Limited enforcement, bubbles and trading in incomplete markets *

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Abstract

Rational bubbles are believed to be fragile and unable to explain the trading frenzy associated to price run-ups. With limited enforcement of credit contracts and endogenous debt limits designed to prevent default and allow for maximal credit expansion, a large class of bubbles can be introduced in asset prices by appropriately tightening agents' debt limits. By not affecting consumption, these bubbles are ideally suited to explain a variety of asset pricing puzzles. They can generate large increases in trade volume until they crash. Nonpositivity of debt limits restricts the potential for bubble injections to assets in zero supply or to equilibria with an infinite present value of aggregate endowment. Such equilibria are common in economies with limited enforcement, where interest rates are low to induce debt repayment (Bidian and Bejan 2012).

*Some of the results in here are based on Chapter 3 in Bidian (2011).

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1 Introduction

A bubble is defined as the price of an asset in excess of the discounted present value of its dividends. While developing (collapsing) price bubbles are a favorite explanation for stock market run-ups (crashes), their existence in standard stochastic dynamic general equilibrium models is possible only under special conditions. Santos and Woodford (1997) showed that bubbles can be ruled out on assets in positive supply, when the present value of aggregate consumption is finite.[1] This is always the case if, for example, there is at least an asset that grows at a (long-run) rate greater or equal to the growth rate of aggregate consumption. The outline of their argument is that an optimizing agent exhausts his financial wealth and does not allow it to exceed the present value of his future consumption. Thus the aggregate financial wealth becomes arbitrarily small in present value terms, which is incompatible with the existence of a bubble on an asset in positive supply. For deterministic economies, the results of Santos and Woodford (1997) were anticipated by Kocherlakota (1992) and later refined by Huang and Werner (2000).[2]

In apparent contradiction with the nonexistence of bubbles results, there exists a well developed literature on *speculative bubbles* in economies with short sale constraints and asymmetric information (Allen, Morris, and Postlewaite 1993) or heterogenous beliefs (Scheinkman and Xiong 2003, Slawski 2008). Such bubbles are possible even with a finite time horizon. Moreover, as emphasized by Scheinkman

[1] They prove that there exists a discount factor (pricing kernel) compatible with the absence of arbitrage opportunities such that the fundamental value of the asset computed under this discount factor equals its price. Moreover, if the agents are sufficiently impatient, in the sense that they are always willing to give up a fixed fraction of all future consumption in exchange for the current aggregate endowment, then the price of an asset in positive supply is always equal to its fundamental value, irrespective of the choice of a discount factor compatible with the absence of arbitrage.

[2] Montrucchio and Privileggi (2001) also show that under mild assumptions on agent's preferences, bubbles cannot exist in a representative agent economy. The absence of bubbles follows even without assuming the existence of a sufficiently productive asset.

and Xiong (2003, 2004), speculative bubbles can generate large volumes of trade, which is a typical feature of bubble episodes (Cochrane 2002) and which, according to them, cannot be explained by *rational bubbles* (bubbles arising without differential information). Crucially, these papers use a different definition of bubbles, which takes as the fundamental value of an asset, following Harrison and Kreps (1978) and Morris (1996), the amount that an agent would be willing to pay if he were forced to maintain the holdings of the asset forever. Such a valuation ignores the "convenience yield" accruing to an agent holding an asset and thus underestimates the fundamental value.[3]

Bidian (2011, Chapter 2) shows that once a unified definition of bubbles is used, speculative bubble also cease to exist under the same conditions under which rational bubbles fail to exist, despite the presence of differential information and short sale constraints. Fundamental values are computed using, whenever possible, discount factors that satisfy the fundamental theorem of asset pricing, that is the asset prices are equal to the expected discounted value of next period dividends and resale price of the assets.

In this paper, we show that rational bubbles are a robust and intrinsic feature of economies where restrictions on debt arise endogenously from enforcement limitations. Agents have the option to default on debt and receive a continuation utility that can be date and state dependent. As in Alvarez and Jermann (2000), we assume that the markets select the largest debt limits so that repayment is always individually rational given future bounds on debt. Rational bubbles enable agents to circumvent tight debt limits and achieve identical allocations to those possible under more relaxed, but still self-enforcing debt limits. Thus, one interpretation of the type of bubbles we construct here is that they develop as a way to correct credit crunches.

We build on the insight of Kocherlakota (2008), who showed that arbitrary bub-

[3] Indeed, with shorting restrictions, an agent keeping inventories of an asset has the option to sell it if its price is high and can better smooth demand shocks, and therefore enjoys a convenience yield (Cochrane 2002). Duffie, Garleanu, and Pedersen (2002) rationalize the convenience yield induced by short sale constraints as the value of lending fees arising from searching for security lenders and bargaining over the terms of lending.

bles can be injected in asset prices, while leaving agents' consumption unchanged, as long as the debt constraints of the agents are allowed to be adjusted upwards by their initial endowment of the assets multiplied with the bubble term. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. Kocherlakota (2008) refers to this result as "the bubble equivalence theorem". The modified debt constraints bind in exactly the same dates and states, and they are again the endogenous bounds allowing for maximal credit expansion and preventing default.

A major limitation of Kocherlakota's (2008) results is the assumption that agents can trade in a full set of state-contingent claims to consumption next period, in addition to the existing long-lived securities. Hence one might infer that bubble injections are associated to knife-edge situations, and they might not apply even to economies with dynamically complete markets (rather than Arrow-Debreu complete). Moreover, a bubble injection influences only agents' holdings of Arrow securities, leaving untouched their portfolios of long-lived securities. Therefore his result cannot justify the trade volume increases associated with the presence of bubbles.

We prove that the bubble equivalence theorem holds even when markets are incomplete. Incomplete markets models with limited enforcement warrant study since they can better reflect the limited extent of risk-sharing in the data (Ábrahám and Cárceles-Poveda 2010). We show that any positive process that does not distort the set of pricing kernels can be injected in the asset prices as a bubble. Gain processes associated to a large class of trading strategies satisfy this condition. We also allow for more general punishments after default than in Kocherlakota (2008). In particular, we cover the case where upon default the agents are forbidden to carry debt (Bulow and Rogoff 1989, Hellwig and Lorenzoni 2009). For this outside option, the agents' continuation utilities after default depend on asset prices, since lending is still allowed.

It should be emphasized that there is no contradiction between the bubble equivalence theorem presented here and the nonexistence of bubbles results. The latter rely on the hidden assumption that the debt limits faced by agents are nonpositive, while the adjusted debt bounds after a bubble injection must become positive eventually, if the asset is in positive supply and the present value of aggregate consumption is finite (interest rates are *high*), even though this may happen with arbitrarily small probability. However, *low interest rates* are the natural result of the existence of enforcement limitations, since in equilibrium the interest rates adjust to a lower level to entice agents to repay their debt and prevent default. In fact, when the default punishment is the interdiction to borrow and markets are complete, Hellwig and Lorenzoni (2009) show that the discounted (that is, multiplied by the pricing kernel) debt limits are martingales, and that the present value of aggregate consumption must be infinite.[4] With complete markets, Bidian and Bejan (2012) show that bubble injections leading to nonpositive debt limits are possible for much more general penalties for default. In particular, bubbles can be sustained in equilibrium also under a permanent interdiction to trade, or just a temporary interdiction to trade.

We show through examples that bubble injections can generate large increases in the volume of trade. First, we analyze the complete markets example studied in Alvarez and Jermann (2001) and Kehoe and Levine (2001), in which agents are not allowed to trade after default. We substitute the one-period Arrow securities used by them with infinitely lived assets that dynamically complete the markets. We focus on two types of dividend structures and on deterministic and stochastic bubbles, and show that a bubble injection in one of the assets can induce a volume of trade increase in all assets, thus causing a large market-wide increase in trade volume. The increase is persistent, even when the bubble lasts arbitrarily long. When the bubble crashes, the volume of trade collapses when compared to the volume levels in the absence of a bubble, and then reverts back to normal. Second, we consider an incomplete markets economy having a Pareto optimal equilibrium. As in Judd, Kubler, and Schmedders (2003), there is no trade after an initial portfolio rebalancing by the agents. A bubble injection generates persistent market-wide increases in the volume of trade.

Bubble injections distort prices (and returns) and can increase their volatility, without affecting consumption (fundamentals). Therefore they are ideally suited to explain the "excess volatility puzzle" - the large volatility of asset prices, with very

[4] A bubble discounted by the pricing kernel is always a nonnegative martingale.

little movements in dividends or consumption (Shiller 1981, LeRoy and Porter 1981). We show that introducing a bubble in an asset increases the conditional expected return, respectively conditional Sharpe ratio of the asset if the rate of growth of the bubble *covaries*, respectively is *correlated more negatively* to fundamentals (the stochastic discount factor) than the initial asset return. Thus bubbles can help explaining the "equity premium puzzle" (Cochrane 2000, Chapter 21). The variability of Sharpe ratios over time without an accompanying variability of the volatility of consumption (the "conditional equity premium puzzle") would also be an immediate consequence of a bubble injection (Cochrane 2000, Chapter 21).

Rational bubbles do not have to be nonstationary, as commonly believed, and therefore at odds with empirical observations. With low interest rates, bubbles can grow at the rate of aggregate endowment and be stationary. Such examples are constructed in Hellwig and Lorenzoni (2009) and Bidian and Bejan (2012). Bidian (2011, Chapter 5) gives an example of a (strictly) stationary bubble for economies with an arbitrary pricing kernel. As explained there, such bubbles are virtually undetectable by standard stationarity tests.

The paper is organized as follows. Section 2 presents the model and the bubble equivalence theorem. Section 3 analyzes the implications of bubble injections on volume of trade and asset returns, and Section 4 concludes. Appendix A gives necessary and sufficient conditions on a process which, if added to asset prices, will not distort the pricing kernels (and the one-period asset spans). Appendix B shows that gain processes associated to a large class of strategies satisfy those conditions.

2 Bubble injections

We consider a stochastic, discrete-time, infinite horizon economy. The time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of σ-algebras on the set of states of the world Ω, generating $\mathcal{F}$, that is such that $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$. We interpret $\mathcal{F}_t$ as the information available at period t. We assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and that $\mathcal{F}_t$ is a finite σ-algebra, for all t. For $\omega \in \Omega$ and $t \in \mathbb{N}$, the set of states that are known to be possible at t if the true state is ω is $\mathcal{F}_t(\omega) := \cap\{A \in \mathcal{F}_t \mid \omega \in A\}$.[5]

A sequence $x = (x_t)_{t\in\mathbb{N}}$ of random variables ($\mathcal{F}$-measurable real-valued functions) is a *stochastic process adapted to* $(\mathcal{F}_t)_{t=0}^{\infty}$ ("process" henceforth) if for each $t \in \mathbb{N}$, x_t is $\mathcal{F}_t$-measurable. We let X be the set of all stochastic processes, and denote by X_+ (respectively X_{++}) the processes $x \in X$ such that $x_t \geq 0$ P-almost surely (respectively $x_t > 0$ P-almost surely) for all $t \in \mathbb{N}$. All statements, equalities, and inequalities involving random variables are assumed to hold only "P-almost surely", and we will omit adding this qualifier. When $K, L \in \mathbb{N} \setminus \{0\}$, let $X^{K\times L}$ be the set of vector (or matrix) processes $(x^{ij})_{1\leq i\leq K, 1\leq j\leq L}$ with $x^{ij} \in X$. For $x \in X^{K\times L}$, we write $x \geq 0$ (respectively $x > 0, x = 0$) if for all $1 \leq i \leq K, 1 \leq j \leq L$ and $t \in \mathbb{N}$, $x_t^{ij} \geq 0$ (respectively $x_t^{ij} > 0, x_t^{ij} = 0$).[6] The set of nonnegative processes $x \in X^{K\times L}$ (that is, such that $x \geq 0$) is denoted by $X_+^{K\times L}$.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has endowments $e^i \in X_+$, and his preferences are represented by a utility $U : X_+ \to \mathbb{R}$ given by $U^i(c) = E\sum_{t=0}^{\infty} u_t^i(c_t)$, where $u_t^i : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing and concave and $E(\cdot)$ is the expectation operator with respect to probability P. The conditional expectation given the information available at t, $\mathcal{F}_t$, is denoted by $E_t(\cdot)$. Since there is no information at period 0, $E_0(\cdot) = E(\cdot)$. The continuation utility of agent i at t provided by a consumption stream $c \in X_+$ is $U_t^i(c) := E_t \sum_{s\geq t} u_s^i(c_s)$.

