))\left(Q^{H1}(c^*) - \frac{1}{1-F(c^*)}\int_{c^*}^{\infty} q\left(Q^{H1}(c^*), c_i\right) dF(c_i)\right)$$
$$\leq (1-F(c^*))\, Q^{H1}(c^*)$$

Thus

$$\Delta\Pi(c^*; c^*) \leq \beta q\left(Q^L(c^*), c^*\right)(1-F(c^*))\left(Q^{H2}(c^*) - \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right)$$

■

Lemma 13. *Let* $\widehat{c} > 0$ *be such that* $q_i(0, \widehat{c}) \leq \left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}}\right)^2$, *where* $\eta > 0$ *satisfies condition* (27) *(such* $\widehat{c}$ *exists by A6 and the fact that* $Q^L(0) > 0$ *by Lemma 9). If* $F(\widehat{c}) < 1$, *then there exists* $c^* \in (0, \overline{c})$ *such that the "min" mechanism with threshold* c^* *is incentive compatible.*

Proof. By Lemma 10, it is enough to show that there exists $c^* \in (0, \overline{c})$ such that $\Delta\Pi(c^*; c^*) = 0$.

Note that $\Delta\Pi(c_i; c^*)$ is continuous in c_i and c^* (since Π_i is continuous in (q_{-i}, c_i), c_i is continuously distributed, and $Q^L(c^*)$, $Q^{H1}(c^*)$, and $Q^{H2}(c^*)$ are continuous in c^* (Lemmas 8 & 9)). Thus it is enough to show that $\Delta\Pi(0; 0) > 0$, and $\Delta\Pi(c^*; c^*) \leq 0$ for some $c^* \in (0, \overline{c})$.

By Lemmas 8 and 9, $Q^{H2}(0) = Q^{NC} > Q^L(0)$. By A2 and A3, $q\left(Q^L(0), 0\right) \geq q(q(0,0), 0) > 0$. Therefore

$$\Delta\Pi(0; 0) = \Pi_i\left(Q^L(0), 0\right) - \Pi_i\left(Q^{H2}(0), 0\right) = \beta\int_{Q^L(0)}^{Q^{H2}(0)} q(q_{-i}, 0)\, dq_{-i} > 0$$

If $q\left(Q^L(\widehat{c}), \widehat{c}\right) = 0$, then $\Pi_i\left(Q^L(\widehat{c}), \widehat{c}\right) = 0$, and thus $\Delta\Pi(\widehat{c}; \widehat{c}) \leq 0$.

Suppose that $q\left(Q^L(\widehat{c}), \widehat{c}\right) > 0$. Note that $Q^L(\widehat{c}) \geq Q^L(0)$ (Lemma 9), and $Q^{H2}(\widehat{c}) \leq q(0, \widehat{c}) \leq \left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}}\right)^2$ by A2.

Thus

$$Q^L(\widehat{c})-Q^{H2}(\widehat{c}) \geq Q^L(0)-\left(\sqrt{\frac{1}{2\eta}+Q^L(0)}-\sqrt{\frac{1}{2\eta}}\right)^2=\sqrt{\frac{2}{\eta}}\left(\sqrt{\frac{1}{2\eta}+Q^L(0)}-\sqrt{\frac{1}{2\eta}}\right)>0$$

Therefore Lemma 12 applies, and

$$\Delta\Pi(\widehat{c};\widehat{c}) \leq \beta q\left(Q^L(\widehat{c}),\widehat{c}\right)(1-F(\widehat{c}))\left(Q^{H2}(\widehat{c})-\frac{\eta}{2}\left(Q^L(\widehat{c})-Q^{H2}(\widehat{c})\right)^2\right)$$
$$\leq \beta q\left(Q^L(\widehat{c}),\widehat{c}\right)(1-F(\widehat{c}))\left(\left(\sqrt{\frac{1}{2\eta}+Q^L(0)}-\sqrt{\frac{1}{2\eta}}\right)^2-\frac{\eta}{2}\left(\sqrt{\frac{2}{\eta}}\left(\sqrt{\frac{1}{2\eta}+Q^L(0)}-\sqrt{\frac{1}{2\eta}}\right)\right)^2\right)$$
$$=0$$

■

Proof of Theorem 3. Follows from Lemmas 8-13. ■

Proof of Theorem 4. By Lemma 9, $Q^L(c^*) \leq Q^{NC}$; therefore $\pi_i(q_i, Q^L(c^*), c_i) \geq \pi_i(q_i, Q^{NC}, c_i)$, for every $q_i \geq 0$ and $c_i \in [0, \overline{c}]$, and $\pi_i(q_i, Q^L(c^*), c_i) > \pi_i(q_i, Q^{NC}, c_i)$ if $q_i > 0$. This implies that $\Pi_i(Q^L(c^*), c_i) \geq \Pi_i(Q^{NC}, c_i)$.

Consider firm i of type c_i. If $c_i < c^*$ and it reports its type truthfully, its interim expected profit equals $\Pi_i(Q^L(c^*), c_i) \geq \Pi_i(Q^{NC}, c_i)$. If $c_i \geq c^*$ and it reports its type truthfully, its interim expected profit equals $F(c^*)\Pi_i(Q^{H1}(c^*), c_i) + (1 - F(c^*))\Pi_i(Q^{H2}(c^*), c_i) \geq \Pi_i(Q^L(c^*), c_i) \geq \Pi_i(Q^{NC}, c_i)$, where the first inequality follows from the incentive compatibility of the "min" mechanism.

By condition A4, $q(q_{-i}, c_i) > 0$, for every $q_{-i} \in [0, q_i(0, 0)]$, $c_i \in [0, \overline{c}]$. Therefore $q(Q^{NC}, c_i) > 0$, so $\Pi_i(Q^{NC}, c_i) < \pi_i(q_i(Q^{NC}, c_i), Q^L(c^*), c_i) \leq \Pi_i(Q^L(c^*), c_i)$. Thus $\max\left\{\Pi_i(Q^L(c^*), c_i), F(c^*)\Pi_i(Q^{H1}(c^*), c_i) + (1 - F(c^*))\Pi_i(Q^{H2}(c^*), c_i)\right\} > \Pi_i(Q^{NC}, c_i)$, and every type is strictly better off under the "min" mechanism than in the Bayesian-Nash equilibrium without communication. ■

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