There is a finite number J of infinitely lived, disposable securities, traded at every date. The dividend and price vector processes are $d = (d^1, \ldots, d^J) \in X_+^{1\times J}$ and $p = (p^1, \ldots, p^J) \in X_{++}^{1\times J}$.

Consumer i has an initial endowment $\theta_{-1}^i \in \mathbb{R}_+^J$ of securities and his trading strategy is represented by a process $\theta^i \in X^{J\times 1}$. Fix some wealth bounds $w^i \in X$ for agent i and define the budget constraint and indirect utility of an agent i from

[5] Using the usual "event tree" terminology, $\mathcal{F}_t(\omega)$ is the date t node containing state ("leaf") ω (for the parallel between the stochastic processes and event tree language, see Leroy and Werner 2001, chapter 21).

[6] We write $x \neq 0$ if there exists t, i, j such that $x_t^{i,j} = 0$ does not hold (that is, $x_t^{i,j}$ differs from zero on a set of positive probability). Similarly $x \gneqq 0$ means that $x \geq 0$ but $x \neq 0$.

period $s \geq 0$ onward, when faced with prices $p \in X_+^{1\times J}$, debt bounds $w^i \in X$ and having an initial wealth $\nu_s : \Omega \to \mathbb{R}$ which is $\mathcal{F}_s$-measurable, as

$$B_s^i(\nu_s, w^i, p) = \{(c^i, \theta^i) \in X_+ \times X^{J\times 1} \mid c_s^i + p_s\theta_s^i \leq e_s^i + \nu_s, \tag{2.1}$$
$$c_t^i + p_t\theta_t^i \leq e_t^i + (p_t + d_t)\theta_{t-1}^i, (p_t + d_t)\theta_{t-1}^i \geq w_t^i, \forall t > s\},$$
$$V_s^i(\nu_s, w, p) = \max_{(c^i,\theta^i)\in B_s^i(\nu_s,w^i,p)} U_s^i(c^i). \tag{2.2}$$

Definition 2.1. *A vector $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I\right)$ consisting of a security price process $\bar{p} \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, debt limits $\bar{w}^i \in X$, a consumption process $\bar{c}^i \in X_+$ and a trading strategy $\bar{\theta}^i \in X^{J\times 1}$ is an equilibrium with exogenous debt limits if the following conditions are met:*[7]

i. The consumption and trading strategies of each agent i are feasible and optimal, that is $(\bar{c}^i, \bar{\theta}^i) \in B_0((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p})$ and $U^i(\bar{c}^i) = V_0^i\left((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p}, d\right)$.

ii. Markets clear: $\sum_{i=1}^I \bar{c}_t^i = \sum_{i=1}^I e_t^i + d_t \cdot \sum_{i=1}^I \theta_{-1}^i, \quad \sum_{i=1}^I \theta_t^i = \sum_{i=1}^I \theta_{-1}^i, \ \forall t \in \mathbb{N}$.

Consider an equilibrium $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ with exogenous debt bounds. Since the utilities of the agents are strictly increasing in consumption at each date and state, prices p exclude arbitrage opportunities. Thus there cannot exist $\theta \in X^{J\times 1}$ and $t \in \mathbb{N}$ such that $p_t\theta_t \leq 0$ and $(p_{t+1} + d_{t+1})\theta_t \geq 0$, with at least one inequality being strict on a set of positive probability. Otherwise consumer i would alter his portfolio θ_t^i at t by adding to it the strategy θ_t, guaranteeing an increase in his consumption at t and $t + 1$, and a strict increase in one of the periods, with positive probability. This modified strategy still satisfies the debt constraints. The absence of arbitrage opportunities is equivalent to the existence of a process $a \in X_{++}$ such that (Leroy and Werner 2001)

$$a_t p_t = E_t\left[a_{t+1}(p_{t+1} + d_{t+1})\right], \forall t \geq 0. \tag{2.3}$$

We denote by $A(p)$ the set of all processes $a \in X$ satisfying equation (2.3), and we

[7] Although debt limits are exogenous up to this point, we include them in the equilibrium outcome for ease of exposition.

call them *deflators*. Strictly positive deflators belonging to $A_{++}(p) := A(p) \cap X_{++}$ will be called *state price densities*, or (interchangeably) *pricing kernels*. Equation (2.3) implies that $p_t = \frac{1}{a_t} E_t \sum_{s>t} a_s d_s + \lim_{T\to\infty} \frac{1}{a_t} E_t a_T p_T$, and

$$b_t(a,p) := \frac{1}{a_t} \lim_{T\to\infty} E_t a_T p_T \tag{2.4}$$

is well defined and nonnegative, and for all $t \in \mathbb{N}$, $a_t b_t(a,p) = E_t a_{t+1} b_{t+1}(a,p)$. Therefore $a \cdot b(a,p)$ is a nonnegative martingale, and $b(a,p) = 0$ if and only if $b_0(a,p) = \frac{1}{a_0} \lim_{t\to\infty} E a_t p_t = 0$. We interpret the discounted present value of dividends d under under the state price density a, that is $f_t(a) := \frac{1}{a_t} E_t \sum_{s>t} a_s d_s$, as the *fundamental value* of d. Hence $b(a,p)$ represents the part of asset prices in excess of fundamental values. Following Santos and Woodford (1997), we say that the equilibrium price process p *ambiguously involves a bubble* if $b_0(a,p) > 0$ for some $a \in A_{++}(p)$, while $b_0(a',p) = 0$ for some other $a' \in A_{++}(p)$. If $b_0(a,p) > 0$ for all $a \in A_{++}(p)$, the equilibrium prices *unambiguously involves a bubble* component.

Kocherlakota (2008) assumed that in addition to trading in long-lived securities, agents can also trade in each period a full set of state-contingent claims to consumption next period. Given an equilibrium without bubbles in which the asset prices are p and the state price density[8] is a, and given an arbitrary process $\varepsilon \in X_+^{1\times J}$ such that $a \cdot \varepsilon$ is a martingale, he showed that an "equivalent" equilibrium with prices $p + \varepsilon$, pricing kernel a and identical consumption paths for the agents can be constructed. Moreover, in the new equilibrium, the debt constraints bind in exactly the same dates and states as in the original equilibrium. He dubbed this result the "bubble equivalence theorem", since the process ε "injected" in the asset prices is the bubble component for the price process $p + \varepsilon$, that is $\varepsilon = b(a, p + \varepsilon)$.

We show that Kocherlakota's (2008) bubble equivalence theorem holds in our incomplete markets framework, if the candidate processes to be injected in asset

[8]The pricing kernel is unique when markets are complete .

prices are nonnegative processes in the set of *span-preserving martingales*

$$M^J(p) := \big\{\varepsilon \in X^{1\times J} \mid \exists \Lambda \in X^{J\times J} \text{ s.t. } \forall t \geq 0,\ \varepsilon_t = p_t\Lambda_t,$$
$$\varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t,\ \text{and}\ \det(\mathbf{I} + \Lambda_t) \neq 0\big\}\,, \tag{2.5}$$

where $\mathbf{I}$ denotes the J-dimensional identity matrix, and $\det(\cdot)$ is the determinant of a matrix. Let $M^J_+(p) := M^J(p) \cap X^{1\times J}_+$. The chosen terminology is justified in Appendix A, Proposition A.2, where it is shown that a process ε belongs to $M^J(p)$ if and only if the set of deflators associated to prices p and $p+\varepsilon$ coincide, that is $A(p) = A(p+\varepsilon)$, or equivalently, if and only if p and $p+\varepsilon$ generate the same one-period asset spans and $a \cdot \varepsilon$ is a martingale for any deflator $a \in A(p)$. Two equilibria with prices p and $p+\varepsilon$ cannot be equivalent *unless* $\varepsilon \in M^J_+(p)$, otherwise the set of pricing kernels would differ, $A(p) \neq A(p+\varepsilon)$. We will show in fact that *any* $\varepsilon \in M^J_+(p)$ can be injected as a bubble in prices and leading to an equivalent equilibrium. The question whether the set $M^J_+(p)$ is nonempty is legitimate. In Appendix B, Proposition B.1, we give sufficient conditions under which the (vector) gain process of J trading strategies belongs to $M^J_+(p)$. In particular, we show that the gain process associated with a buy-and-hold portfolio of the shares of the J assets belongs to $M^J_+(p)$, when the gains are discounted using the returns on arbitrary strategies with long positions in the assets. Also as a special case, gain process vectors with only one nonzero component belong to the set $M^J_+(p)$ under very mild conditions (they correspond to a bubble in only one of the assets). Therefore $M^J_+(p)$ is a large set. For any $\varepsilon \in M^J(p)$, the set of portfolios "spanning" ε is denoted by

$$\Lambda(\varepsilon, p) := \big\{\Lambda \in X^{J\times J} \mid \forall t \geq 0, \varepsilon_t = p_t\Lambda_t, \varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t, \det(\mathbf{I} + \Lambda_t) \neq 0\big\}\,.$$

We prove first that agents' feasible consumption paths remain unchanged when prices are inflated by a bubble in $M^J_+(p)$, if the debt limits are tightened appropriately.

Proposition 2.1. *Consider an agent i starting period t with wealth equal to ν_t ($\mathcal{F}_t$-*

measurable). Then for any $\bar{\theta}_{-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_t$-measurable and any $\varepsilon \in M_+^J(p)$,

$$(c^i, \theta^i) \in B_t^i\left(\nu_t, w^i, p, d\right) \Longleftrightarrow (c^i, \hat{\theta}^i) \in B_t^i\left(\nu_t + \varepsilon_t\bar{\theta}_{-1}, w^i + \varepsilon\bar{\theta}_{-1}, p + \varepsilon, d\right),$$

where $\hat{\theta}_s^i = (\mathbf{I} + \Lambda_s)^{-1}\left(\theta_s^i + \Lambda_s\bar{\theta}_{-1}\right)$ for every $s \geq t$, and $\Lambda \in \Lambda(\varepsilon, p)$.

Proof. By Lemma A.1, $\varepsilon_s = p_s\Lambda_s$, for all $s \geq t$. It follows that

$$\nu_t + \varepsilon_t\bar{\theta}_{-1} - (p_t + \varepsilon_t)\hat{\theta}_t^i = \nu_t + \varepsilon_t\bar{\theta}_{-1} - p_t\theta_t^i - p_t\Lambda_t\bar{\theta}_{-1} = \nu_t - p_t\theta_t^i$$

and for $s \geq t+1$, $(p_s + \varepsilon_s)\hat{\theta}_s^i = p_s(\theta_s^i + \Lambda_s\bar{\theta}_{-1}) = p_s\theta_s^i + \varepsilon_s\bar{\theta}_{-1}$ and

$$(p_s + d_s + \varepsilon_s)\hat{\theta}_{s-1}^i = (p_s + d_s)(\theta_{s-1}^i + \Lambda_{s-1}\bar{\theta}_{-1}) = (p_s + d_s)\theta_{s-1}^i + \varepsilon_s\bar{\theta}_{-1}.$$

Therefore for $s \geq t+1$, $(p_s+d_s+\varepsilon_s)\hat{\theta}_{s-1}^i - (p_s+\varepsilon_s)\hat{\theta}_s^i = (p_s+d_s)\theta_{s-1}^i - p_s\theta_s^i$. Moreover, $(p_s + d_s + \varepsilon_s)\hat{\theta}_{s-1}^i \geq w_s^i + \varepsilon_s\theta_{-1}^i$ if and only if $(p_s + d_s)\theta_{s-1}^i \geq w_s^i$. □

The intuition for the proposition is as follows. With bubble-inflated prices, the initial owners of the asset receive a windfall in the form of higher initial wealth. Tightening their future debt bounds by the bubble weighted by initial asset holdings will force them to save the initial windfall in order to meet the more stringent borrowing requirements, leading thus to equivalent budget constraints.

Given an equilibrium with asset prices p which do not contain bubbles, for any process $\varepsilon \in M_+^J(p)$, we show that there is an equivalent equilibrium with prices $p+\varepsilon$, identical consumption and state price densities, and in which the debt constraints bind in exactly the same date and states (even though they differ). Moreover ε is the bubble component in the prices $p+\varepsilon$ for any state price density $a \in A(p+\varepsilon)(= A(p))$, that is $\varepsilon = b(a, p+\varepsilon)$, hence the new equilibrium unambiguously involves a bubble.

Theorem 2.2. *Let $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ be an equilibrium (with exogenous debt limits) and without bubbles. Choose $\varepsilon \in M_+^J(p)$ and $\Lambda \in \Lambda(\varepsilon, p)$. Then $\left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i, \hat{\theta}^i)_{i=1}^I\right)$ is an equilibrium with (unambiguous) bubble ε, where*

$$\hat{p} = p + \varepsilon, \quad \hat{\theta}^i = (\mathbf{I} + \Lambda)^{-1}\left(\theta^i + \Lambda\theta_{-1}^i\right), \quad \hat{w}^i = w^i + \varepsilon\theta_{-1}^i. \tag{2.6}$$

Proof. Optimality of $(c^i, \hat{\theta}^i)$ in the set $B_0^i((\hat{p}_0 + d_0)\theta_{-1}^i, \hat{w}, \hat{p})$ follows from the optimality of (c^i, θ^i) in $B_0^i((p_0 + d_0)\theta_{-1}^i, w, p)$, and the equality of these two budgets (Proposition 2.1). Notice that $\sum_i \hat{\theta}_t^i = (\mathbf{I} + \Lambda)^{-1}(\mathbf{1} + \Lambda\mathbf{1}) = \mathbf{1}$, since $\sum_i \theta_t^i = \sum_i \theta_{-1}^i = \mathbf{1}$. Thus the market clearing conditions are satisfied. □

Proposition 2.1 and Theorem 2.2 are valid, without changes, if agents are subject to borrowing constraints rather than debt constraints, that is if the constraints $(p_t + d_t)\theta_{t-1}^i \geq w_t^i$ in (2.1) are replaced by $p_t\theta_t^i \geq w_t^i$. This follows, with the notation in (2.6), from the identity $\hat{p}_t\hat{\theta}_t^i = p_t(\mathbf{I} + \Lambda_t)(\mathbf{I} + \Lambda_t)^{-1}(\theta_t^i + \Lambda_t\theta_{-1}^i) = p_t\theta_t^i + \varepsilon_t\theta_{-1}^i$, where we used twice the equality $\varepsilon_t = p_t\Lambda_t$.

The "bubble equivalence" theorem above compares equilibria with different debt constraints. This seems artificial, if the debt limits are viewed as exogenously given. We allow for the *endogenous* determination of debt constraints driven by limited commitment/imperfect enforcement as in Alvarez and Jermann (2000), and show that the bubble inflated debt bounds in the equivalent equilibrium are also compatible with the endogenous mechanism determining debt limits.

Assume that at any period t, when facing prices p (and dividends d), consumer i can choose to default on his beginning of period debt[9] and leave the economy, receiving a continuation utility after default $\widetilde{V}_t^i(p)$ ($\mathcal{F}_t$-measurable). We allow this continuation utility to depend on exogenous variables such as endowments and dividends, but we make explicit only the functional dependence on prices, which are endogenous. Thus the *default penalty* for each agent i is described by a mapping $\widetilde{V}^i : X_+^{1\times J} \to X$. Alvarez and Jermann (2000), following Kehoe and Levine (1993), worked under the assumption that agents are banned from trading following default, hence for each agent i,

$$\widetilde{V}_t^i(p) := U_t^i(e^i). \tag{2.7}$$

Alternatively, Hellwig and Lorenzoni (2009), building on the work of Bulow and Rogoff (1989), assume that agents are subject to a milder punishment than (2.7).

[9]This is equal to $(p_t + d_t)\theta_{t-1}^i$ if his trading strategy is $\theta^i \in X^{J\times 1}$.

Agents can continue to lend but not to borrow following default,

$$\widetilde{V}_t^i(p) := V_t^i(0, 0, p), \tag{2.8}$$

where the second argument in $V_t^i(0, 0, p)$ is the process in X identically equal to zero. As in Alvarez and Jermann (2000), the option to default endogenizes the debt limits to the maximum level so that repayment is always individually rational given future debt limits. This leads to the notion of debt limits that are *not-too-tight*.

Definition 2.2. *Debt limits w^i faced by agent i are not-too-tight (NTT) given prices p and penalties $\widetilde{V}^i : X_+^{1\times J} \times X_+^{1\times J} \to X$ if $V_t^i(w_t^i, w^i, p) = \widetilde{V}_t^i(p), \forall t$.*

The definition captures the idea that the bounds w^i have to be "tight enough" to prevent default, that is to be "self-enforcing" ($V_t^i(w_t^i, w^i, p) \geq \widetilde{V}_t^i(p)$), but they should allow for maximum credit expansion (thus one should not have $V_t^i(w_t^i, w^i, p) > \widetilde{V}_t^i(p)$ on a positive probability set). One can envision the NTT debt limits as being set by competitive financial intermediaries, with agents unable to trade directly with each other. The intermediaries set debt limits such that default is prevented, but credit is not restricted unnecessarily, since competing intermediaries could relax them and increase their profits (see Ábrahám and Cárceles-Poveda (2010) for such a model in an economy with production).

We extend our definition of equilibrium to allow for the endogenous determination of debt constraints, in the presence of an outside option to default. An *Alvarez-Jermann equilibrium* (AJ-equilibrium, for short) $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ consists of a security price process $\bar{p} \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, debt limits $\bar{w}^i \in X$, a consumption process $\bar{c}^i \in X_+$, a trading strategy $\bar{\theta}^i \in X^{J\times 1}$ and a mapping $\widetilde{V}^i$ from prices and dividends into continuation utilities after default such that $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I\right)$ is an equilibrium (with exogenous debt limits), and $\bar{w}^i$ are not-too-tight given penalties $\widetilde{V}^i(\bar{p})$ for default.

Existence of AJ-equilibria is a delicate problem, due to the presence of incomplete markets, real (long-lived) securities and infinite horizon, which creates existence problems even for equilibria with exogenous debt limits as in definition 2.1.[10] When markets are complete and the punishment for default is given by (2.7), the existence of the AJ-equilibrium is established by Kehoe and Levine (1993) and Alvarez and Jermann (2000). With incomplete markets, Hernandez and Santos (1996) show that in our environment, an equilibrium with exogenous debt limits exists for a dense subset of endowment and dividend processes, if agents are "sufficiently impatient" (see footnote 1), have a nonnegative initial holding of securities, and if their debt is restricted by the present value of future endowments,[11]

$$w_t^i = -\inf_{a \in A_{++}(p)} E_t \sum_{s \geq t} \frac{a_s}{a_t} e_s^i. \tag{2.9}$$

The debt limits in (2.9) are chosen equal to the maximum amount that an agent can borrow, if he must hold nonnegative wealth after some finite date. With complete markets, they are the NTT debt limits when the punishment for default is the confiscation of endowment, as it can be seen immediately.

An equilibrium $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ with exogenous debt limits, can be transformed into an AJ-equilibrium by appropriately choosing the continuation utilities. Indeed, $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ with $\widetilde{V}_t^i(p) := V_t^i(w_t^i, w^i, p)$ is trivially an AJ-equilibrium (if the indirect utility $V_t^i(w_t^i, w^i, p)$ is well defined). Similarly, by setting $\widetilde{V}_t^i := U_t^i(c^i)$ and $\bar{w}_t^i := (p_t + d_t)\theta_{t-1}^i$, the initial equilibrium with exogenous debt bounds becomes an AJ-equilibrium.

[10]The dependence of the rank of the matrix of returns (at each date and state) on asset prices can create discontinuities in demand and lead to existence failures (for a two period environment where an equilibrium does not exist, due to the "drop in rank" problem, see Hart 1975).

[11]Hernandez and Santos (1996) actually work with *borrowing constraints* that limit end of period wealth, that is an agent's i trading strategy θ^i must satisfy

$$p_t \theta_t^i \geq -\inf_{a \in A_{++}(p)} E_t \sum_{s \geq t+1} \frac{a_s}{a_t} e_s^i,$$

when faced with prices p. Florenzano and Gourdel (1993) show that agents subject to these borrowing constraints have identical budget constraints to the situation where they are subjected to *debt constraints* $(p_t + d_t)\theta_{t-1}^i \geq w_t^i$ that limit the beginning of period wealth, with w^i given by (2.9). Therefore all the results of Hernandez and Santos (1996) apply to the corresponding environment with debt constraints (2.9).

We show next that bubble injections as in Theorem 2.2 preserve the NTT condition on debt limits, under a mild assumption on the form of penalties for default.

Theorem 2.3. *Let* $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *be an AJ-equilibrium. Choose* $\varepsilon \in M_+^J(p)$ *and* $\Lambda \in \Lambda(\varepsilon, p)$*. If* $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$ *for all agents* $i \in \{1, \dots, I\}$*, then* $\left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i, \hat{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *is an AJ-equilibrium, with* $\hat{p}, \hat{\theta}^i, \hat{w}^i$ *given by* (2.6).

Proof. By Proposition 2.1, $\hat{w}^i := w^i + \varepsilon\theta_{-1}^i$ are not-too-tight for prices $\hat{p}$, since

$$\widetilde{V}_t^i(\hat{p}) = \widetilde{V}_t^i(p) = V_t^i(w_t^i, w^i, p) = V_t^i(w_t^i + \varepsilon_t\theta_{-1}^i, w^i + \varepsilon\theta_{-1}^i, p+\varepsilon) = V_t^i(\hat{w}_t^i, \hat{w}^i, \hat{p}).$$

The conclusion follows from Theorem 2.2. $\square$

Condition $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$ holds when the continuation utilities after default are of the form (2.7) since in this case $\widetilde{V}^i$ does not depend on prices. It holds also for the penalties (2.8). In fact, assume more generally that after default agent i is subjected to some exogenous debt limits $\widetilde{w}^i$ (equal to zero for the case in (2.8)). By Proposition 2.1, with $\nu_t := \widetilde{w}_t^i$, $\theta_{-1} := 0 \in \mathbb{R}^J$ and $w^i := 0$, $V_t^i(\widetilde{w}_t^i + \varepsilon_t \cdot 0, \widetilde{w}^i, p+\varepsilon) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p)$, and therefore

$$\widetilde{V}_t^i(p+\varepsilon) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p+\varepsilon) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p) = \widetilde{V}_t^i(p). \tag{2.10}$$

In order for the modified debt limits $(\hat{w}^i)$ of the bubble-equivalent equilibrium of Theorem 2.3 to remain nonpositive (assuming that the initial constraints (w^i) were nonpositive), it must be the case that the present value of the aggregate consumption is infinite under at least one pricing kernel. This follows by adapting Santos and Woodford's (1997) results to our framework with debt constraints rather than borrowing constraints (Bidian 2011, Chapter 2). Infinite present value of consumption caused by low interest rates is the natural result of the existence of enforcement limitations, since in equilibrium interest rates adjust to a lower level to induce agents to repay their debt and prevent default.

In fact, when the default punishment is the interdiction to borrow and markets are complete, Hellwig and Lorenzoni (2009) show that the discounted (that is, multiplied by the pricing kernel) debt limits are martingales, and that the present value of aggregate consumption must be infinite. Moreover, Bidian and Bejan (2012) prove that bubble injections leading to nonpositive debt limits are possible for much more general penalties for default, where agents are allowed to borrow arbitrary fractions (possible zero) of their endowments upon default. They also show that bubbles can also be sustained in equilibrium under a permanent interdiction to trade (Kehoe and Levine 1993, Alvarez and Jermann 2000), or just a one-period interdiction to trade. However, an interdiction to trade as punishment for default does not preclude equilibria with high interest rates, where bubble injections with nonpositive debt limits cannot exist. In fact, these were the equilibria that the previous literature almost exclusively focused on (Kehoe and Levine 1993, Kehoe and Levine 2001, Alvarez and Jermann 2000, Krueger and Perri 2006, etc). The exceptions are Antinolfi, Azariadis, and Bullard (2007) and Bloise, Reichlin, and Tirelli (2009).

While the injection of the bubble leaves agents' consumption unchanged, it affects asset prices and returns, and the volume of trade. Therefore bubbles can reconcile the high volatility of prices relative to consumption and dividends (the "excess volatility puzzle") and can induce high risk premia and high and time varying Sharpe ratios (the unconditional/conditional "equity premium puzzle"). The injection of a bubble can generate large increases in the volume of trade, feat that rational bubbles were thought unable to accomplish, as explained in the introduction. These ideas are pursued next.

3 Effects of bubbles on returns and trade volume

We investigate the effect of a bubble injection on returns and trade volumes. We compare the two "equivalent" equilibria of Theorem 2.2, the bubble-free equilibrium $\left(p,(w^i)_{i=1}^I,(c^i)_{i=1}^I,(\theta^i)_{i=1}^I\right)$ and the bubbly equilibrium $\left(\hat{p},(\hat{w}^i)_{i=1}^I,(c^i,\hat{\theta}^i)_{i=1}^I\right)$. For concreteness, we focus on the case when a bubble is introduced in the first asset, that is $\varepsilon=(\varepsilon^1,0,\ldots,0)\in X_+^{1\times J}$. The period $t+1$ gross returns on the first asset in the two equilibria are $R_{t+1}^1:=(p_{t+1}^1+d_{t+1}^1)/p_t^1$, $\hat{R}_{t+1}^1:=(\hat{p}_{t+1}^1+d_{t+1}^1)/\hat{p}_t^1$. Since $\hat{p}^1=p^1+\varepsilon^1$, the return in the bubbly equilibrium is a convex combination of the initial return and the rate of growth of the bubble,

$$\hat{R}^1_{t+1} = \frac{p^1_t}{\hat{p}^1_t} R^1_{t+1} + \left(1 - \frac{p^1_t}{\hat{p}^1_t}\right) \frac{\varepsilon^1_{t+1}}{\varepsilon^1_t}. \tag{3.1}$$

We show that a bubble injection on an asset increases its risk premium, respectively Sharpe ratio, if the bubble growth rate has a higher risk premium, respectively higher Sharpe ratio. If Z_1, Z_2 are F_{t+1}-measurable, $Cov_t(Z_1, Z_2)$ and $\rho_t(Z_1, Z_2)$ are their conditional covariance and conditional correlation given $\mathcal{F}_t$, while $\sigma_t(Z_1)$ is the conditional standard deviation of Z_1 given $\mathcal{F}_t$.

Proposition 3.1. *Let $a \in A_{++}(p)$ and let $m_{t+1} := a_{t+1}/a_t$ be the associated stochastic discount factor and $R^f_{t+1} := 1/E_t m_{t+1}$ the risk free rate. The following hold:*

(i)

$$E_t \hat{R}^1_{t+1} - R^f_{t+1} \geq E_t R^1_{t+1} - R^f_{t+1}\ (\geq 0) \Leftrightarrow E_t \frac{\varepsilon^1_{t+1}}{\varepsilon^1_t} - R^f_{t+1} \geq E_t R^1_{t+1} - R^f_{t+1}\ (\geq 0)$$
$$\Leftrightarrow Cov_t\left(m_{t+1}, \frac{\varepsilon^1_{t+1}}{\varepsilon^1_t}\right) \leq Cov_t(m_{t+1}, R^1_{t+1})\ (\leq 0).$$

(ii)

$$\frac{E_t \frac{\varepsilon^1_{t+1}}{\varepsilon^1_t} - R^f_{t+1}}{\sigma_t\left(\frac{\varepsilon^1_{t+1}}{\varepsilon^1_t}\right)} \geq \frac{E_t R^1_{t+1} - R^f_{t+1}}{\sigma(R^1_{t+1})} \geq 0 \Leftrightarrow \rho_t\left(m_{t+1}, \frac{\varepsilon^1_{t+1}}{\varepsilon^1_t}\right) \leq \rho_t(m_{t+1}, R_{t+1}) \leq 0 \Rightarrow$$
$$\Rightarrow \frac{E_t \hat{R}^1_{t+1} - R^f_{t+1}}{\sigma_t(\hat{R}^1_{t+1})} \geq \frac{E_t R^1_{t+1} - R^f_{t+1}}{\sigma_t(R^1_{t+1})} \geq 0.$$

Proof. For an arbitrary return R_{t+1} and in particular for the bubble growth rate $\varepsilon^1_{t+1}/\varepsilon^1_t$, the (conditional) risk premium and Sharpe ratio satisfy

$$E_t(R_{t+1} - R^f_{t+1}) = -R^f_{t+1} Cov_t(m_{t+1}, R_{t+1}), \tag{3.2}$$
$$\frac{E_t R_{t+1} - R^f_{t+1}}{\sigma_t(R_{t+1})} = \frac{-R^f_{t+1} Cov_t(m_{t+1}, R_{t+1})}{\sigma_t(R_{t+1})} = -R^f_{t+1} \sigma_t(m_{t+1}) \cdot \rho_t(m_{t+1}, R_{t+1}). \tag{3.3}$$

The first part now follows from (3.1), while the second part follows from

$$\sigma_t(\hat{R}^1_{t+1}) \leq \frac{p^1_t}{\hat{p}^1_t}\sigma_t(R^1_{t+1}) + \left(1 - \frac{p^1_t}{\hat{p}^1_t}\right)\sigma_t\left(\frac{\varepsilon^1_{t+1}}{\varepsilon^1_t}\right).$$

□

We proved that a bubble injected in an asset increases the conditional expected return, respectively conditional Sharpe ratio of the asset if the rate of growth of the bubble covaries, respectively is correlated *more negatively* to fundamentals (the stochastic discount factor) than the initial asset return. Such bubbles can help explain the *equity premium puzzle*, as they increase the Sharpe ratios without affecting agents' consumption. By (3.1) and (3.3), the conditional Sharpe ratio of the bubble-inflated asset varies over time even when the conditional expectations $E_t R_{t+1}, E_t \varepsilon^1_{t+1}/\varepsilon_t$ and conditional volatilities $\sigma_t R_{t+1}, \sigma_t\left(\varepsilon^1_{t+1}/\varepsilon_t\right)$ of the initial return and bubble rate of growth are constant, as the "weight" $p_t/\hat{p}_t$ of the initial return in the bubbly return varies with the size of the bubble. This variability of the Sharpe ratio over time without an accompanying variability of the volatility of consumption represents a resolution to the *conditional equity premium puzzle* (Cochrane 2000, Chapter 21). As bubble injections distort prices and can increase their volatility, without affecting fundamentals, they also provide a natural explanation for the *excess volatility puzzle* (Shiller 1981). Indeed,

$$\frac{\sigma^2_t(\hat{p}_{t+1})}{\sigma^2_t(p_{t+1})} = 1 + \frac{\sigma_t(\varepsilon_{t+1})}{\sigma_t(p_{t+1})}\left(\frac{\sigma_t(\varepsilon_{t+1})}{\sigma_t(p_{t+1})} + 2\rho_t(\varepsilon_{t+1}, p_{t+1})\right),$$

and therefore the (conditional) volatility of prices increases whenever the (conditional) volatility of the bubble is high enough (it is sufficient to be twice as high as volatility of prices), or if the correlation between the bubble and prices is high enough (it is sufficient to be positive).

We focus next on the volume of trade effects of bubble injections. As mentioned in the introduction, bubbles in an asset are typically associated with large increases in the volume of trading in that asset (Cochrane 2002). In the bubble-free equilibrium, the number of shares of each asset bought, respectively sold, by agent i at t is $(\theta_t^i - \theta_{t-1}^i)^+$, respectively $(\theta_t^i - \theta_{t-1}^i)^-$ (the positive part and the negative part of the change in portfolio are applied component-wise). Notice that the total number of shares of each asset bought and sold at t are equal, since

$$\sum_i (\theta_t^i - \theta_{t-1}^i)^+ = \sum_i (\theta_t^i - \theta_{t-1}^i)^- = \frac{1}{2}\sum_i |\theta_t^i - \theta_{t-1}^i|.$$

Thus we can measure the *share volume of trade* at t in each asset as

$$SV_t = (SV_t^1, \ldots, SV_t^J)' := \frac{1}{2}\sum_i |\theta_t^i - \theta_{t-1}^i|, \tag{3.4}$$

and the *dollar volume of trade* for asset j as $DV_t^j := p_t^j SV_t^j$. The share and dollar volume of trade in the bubbly equilibrium are $\hat{SV}_t := \frac{1}{2}\sum_i |\hat\theta_t^i - \hat\theta_{t-1}^i|$, $\hat{DV}_t^j := (p_t^j + \varepsilon_t^j)\hat{SV}_t^j, \forall j$, where $\hat\theta_t^i = (\mathbf{I} + \Lambda_t)^{-1}(\theta_t^i + \Lambda_t \theta_{-1}^i)$, for all $t \geq -1$ and $\Lambda \in \Lambda(\varepsilon, p)$.

When $\varepsilon = (\varepsilon^1, 0, \ldots, 0) \in X_+^{1\times J}$, then $\Lambda = (\Lambda^1, 0, \ldots, 0) \in \Lambda(\varepsilon, p) \subset X^{J\times J}$, and

$$\hat\theta_t^i = \left(\mathbf{I} - \frac{\Lambda_t}{1+\Lambda_t^{11}}\right)(\theta_t^i + \Lambda_t \theta_{-1}^i) = \left(\mathbf{I} - \frac{\Lambda_t}{1+\Lambda_t^{11}}\right)\theta_t^i + \frac{\Lambda_t}{1+\Lambda_t^{11}}\theta_{-1}^i. \tag{3.5}$$

It is difficult to study the volume of trade effects of bubbles at this level of generality. We focus therefore on examples having several features that make the problem tractable. The examples have only two agents. Hence the portfolio of one agent determines fully the volume of trade in each security. In other words, for each agent $i \in \{1, 2\}$ and security j, $SV_t^j = |\theta_t^{i,j} - \theta_{t-1}^{i,j}|$ and $\hat{SV}_t^j = |\hat\theta_t^{i,j} - \hat\theta_{t-1}^{i,j}|$. Furthermore, we assume that there are only two assets, and (3.5) becomes

$$\hat\theta_t^{i,1} = \frac{\theta_t^{i,1} - \theta_{-1}^i}{1+\Lambda_t^{11}} + \theta_{-1}^i; \quad \hat\theta_t^{i,2} = -\frac{\Lambda_t^{21}}{1+\Lambda_t^{11}}\theta_t^{i,1} + \theta_t^{i,2}. \tag{3.6}$$

We focus first on a complete markets economy where the penalty for default is the interdiction to trade (2.7), and then on an incomplete markets economy with an interdiction to borrow (2.8) as penalty for default.

3.1 Example: complete markets

The uncertainty is described by a time homogeneous Markov process $(s_t)_{t\in\mathbb{N}}$ with states $s_t \in \{1,2\}$, and with a probability of reversal equal to $\pi \in (0,1]$. Thus for any t, $s_{t+1} \neq s_t$ with probability π. The case $\pi = 1$ generates a deterministic economy. There are two agents $\{1,2\}$ with identical utilities $U(c) = E\sum_{t\geq 0}\beta^t u(c_t)$, where u is strictly increasing and concave. At each period t, agent i receives an income $e_t^i := y^H$ if $s_t = i$ and $e_t^i := y^L$ otherwise, with $y^H > y^L$. At any period, the agent with income y^H is referred to as high-type, and the agent with income y^L is the low-type. The penalty for default is the interdiction to trade (2.7).

When agents can trade in one period Arrow securities in zero supply, the stationary equilibria in this framework were studied by Kehoe and Levine (2001) and Alvarez and Jermann (2001). We present these equilibria, support them with infinitely lived assets that dynamically complete the markets rather than with Arrow securities, and then analyze the effect of bubble injections on the volume of trade.

There exists a unique stationary equilibrium. For the high (low) type agent, consumption is c^H (c^L), wealth level (beginning of each period) is $-w$ (w), and the unique pricing kernel a is such that $\frac{a_{t+1}}{a_t} = q_c$ if $s_t \neq s_{t+1}$ and $\frac{a_{t+1}}{a_t} = q_{nc}$ if $s_t = s_{t+1}$. Moreover, if the initial levels of wealth do not coincide with the steady state levels, in particular if agents start with no wealth, as we will assume, the transition to the steady state is complete at the first state reversal. During the transition, the agents' consumption is constant, but different from the steady state levels. Steady state consumptions satisfy the market clearing condition $c^L + c^H = y^L + y^H$, and the high type agent is indifferent between defaulting or not, which gives

$$\frac{(1-\beta(1-\pi))u(c^H) + \beta\pi u(c^L)}{(1-\beta)(1-\beta+2\pi\beta)} = \frac{(1-\beta(1-\pi))u(y^H) + \beta\pi u(y^L)}{(1-\beta)(1-\beta+2\pi\beta)}. \tag{3.7}$$

The pricing kernel follows from the Euler conditions of an (unconstrained) high-type,

$$q_{nc} = \beta, \quad q_c = \beta u'(c^L)/u'(c^H). \tag{3.8}$$

Let $\bar{q}_c := \pi q_c$ and $\bar{q}_{nc} := (1-\pi)q_{nc}$. The beginning of period wealth level for the low-type agent is

$$w = (y^H - c^H)/(1 + \bar{q}_c - \bar{q}_{nc}), \tag{3.9}$$

(and for a high-type is $-w$), while the NTT debt limits for a high-type agent are $\phi^H := -w$ and for a low-type agent are $\phi^L := -\bar{q}_c w/(1 - \bar{q}_{nc})$. As shown in Alvarez and Jermann (2001), the quantities and prices outlined above are an equilibrium (with imperfect risk sharing) if and only if $y^L < c^L < c^H < y^H$. Notice that (3.7) amounts to $u(c^H) + \tilde{\beta}u(c^L) = u(y^H) + \tilde{\beta}u(y^L)$, where $\tilde{\beta} = \frac{\beta\pi}{1-\beta(1-\pi)}$. Bidian and Bejan (2012) show that $y^L < c^L$ and $c^H < y^H$ if and only if $\tilde{\beta}u'(y^L)/u'(y^H) > 1$, and if this assumption holds, then $c^L < c^H$ if and only if

$$(1 + \tilde{\beta})u\left(\frac{y^H + y^L}{2}\right) \leq u(y^H) + \tilde{\beta}u(y^L). \tag{3.10}$$

Condition (3.10) can be understood as requiring that the first best symmetric allocation in which each agent consumes half of the aggregate endowment does not satisfy the participation constraints of the high type agents. It can be verified immediately that the price $\bar{q}$ of a riskless asset is less than 1, $\bar{q} := E_t a_{t+1}/a_t = \bar{q}_c + \bar{q}_{nc} < 1$, and therefore interest rates are "high".[12]

We substitute the Arrow securities with two infinitely lived assets, which dynamically complete the markets. We analyze two types of dividend structure. In the first case, one of the assets pays dividends contingent on a reversal having occurred, and the other way around for the other asset. In the second case, asset $j \in \{1, 2\}$ pays dividends at a given period if and only if state j occurred at that period.[13] We

[12] For the case (2.8) where the penalty for default is the *interdiction to borrow*, the equilibria can be described along the same lines. The steady state pricing kernel and consumption are determined from (3.8), the market clearing condition $c^L + c^H = y^L + y^H$ and the property that the risk free rate is zero (or equivalently, $\bar{q} = 1$). Wealth for the low-type agent is given by (3.9), and for a high-type is $-w$, while the NTT debt limits for both types are $-w$. The transition to the steady state is completed at the first state reversal (Hellwig and Lorenzoni 2009).

[13] When the penalty for default is the interdiction to borrow, the interest rates are low, and therefore the discounted present value of the infinite stream of dividends will be infinite whenever the dividend stream is bounded from below infinitely often. We could assume stationary dividends up to a decreasing time trend, but in this case the volume of trade needed to generate constant wealth transfers between agents (as needed in the equilibrium described in footnote 12) explodes to infinity (even without bubbles), as prices of the assets converge to zero at the rate of the time

assume that agents start with no endowment of securities, hence bubble injections will not affect agents' debt limits (see Theorem 2.2).

3.1.1 Dividends depending on state reversal

There are two infinitely lived assets $\{1, 2\}$ in zero supply with dividends $d_t^1 = \lambda \mathbf{1}_{s_t = s_{t-1}}$, $d_t^2 = \lambda \mathbf{1}_{s_t \neq s_{t-1}}$ for $t > 0$ and equal to zero at $t = 0$, where $\lambda > 0$ and $\mathbf{1}$ is the indicator function (for $A \subset \Omega$ and $\omega \in \Omega$, $\mathbf{1}_A(\omega)$ is 1 if $\omega \in A$ and 0 if $\omega \notin A$). Thus the first asset pays dividends if there is no change in state, while the second asset pays dividends after a reversal of state. We focus on a period $t \geq 1$ *after* the economy has reached steady state, which happens on the first state reversal. Thus if the steady state is reached at $T > 0$ (which is an a.s. finite stopping time), then any variable below with a subscript t is to be understood as referring to period $T + t$. The fundamental values of the assets are

$$p_t^1 = \lambda \sum_{s > t} \bar{q}^{s-t-1} (\bar{q}_c \cdot 0 + \bar{q}_{nc} \cdot 1) = \frac{\lambda \bar{q}_{nc}}{1 - \bar{q}}, \; p_t^2 = \lambda \sum_{s > t} \bar{q}^{s-t-1} (\bar{q}_c \cdot 1 + \bar{q}_{nc} \cdot 0) = \frac{\lambda \bar{q}_c}{1 - \bar{q}}.$$

We replicate agents' wealth levels with portfolios of long-lived securities, eliminating the need for Arrow securities. Thus we construct portfolios θ^i for each agent such that, given the asset prices computed before, $(p_t + d_t)\theta_{t-1}^i = (-1)^{\mathbf{1}_{s_t = i}} w$. Denote by $\bar{\theta}_{j,t-1}$ the holdings of a low type at $t-1$ of security j. If the state changes from $t-1$ to t, $(p_t^1 + 0)\bar{\theta}_{1,t-1} + (p_t^2 + \lambda)\bar{\theta}_{2,t-1} = -w$, while if there is no change, $(p_t^1 + \lambda)\bar{\theta}_{1,t-1} + (p_t^2 + 0)\bar{\theta}_{2,t-1} = w$. Solving this system of equations, we obtain

$$\bar{\theta}_{1,t-1} = w\lambda^{-1}(2\bar{q}_c + 1 - \bar{q}), \quad \bar{\theta}_{2,t-1} = -w\lambda^{-1}(2\bar{q}_{nc} + 1 - \bar{q}).$$

Notice that the asset prices and portfolios are time invariant after the economy reaches the steady state, and we can *omit* the time subscript.

We consider bubbles ε which do not crash before the steady state is reached (before T), and their value at T is some positive $\bar{\varepsilon}$. After the steady state is reached,

trend. This is the reason why we focused on the case where agents are not allowed to trade after default.

they grow at the rate $\varepsilon_c \geq 0$ if there is a state change, respectively $\varepsilon_{nc} \geq 0$ if there is no state change, that is $\varepsilon_t = \varepsilon_{t-1} \cdot \left(\varepsilon_c \mathbf{1}_{s_t \neq s_{t-1}} + \varepsilon_{nc} \mathbf{1}_{s_t = s_{t-1}}\right)$. A value $\varepsilon_c = 0$ implies that the bubble crashes on the first state change, while if $\varepsilon_{nc} = 0$, the bubble crashes if the state does not change (after the steady state is reached). The bubble spanning portfolios follow from

$$(p_t^1 + 0)\Lambda_{t-1}^{11} + (p_t^2 + \lambda)\Lambda_{t-1}^{21} = \varepsilon_{t-1}\varepsilon_c, \; (p_t^1 + \lambda)\Lambda_{t-1}^{11} + (p_t^2 + 0)\Lambda_{t-1}^{21} = \varepsilon_{t-1}\varepsilon_{nc},$$

and therefore

$$\Lambda_{t-1}^{11} = \lambda^{-1}(\varepsilon_{nc} - 1) \cdot \varepsilon_{t-1}; \quad \Lambda_{t-1}^{21} = \lambda^{-1}(\varepsilon_c - 1) \cdot \varepsilon_{t-1}. \tag{3.11}$$

Denote by $SV_t^j(c)$, respectively $SV_t^j(nc)$ the share volumes in asset j if state changes (c), respectively it does not change (nc) from $t-1$ to t, and similarly for dollar volumes, and the share and dollar volumes after the bubble injection. Notice that $SV_t^1(c) = 2\bar{\theta}_1 > 0$, $SV_t^1(nc) = 0$, $SV_t^2(c) = -2\bar{\theta}_2 > 0$, $SV_t^2(nc) = 0$ and

$$\hat{SV}_t^1(c) = \bar{\theta}_1 \left| \frac{1}{1 + \Lambda_{t-1}^{11}} + \frac{1}{1 + \Lambda_t^{11}} \right|, \; \hat{SV}_t^1(nc) = \bar{\theta}_1 \left| \frac{1}{1 + \Lambda_{t-1}^{11}} - \frac{1}{1 + \Lambda_t^{11}} \right|,$$
$$\hat{SV}_t^2(c) = \left| -2\bar{\theta}_2 + \bar{\theta}_1 \frac{\Lambda_{t-1}^{21}}{1 + \Lambda_{t-1}^{11}} + \bar{\theta}_1 \frac{\Lambda_t^{21}}{1 + \Lambda_t^{11}} \right|, \; \hat{SV}_t^2(nc) = \bar{\theta}_1 \left| \frac{\Lambda_{t-1}^{21}}{1 + \Lambda_{t-1}^{11}} - \frac{\Lambda_t^{21}}{1 + \Lambda_t^{11}} \right|.$$

Therefore a bubble injection in the first asset *increases* the share volume of trade in *both* assets at periods when there is *no reversal*, since the volume of trade jumps from zero to a positive value. If the state changes, the effect of a bubble injection depends on the type of bubble introduced. For concreteness, in what follows we focus on two types of bubbles: a *deterministic* bubble and a bubble that *crashes on the first reversal*.

For a *deterministic* bubble, $\varepsilon_c = \varepsilon_{nc} = \bar{q}^{-1}$, $\Lambda_{t-1}^{11} = \Lambda_{t-1}^{21} = \lambda^{-1}(1 - \bar{q})\bar{q}^{-t}\bar{\varepsilon}$. If there is no reversal from $t-1$ to t, the bubble increases the volume of trade in both securities, as seen before. However, asymptotically these increases vanish except for the dollar volume of trade in first asset. Indeed, for large t, since $\Lambda_t^{11} \nearrow \infty$, it follows that $\hat{SV}_t^1(nc), \hat{SV}_t^2(nc), \hat{DV}_t^2(nc) \approx 0$, while $\hat{DV}_t^1(nc) \approx \lambda\bar{\theta}_1$. When there is a reversal from $t-1$ to t, the share volume of trade in the first security decreases, while it increases for the second security. The dollar volume of trade increases however even for the first security,

$$\hat{DV}^1_t(c) = \left(p^1_t + \varepsilon_t\right) \hat{SV}^1_t(c) = \left(p^1_t(1+\Lambda^{11}_t) + p^2_t \Lambda^{21}_t\right) \hat{SV}^1_t(c) > p^1_t \cdot 2\bar{\theta}_1 = DV^1_t.$$

For large t, $\hat{SV}^1_t(c) \approx 0$, $\hat{SV}^2_t(c) \approx 2\bar{\theta}_1 - 2\bar{\theta}_2$, $\hat{DV}^2_t(c) \approx 2p^2(\bar{\theta}_1 - \bar{\theta}_2)$, while $\hat{DV}^1_t(c) \approx \lambda\bar{\theta}_1(1+\bar{q})/(1-\bar{q})$. Thus a deterministic bubble in the first asset always increases the dollar volumes of trade in both assets. The increase in the dollar volume of trade in the first asset is persistent.

For a *stochastic* bubble that *crashes on the first reversal* (after the steady state is reached), $\varepsilon_c = 0$, $\varepsilon_{nc} = \bar{q}^{-1}_{nc}$, $\Lambda^{11}_{t-1} = \lambda^{-1}(\varepsilon_{nc} - 1)\varepsilon_{t-1} > 0$, $\Lambda^{21}_{t-1} = -\lambda^{-1}\varepsilon_{t-1}$. If there is no reversal from $t-1$ to t, the bubble increases the volume of trade in both securities. If t is large, the share volume of trade in both securities is close to zero, but the dollar volume in the first security $\hat{DV}^1_t(nc)$ is bounded away from zero, as it approaches $\lambda\bar{\theta}_1$. If the state changes from $t-1$ to t, the bubble crashes and $\Lambda^{11}_t = \Lambda^{21}_t = 0$. Therefore the share and dollar volume of trade in the first asset decrease in the period when the bubble crashes, as

$$\hat{SV}^1_t(c) = \bar{\theta}_1 \left(1 + \frac{1}{1+\Lambda^{11}_{t-1}}\right) < 2\bar{\theta}_1 = SV^1_t(c),$$

and $\hat{DV}^1_t(c) = p^1 \hat{SV}^1_t(c) < p^1 SV^1_t(c) = DV^1_t(c)$. The volume of trade in the second security also decreases when the bubble crashes, since it can be checked that

$$\hat{SV}^2_t(c) = \left| -2\bar{\theta}_2 + \bar{\theta}_1 \frac{\Lambda^{21}_{t-1}}{1+\Lambda^{11}_{t-1}} \right| < -2\bar{\theta}_2 = SV^2_t(c).$$

In summary, as long as the stochastic bubble is running, the share and dollar volume of trade are higher than normal. The dollar volume of trade in the first security is bounded away from zero. When the bubble collapses, the volume of trade shrinks to levels lower than normal. After the crash, the trade volume reverts back to normal.

We calibrate the example using the parameters already employed by Alvarez and Jermann (2001) in their analysis of the volatility of the pricing kernel in this model. Thus $\beta = 0.65$, $y^H = 0.641$, $y^L = 0.359$, $\pi = 0.25$, and $u(c) = c^{1-\gamma}/(1-\gamma)$, with $\gamma = 2$. Finally, we take $\lambda = 0.03$ as being the average ratio of US net corporate dividends to gross domestic product (GDP) for 1947-2011 (Federal Reserve Economic Data). It follows that $c^H \approx 0.639$, $c^L \approx 0.361$, $\bar{q}_{nc} \approx 0.487$, $\bar{q}_c \approx 0.507$, $p^1 \approx 1.772$, $p^2 \approx 1.843$, $\bar{\theta}_1 \approx 0.124$, $\bar{\theta}_2 \approx -0.12$. The value of the stochastic bubble when the economy enters the steady state is assumed to be $\bar{\varepsilon} = 0.001$. We compare the dollar trade volumes after the bubble with their levels without the bubble. The increase in the dollar volume of trade in the first period if the bubble has not crashed yet is 0.007 for each asset. Therefore a very small initial bubble, equal to 0.1% of the GDP, generates an initial increase in the total trade volume of 14 times its size. Conditional on the bubble not having crashed, the increase in trade volume continues to grow (for 5 periods) and reaches a maximum of 7.81% of GDP, and then tapers off, approaching 0.25% of GDP if the bubble runs for a long time. If the bubble crashes in the first period, the drop in trade volume equals 0.55% of GDP, while if the bubble is sustained for a long time and then crashes, the drop in trade volume is 16.2% of GDP. In relative terms, when compared to the no-bubble case, the total trade volume drops by 1.68% if the bubble crashes in the first period, and by 49.7% if the bubble crashes after a long run. Thus small bubbles can produce disproportionately large increases in the volume of trade, and subsequent large collapses in trade volume when they crash.

3.1.2 Dividends depending on current state

Assume that the dividends of the two securities are $d_t^i = \lambda \mathbf{1}_{s_t = i}$ for $t > 0$, and zero at $t = 0$. Thus asset $j \in \{1, 2\}$ pays dividends λ at t if state j is realized at t, and zero otherwise. It is immediate to see that asset prices depend only on the realization of the current state, thus $p_t^j = p^j(s_t)$. The fundamental valuation equation gives

$$p^1(1) = \bar{q}_c \left(p^1(2) + 0\right) + \bar{q}_{nc} \left(p^1(1) + \lambda\right), \; p^1(2) = \bar{q}_c \left(p^1(1) + \lambda\right) + \bar{q}_{nc} \left(p^1(2) + 0\right),$$

hence

$$p^1(1) = \frac{1}{2} \cdot \frac{\lambda \bar{q}}{1-\bar{q}} + \frac{1}{2} \cdot \frac{\lambda(\bar{q}_{nc} - \bar{q}_c)}{1-(\bar{q}_{nc} - \bar{q}_c)},\ p^1(2) = \frac{1}{2} \cdot \frac{\lambda \bar{q}}{1-\bar{q}} - \frac{1}{2} \cdot \frac{\lambda(\bar{q}_{nc} - \bar{q}_c)}{1-(\bar{q}_{nc} - \bar{q}_c)}.$$

By symmetry, $p^2(1) = p^1(2)$, $p^2(2) = p^1(1)$. Let $\theta^i_{t-1}(k)$ denote the portfolio of agent i at $t-1$ if the state realized at $t-1$ is k. It follows that

$$(p^1(1)+\lambda)\theta^{1,1}_{t-1}(1) + p^2(1)\theta^{1,2}_{t-1}(1) = -w, \quad p^1(2)\theta^{1,1}_{t-1}(1) + (p^2(2)+\lambda)\theta^{1,2}_{t-1}(1) = w,$$

and therefore $\theta^{1,1}_{t-1}(1) = -\theta^{1,2}_{t-1}(1) = -w/(p^1(1) - p^1(2) + \lambda) < 0$. A similar reasoning shows that $\theta^{1,1}_{t-1}(2) = -\theta^{1,2}_{t-1}(2) = \theta^{1,1}_{t-1}(1)$. Since the steady state portfolios are time invariant and do not depend on the state process, we can *drop* the time subscripts and the state arguments. The agents hold balanced amounts of the two securities, equal in absolute value, but of opposite sign. The share and dollar volume of trade in both securities are zero (after the steady state is reached), $SV^j_t = DV^j_t = 0$, $j \in \{1, 2\}$. Therefore an (arbitrary) bubble injection *increases* the share and dollar volume of trade in *all* securities.

Consider first a *deterministic* bubble (ε_t) (in the first asset). The process $\Lambda \in X^{2\times 1}$ satisfying $(p_t + d_t)\Lambda_{t-1} = \varepsilon_t$ is $\Lambda^{11}_{t-1} = \Lambda^{21}_{t-1} = \bar{\varepsilon}\bar{q}^{-t}(p^1(1) + p^2(1) + \lambda)^{-1}$. The volume of trade t periods *after* the economy reaches the steady state is

$$\begin{aligned}
\hat{SV}^1_t &= \left| \frac{1}{1+\Lambda^{11}_t}\theta^{1,1} - \frac{1}{1+\Lambda^{11}_{t-1}}\theta^{1,1} \right| = |\theta^{1,1}_t| \frac{\Lambda^{11}_t(1-\bar{q})}{(1+\Lambda^{11}_t)(1+\bar{q}\Lambda^{11}_t)} \to_{t\to\infty} 0, \\
\hat{SV}^2_t &= \left| \theta^{1,2} - \frac{\Lambda^{21}_t}{1+\Lambda^{11}_t}\theta^{1,1} - \theta^{1,2} + \frac{\Lambda^{21}_{t-1}}{1+\Lambda^{11}_{t-1}}\theta^{1,1} \right| = \hat{SV}^1_t \to_{t\to\infty} 0, \\
\hat{DV}^1_t &= (p^1(s_t) + \bar{\varepsilon}\bar{q}^{-t})\hat{SV}^1_t \to_{t\to\infty} |\theta^{1,1}|(1-\bar{q})(p^1(1) + p^1(2) + \lambda).
\end{aligned}$$

It follows that the market-wide increase in the share volume of trade induced by a deterministic bubble (in the first asset) vanishes asymptotically, while the increase in the dollar volume of trade in the first asset is persistent.

The effects of a *stochastic* bubble can be analyzed in a similar fashion. Consider (as in Section 3.1.1) a bubble in the first asset that *crashes on the first reversal* (after the steady state is reached). For concreteness, assume that the economy starts in state 2 and therefore the steady state is reached when the state switches to 1 for the first time. Thus $\varepsilon_c = 0$, $\varepsilon_{nc} = \bar{q}_{nc}^{-1}$. The bubble spanning portfolios satisfy

$$(p^1(1)+\lambda)\Lambda_{t-1}^{11} + p^2(1)\Lambda_{t-1}^{21} = \varepsilon_{nc}\varepsilon_{t-1}, \quad p^1(2)\Lambda_{t-1}^{11} + (p^2(2)+\lambda)\Lambda_{t-1}^{21} = 0,$$

and therefore

$$\Lambda_{t-1}^{11} = \frac{\varepsilon_{t-1}\bar{q}_{nc}^{-1}(p^1(1)+\lambda)}{(p^1(1)+\lambda)^2 - (p^1(2))^2}, \quad \Lambda_{t-1}^{21} = -\frac{\varepsilon_{t-1}\bar{q}_{nc}^{-1}p^1(2)}{(p^1(1)+\lambda)^2 - (p^1(2))^2}.$$

It can be checked immediately that Λ_{t-1}^{11} is positive and grows at the rate $\bar{q}_{nc}^{-1} > 1$ as long as the bubble does not crash. As was the case for the deterministic bubble, the share volume of trade in both securities approaches zero if the bubble runs for a long time, but the dollar volume of trade in the first asset is bounded away from zero, and approaches $|\theta^{1,1}|(1-\bar{q}_{nc})((p^1(1)+\lambda)^2 - (p^1(2))^2)/(p^1(1)+\lambda)$.

With the numerical calibration of the previous section, $p^1(1) \approx 2.711$, $p^1(2) \approx 2.712$, $\theta^{1,1} \approx -0.083$. A stochastic bubble (crashing on the first reversal) of size 0.1% of the GDP at the period when the economy reaches the steady state increases the total trade volume by 1.5% of GDP in the first period. The trade volume increase continues to grow initially, reaching a maximum of 7.97% of GDP after 5 periods, and then starts to drop, but nevertheless the increase is persistent and approaches 0.25% of GDP if the bubble runs for a long time.

We wrap up the example of Section 3.1 by discussing the effect of bubble injections on risk premia and Sharpe ratios. This can be done without making specific assumptions on dividends. Since at any period there are only two possible states in the next period, it follows that conditional on current information, the period ahead stochastic discount factor (SDF) is fully correlated (either positively or negatively) with any nondeterministic bubble or with some risky return. Assume that the first asset has a positive risk premium, hence it has a (conditional) correlation of -1 with the SDF. By Proposition 3.1, the positive risk premium of the (first) asset can be increased by a bubble injection if and only if the volatility of the bubble growth rate exceeds the volatility of the return, $\sigma_t(\varepsilon_{t+1}/\varepsilon_t) > \sigma_t(R^1_{t+1})$.[14] The Sharpe ratio cannot increase however, since the growth rate of any bubble cannot have a lower correlation to the SDF (see Proposition 3.1). The two-state Markov process underlying uncertainty cannot generate rich enough correlations between returns/bubbles and the SDF to produce interesting effects of bubbles on Sharpe ratios. Kurtosis of returns is unaffected by bubbles due to the same reason, thus bubbles cannot generate fat tails in this example.

3.2 Example: incomplete markets

The uncertainty is described by a time homogeneous Markov process $(s_t)_{t\in\mathbb{N}}$ with $s_t \in \{1, 2, 3\}$, having a transition probability matrix π with strictly positive entries. There are two agents $\{1, 2\}$ with utilities $U^i(c) = E\sum_{t\geq 0}\beta^t u(c_t)$, where u is strictly increasing and strictly concave. There are two assets in unit supply. In each period, the first asset pays $y > 0$ if the current state is 1, $y/2$ in state 2, and 0 in state 3. The second asset pays 0 in state 1, $y/2$ in state 2, and y in state 3. We assume we have a security markets economy, in that agents' only income is generated by dividends resulting from their asset holdings. Agent $i \in \{1, 2\}$ has an initial endowment of security i equal to 1, and a zero endowment of the other security. The agents face zero debt limits, which of course are NTT when the penalty for default is the interdiction to borrow (2.8). Since agents' wealth originates solely from financial wealth, these debt limits are also given by (2.9), where debt is restricted by the present value of future endowments.

We construct a Pareto optimal equilibrium in which expected (gross) returns are equal to β^{-1} and agents have constant consumption. Thus the securities are fairly priced in that their price equals the expected value of dividends discounted at the risk free rate β^{-1}. There will be no trade after the initial period, when the portfolios are adjusted once and for all. This is not surprising as markets are *effectively complete*

[14] For example, it can be checked that the stochastic bubble of Section 3.1.1 satisfies this condition if and only if $\beta > (1 - \bar{q})/(1 - \bar{q}_{nc})$.

(Leroy and Werner 2001),[15] and Judd, Kubler, and Schmedders (2003) show that no trade obtains generically with complete markets in this type of Markov environment.

Asset prices are the present value of future dividends,

$$p_t^j = E_t \sum_{s>t} \beta^{s-t} d_s^j, \quad \forall j \in \{1,2\}, \forall t \geq 0. \tag{3.12}$$

Beginning of period wealth levels are obtained from the intertemporal budgets,

$$(p_t + d_t)\theta_{t-1}^i = E_t \sum_{s \geq t}^{\infty} \beta^{s-t} c^i = \frac{c^i}{1-\beta}, \quad \forall i \in \{1,2\}, \forall t \geq 0. \tag{3.13}$$

By writing (3.13) at $t = 0$ we obtain the (constant) consumption levels,

$$c^i = (1-\beta)(p_0^i + d_0^i) = (1-\beta) \sum_{t=0}^{\infty} \beta^t \cdot E d_t^i, \quad \forall i \in \{1,2\}. \tag{3.14}$$

Notice that $p_t^1 + p_t^2 = \beta y/(1-\beta)$ and $p_t^1 + d_t^1 + p_t^2 + d_t^2 = y/(1-\beta)$. Therefore, generically in π, (3.13) admits only the solution

$$\theta_t^{i,1} = \theta_t^{i,2} = \frac{c^i}{y}, \quad \forall i \in \{1,2\}, \forall t \geq 0. \tag{3.15}$$

To check that the allocations, portfolios and prices described in (3.12)-(3.15) form an equilibrium, it is enough to prove that the consumptions and portfolios are optimal for each agent, as the market clearing conditions are clearly satisfied. But this is true, since the given consumptions and portfolios satisfy the necessary and sufficient Kuhn-Tucker and transversality conditions for agents' utility maximization problems (Forno and Montrucchio 2003, Th. 3.6 and Prop. 3.9):

$$p_t = E_t \frac{\beta u'(c^i)}{u'(c^i)}(p_{t+1} + d_{t+1}), \forall t \geq 0, \text{ and } \lim_{t\to\infty} E \beta^t u'(c^i) p_t \theta_t^i = 0.$$

[15] The aggregate endowment is constant and in all Pareto optimal allocations agents receive a constant consumption stream, which can be replicated by (balanced) portfolios with equal amounts of the two assets.

Given the zero volume of trade following the initial period, *arbitrary* bubble injections *increase* the share and dollar volume of trade.[16] To analyze further the volume of trade effects, we focus for concreteness on a *deterministic* bubble with an initial value $\bar{\varepsilon}$, injected in the first asset. Thus $\varepsilon_t = \beta^{-t}\bar{\varepsilon}$. The bubble spanning portfolios are generically (in π) unique and given by

$$\Lambda^{11}_{t-1} = \Lambda^{21}_{t-1} = \bar{\varepsilon}\beta^{-t}y^{-1}(1-\beta). \tag{3.16}$$

By (3.6) and (3.15) the share volume of trade in both assets increases, but this increase vanishes asymptotically, as $-\frac{\Lambda^{21}_t}{1+\Lambda^{11}_t} = \frac{1}{1+\Lambda^{11}_t} - 1$ and

$$\hat{SV}^1_t = |\hat{\theta}^{i,1}_t - \hat{\theta}^{i,1}_{t-1}| = \left| \frac{\theta^{i,1}_t - \theta^{i,1}_{-1}}{1+\Lambda^{11}_t} - \frac{\theta^{i,1}_{t-1} - \theta^{i,1}_{-1}}{1+\Lambda^{11}_{t-1}} \right| = \frac{\theta^{2,1}_t(1-\beta)\Lambda^{11}_t}{(1+\Lambda^{11}_t)(1+\beta\Lambda^{11}_t)} \to_{t\to\infty} 0,$$

$$\hat{SV}^2_t = |\hat{\theta}^{i,2}_t - \hat{\theta}^{i,2}_{t-1}| = \left| \left(\frac{1}{1+\Lambda^{11}_t} - 1\right)\theta^{i,1}_t - \left(\frac{1}{1+\Lambda^{11}_{t-1}} - 1\right)\theta^{i,1}_{t-1} \right| = \hat{SV}^1_t \to_{t\to\infty} 0.$$

However the increase in the dollar volume of trade in the first asset is persistent, as

$$\hat{DV}^1_t = (p^1_t + \varepsilon_t)\hat{SV}^1_t \to \lim \bar{\varepsilon}\beta^{-t}\hat{SV}^1_t = c^2.$$

4 Conclusion

We showed that any nonnegative process which does not change the set of pricing kernels can be introduced as a bubble in asset prices, leading to an equivalent equilibrium with identical consumption for the agents, but tighter debt limits. Moreover, with enforcement limitations, if the debt bounds are endogenized as in Alvarez and Jermann (2000) to prevent default but to allow for maximal credit expansion, the modified debt limits in the equilibrium with bubbles still arise endogenously from the existing enforcement limitations. If the underlying creditworthiness of the agents

[16] However, due to constant expected returns, the expected rate of growth of bubble coincides with the expected return on any asset, hence the equity premium is unaffected by the presence of bubbles, and remains zero.

is unchanged but nevertheless a credit tightening occurs, bubbles help agents keep their consumption at the same level as before the credit crunch.

Therefore, with enforcement limitations, rational bubbles are robust. They can cause large increases in the volume of trade while they run and large collapses upon their crash, compared to trade volumes in the absence of bubbles. The bubbles constructed here affect prices and returns, without affecting consumption. By creating a disconnect between the financial and the real side of the economy, they are ideally suited to resolve long-standing asset pricing puzzles: the excess volatility puzzle, and the conditional/unconditional equity premium puzzle.

A Span-preserving martingales

We characterize the set of span-preserving martingales $M^J(p)$.

Lemma A.1. *Let $p, d \in X_+^{1\times J}$ such that $A_{++}(p) \neq \emptyset$ and $\varepsilon \in X^{1\times J}$. The following are equivalent:*

(i) There exists $\Lambda \in X^{J\times J}$ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for all $t \geq 1$ and there exists $a \in A_{++}(p)$ such that $a \cdot \varepsilon$ is a martingale.

(ii) There exists $\Lambda \in X^{J\times J}$ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for all $t \geq 1$ and $\varepsilon_t = p_t\Lambda_t$, for all $t \geq 0$.

(iii) $A(p) \subset A(p + \varepsilon)$

(iv) For each $a \in A(p)$, $a \cdot \varepsilon$ is a martingale.

Proof. (i) $\Rightarrow$ (ii) The conclusion is immediate, since for all $t \geq 0$,

$$\varepsilon_t = E_t \frac{a_{t+1}}{a_t} \varepsilon_{t+1} = E_t \frac{a_{t+1}}{a_t} (p_{t+1} + d_{t+1})\Lambda_t = p_t\Lambda_t.$$

(ii) $\Rightarrow$ (iii) Let $a \in A(p)$. The conclusion follows, since

$$E_t \frac{a_{t+1}}{a_t} (p_{t+1} + d_{t+1} + \varepsilon_{t+1}) = E_t \frac{a_{t+1}}{a_t} (p_{t+1} + d_{t+1})(\mathbf{I} + \Lambda_t) = p_t(\mathbf{I} + \Lambda_t) = p_t + \varepsilon_t.$$

(iii) $\Rightarrow$ (iv) Let $a \in A(p)$. Thus $a \in A(p+\varepsilon)$. It follows that for all $t \geq 0$,

$$p_t + E_t \frac{a_{t+1}}{a_t} \varepsilon_{t+1} \overset{a \in A(p)}{=} E_t \frac{a_{t+1}}{a_t} (p_{t+1} + d_{t+1} + \varepsilon_{t+1}) \overset{a \in A(p+\varepsilon)}{=} p_t + \varepsilon_t,$$

and thus $\varepsilon_t = E_t \frac{a_{t+1}}{a_t} \varepsilon_{t+1}$.

(iv) $\Rightarrow$ (i) Assume that $m \in X$ is such that $a \cdot m$ is a martingale, for any $a \in A(p)$. Pick an arbitrary date t event $\mathcal{F}_t(\omega)$ with $\omega \in \Omega$, with $\mathcal{F}_t(\omega)$ representing the cell of the partition $\mathcal{F}_t$ containing ω) (in other words pick a date t node in the uncertainty tree). Assume that there $\mathcal{F}_{t+1}$ has S subsets of $\mathcal{F}_t(\omega)$ (i.e. there are S branches leaving the fixed node). Then the returns r_{t+1} conditional on the event $\mathcal{F}_t(\omega)$ can be viewed as an $S \times J$ matrix R. Similarly m_{t+1}/m_t conditional on $\mathcal{F}_t(\omega)$ is represented by a vector $M \in \mathbb{R}^S$. If $\mu \in \mathbb{R}^S$ is interpreted as conditional state price process a_{t+1}/a_t times conditional probabilities, it follows that for any $\mu \in \mathbb{R}^S$ such that $\mathbf{1}' = \mu' R$, it must be the case that $1 = \mu' M$. Therefore there cannot exist $\mu \in \mathbb{R}^S$ such that

$$\begin{cases} \mu'(R^1 - M) < 0 \\ \mu'(R^1 - R^j) = 0, \quad j \in \{2, \ldots, J\}. \end{cases}$$

By Motzkin's alternative theorem (Motzkin 1951), there exist $\alpha_2, \ldots, \alpha_J \in \mathbb{R}$ such that $R^1 - M = \sum_{j=2}^J \alpha_j (R^1 - R^J)$. Therefore M can be written as a linear combination of the columns of R and there exists $\lambda \in X^{J \times 1}$ such that $m_t = (p_t + d_t)\lambda_{t-1}$, for all $t \geq 1$.

Each component ε^j of $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^J)$ is a martingale when deflated by any $a \in A(p)$. As proven above, for each j there exists $\lambda^j \in X^{J \times 1}$ such that $\varepsilon^j_t = (p_t + d_t)\lambda^j_{t-1}$ for all $t \geq 1$. The conclusion follows by setting $\Lambda = (\lambda^1, \ldots, \lambda^J)$. $\square$

For each $t \geq 1$, let $\mathcal{S}_t(p)$ be the set of attainable payoffs at t given the price and dividend processes $p, d \in X_+^{1 \times J}$:

$$\mathcal{S}_t(p) := \{(p_t + d_t)\lambda \mid \lambda : \Omega \to \mathbb{R}^J \text{ and } \lambda \text{ is } \mathcal{F}_{t-1} - \text{measurable}\}. \tag{A.1}$$

We refer to $\mathcal{S}_t(p)$ as the period t *asset span*. We say that *there are no redundant securities* at $t-1$, given prices p, if there is no $\lambda : \Omega \to \mathbb{R}^J$ such that λ is $\mathcal{F}_{t-1}-$measurable,

$\lambda \neq 0$ and $(p_t + d_t)\lambda = 0$. We justify now the name "span-preserving martingales" for the set $M^J(p)$ by showing that elements of this set are martingales when deflated by pricing kernels and that they do not affect the pricing kernels if added to asset prices, or equivalently, they do not change the asset span.

Proposition A.2. *Assume that there are no redundant securities at any period t. The following are equivalent:*

(i) $\varepsilon \in M^J(p)$.

(ii) $A(p) = A(p+\varepsilon)$.

(iii) $\mathcal{S}_t(p) = \mathcal{S}_t(p+\varepsilon), \forall t \in \mathbb{N}$ *and* $a \cdot \varepsilon$ *is a martingale, for some* $a \in A(p)$.

Proof. (i) $\Rightarrow$ (ii) Notice that Lemma A.1 implies that $A(p) \subset A(p+\varepsilon)$. Choose $a \in A(p+\varepsilon)$. Then for any $t \geq 0$,

$$E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}) = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1})(\mathbf{I} + \Lambda_t)^{-1} = (p_t + \varepsilon_t)(\mathbf{I} + \Lambda_t)^{-1}$$
$$= (p_t + p_t\Lambda_t)(\mathbf{I} + \Lambda_t)^{-1} = p_t(\mathbf{I} + \Lambda_t)(\mathbf{I} + \Lambda_t)^{-1} = p_t.$$

(ii) $\Rightarrow$ (iii) By Lemmma A.1, $A(p) \subset A(p+\varepsilon)$ implies the existence of Λ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$, for all $t > 0$. Moreover, $\mathbf{I} + \Lambda_{t-1}$ is non-singular. Indeed, as $A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$, there exists Γ such that $\varepsilon_t = (p_t + d_t)\Gamma_{t-1}$, and it follows that $(\mathbf{I} + \Lambda_{t-1})(\mathbf{I} - \Gamma_{t-1}) = \mathbf{I}$, hence $\mathbf{I} + \Lambda_{t-1}$ is indeed non-singular. To show that $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$, notice that for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_{t-1}$-measurable, $(p_t + d_t + \varepsilon_t)\lambda_{t-1} = (p_t + d_t)(\mathbf{I} + \Lambda_{t-1})\lambda_{t-1} \in \mathcal{S}_t(p)$. The inclusion $\mathcal{S}_t(p) \subset \mathcal{S}_t(p+\varepsilon)$ also holds, since for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ that is $\mathcal{F}_{t-1}$-measurable,

$$(p_t + d_t)\lambda_{t-1} = (p_t + d_t + \varepsilon_t)(\mathbf{I} + \Lambda_{t-1})^{-1}\lambda_{t-1} \in \mathcal{S}_t(p+\varepsilon).$$

(iii) $\Rightarrow$ (i) We show first that the inclusion $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$ implies the existence of $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for all $t > 0$. Indeed, for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_{t-1}$-measurable, there exists $\lambda'_{t-1} : \Omega \to \mathbb{R}^J$, $\mathcal{F}_{t-1}$-measurable, such that $(p_t + d_t + \varepsilon_t)\lambda_{t-1} = (p_t + d_t)\lambda'_{t-1}$. It follows that $\varepsilon_t\lambda_{t-1} = (p_t + d_t)(\lambda'_{t-1} - \lambda_{t-1})$, and since λ_{t-1} was arbitrary, we conclude that each of the J components of ε_t belongs to $\mathcal{S}_t(p)$. Thus $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for some $\mathcal{F}_{t-1}$-measurable $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$.

From the inclusion $\mathcal{S}_t(p) = \mathcal{S}_t(p+\varepsilon+(-\varepsilon)) \subset \mathcal{S}_t(p+\varepsilon)$, by the above reasoning, there exists $\Gamma_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable and such that $\varepsilon_t = (p_t + d_t + \varepsilon_t)\Gamma_{t-1}$. Therefore

$$\varepsilon_t(\mathbf{I} - \Gamma_{t-1}) = (p_t + d_t)\Gamma_{t-1} = (p_t + d_t)\left(\Lambda_{t-1}(\mathbf{I} - \Gamma_{t-1}) - \Gamma_{t-1}\right) = 0.$$

Since there are no redundant securities, we conclude that $\Lambda_{t-1}(\mathbf{I} - \Gamma_{t-1}) - \Gamma_{t-1} = 0$, which is equivalent to $(\mathbf{I} + \Lambda_{t-1})(\mathbf{I} - \Gamma_{t-1}) = \mathbf{I}$, hence $\mathbf{I} + \Lambda_{t-1}$ is non-singular. Therefore $\varepsilon_t = E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})\Lambda_t = p_t\Lambda_t$ and $\varepsilon \in M^J(p)$. $\square$

B Gain processes as span-preserving martingales

We give conditions under which the *gain process* associated to a J-dimensional vector of trading strategies $\Theta = (\Theta^1, \ldots, \Theta^J) \in X^{J\times J}$ belongs to $M^J(p)$. Fix a trading strategy $\bar{\theta} \in X_+^{J\times 1}$ having a return $\bar{r}$,

$$\bar{r}_{t+1} := \frac{(p_{t+1} + d_{t+1})\bar{\theta}_t}{p_t\bar{\theta}_t}, \forall t \geq 0.$$

Since dividends and prices are positive, $\bar{r}$ is positive. Define the *discount factor* process $\bar{\rho}$ induced by $\bar{\theta}$ as $\bar{\rho}_t = \prod_{s=1}^t \bar{r}_s^{-1}$, for all $t > 0$. The *gain process* $g(\Theta) \in X^{1\times J}$ associated to the trading strategy vector Θ is defined as (Leroy and Werner 2001, p. 259)

$$g_t(\Theta) := p_t\Theta_t + \bar{\rho}_t^{-1} \sum_{s=1}^{t} \bar{\rho}_s \left((p_s + d_s)\Theta_{s-1} - p_s\Theta_s\right), \forall t. \tag{B.1}$$

Thus $g_t^j(\Theta)$ $(= g_t(\Theta^j))$ represents the gain realized by the trading strategy Θ^j from date 0 to date t measured in units of date t consumption; it is computed as the sum of payoffs of the strategy Θ^j up to date t reinvested in each period at the rate of return generated by $\bar{\theta}$. In particular, when $\Theta := \mathbf{I}$, then $g(\mathbf{I})$ represents the gain process of a buy-and-hold portfolio consisting of a unit of each security,

$$g_t(\mathbf{I}) = p_t + \bar{\rho}_t^{-1} \sum_{s=1}^{t} \bar{\rho}_s d_s \geq 0. \tag{B.2}$$

We establish conditions under which the gain process associated to Θ belongs to $M^J(p)$. For all $t > 0$, let $\Lambda_{t-1} := \Theta_{t-1} + \bar{\theta}_{t-1}\lambda'_{t-1}(\Theta)$, where

$$\lambda_{t-1}(\Theta) := \left((p_{t-1}\bar{\theta}_{t-1})^{-1} \bar{\rho}_{t-1}^{-1} \sum_{s=1}^{t-1} \bar{\rho}_s \left((p_s + d_s)\Theta_{s-1} - p_s\Theta_s \right) \right)'.$$

Lemma B.1. *The following hold:*

(i) $a \cdot g(\Theta)$ *is a martingale for any* $a \in A(p)$.

(ii) $g(\Theta) \in M^J(p) \Leftrightarrow \det(\mathbf{I} + \Theta_{t-1}) \neq 0,\ 1 + \lambda'_{t-1}(\Theta)(\mathbf{I} + \Theta_{t-1})^{-1}\bar{\theta}_{t-1} \neq 0, \forall t > 0.$

(iii) If $\Theta = \mathbf{I}$, *then* $g(\Theta) \in M^J_+(p)$.

(iv) If $\Theta = (\Theta^1, 0, \ldots, 0)$, *then* $g(\Theta) \in M^J(p) \Leftrightarrow 1 + \Theta^{11} \neq 0,\ 1 + \frac{\bar{\theta}^1}{1+\Theta^{11}} \neq 0.$

Proof. (i) For all $a \in A(p)$,

$$E_t a_{t+1} g_{t+1}(\Theta) = E_t a_{t+1} p_{t+1} \Theta_{t+1} + \bar{\rho}_t (g_t(\Theta) - p_t\Theta_t) E_t a_{t+1} \bar{\rho}_{t+1}^{-1} +$$
$$+ E_t a_{t+1} \left((p_{t+1} + d_{t+1})\Theta_t - p_{t+1}\Theta_{t+1} \right) = \bar{\rho}_t (g_t(\Theta) - p_t\Theta_t) \bar{\rho}_t^{-1} a_t + a_t p_t \Theta_t = a_t g_t(\Theta).$$

(ii) Notice that

$$g_t(\Theta) = (p_t + d_t)\Theta_{t-1} + \bar{r}_t \bar{\rho}_{t-1}^{-1} \sum_{s=1}^{t-1} \bar{\rho}_s \left((p_s + d_s)\Theta_{s-1} - p_s\Theta_s \right) = (p_t + d_t)\Lambda_{t-1}.$$

The existence of such a Λ was guaranteed by the previous part and Lemma A.1. We need to show that $\mathbf{I}+\Lambda_{t-1}$ is nonsigular, under the conditions given in the proposition.

This is indeed the case, as it can be checked that

$$(\mathbf{I}+\Lambda_{t-1})^{-1} = (\mathbf{I}+\Theta_{t-1})^{-1} - \frac{(\mathbf{I}+\Theta_{t-1})^{-1}\bar{\theta}_{t-1}\lambda'_{t-1}(\Theta)(\mathbf{I}+\Theta_{t-1})^{-1}}{1+\lambda'_{t-1}(\Theta)(\mathbf{I}+\Theta_{t-1})^{-1}\bar{\theta}_{t-1}}.$$

(iii) For $\Theta = I$, $g(\Theta) \geq 0$ from (B.2). Moreover,

$$(\mathbf{I}+\Lambda_{t-1})^{-1} = \frac{1}{2}\mathbf{I} + \frac{1}{4}\frac{\bar{\theta}_{t-1}\lambda'_{t-1}(I)}{1+\lambda'_{t-1}(I)\bar{\theta}_{t-1}/2},$$

and the conclusion follows, since $1+\lambda'_{t-1}(I)\bar{\theta}_{t-1}/2 \neq 0$, taking into account that $\lambda'_{t-1}(I) = (p_{t-1}\bar{\theta}_{t-1})^{-1}\bar{\rho}_{t-1}^{-1}\sum_{s=1}^{t-1}\bar{\rho}_s d_s \geq 0$ and $\bar{\theta} \in X_+^{J\times 1}$. We would have arrived at the same conclusion by computing $\det(\mathbf{I}+\Lambda_{t-1})$, which equals $2^J(1+\lambda'_{t-1}(I)\bar{\theta}_{t-1}/2)$.

(iv) It is immediate to check that $(\mathbf{I}+\Theta_{t-1})^{-1} = \mathbf{I} - \Theta_{t-1}/(1+\Theta_{t-1}^{11})$, and that $1+\lambda'_{t-1}(\Theta)(\mathbf{I}+\Theta_{t-1})^{-1}\bar{\theta}_{t-1} = 1+\bar{\theta}_{t-1}^{1}/(1+\Theta_{t-1}^{11})$. Then we use part (i). $\square$

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