Platform Differentiation in Two-Sided Markets

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Abstract

In this paper we investigate a two-sided market platform model. In the standard framework, firms interact with consumers through one or more platform providers. For example, video game providers (firms) develop games for a gaming console (platform), and end users (consumers) play these games on the console. Consumer utility may depend on the number of other consumers who interact on one side of the platform and the number of firms that interact on the other side of the platform. We develop a general monopoly platform model and derive general and closed form solutions which are consistent with the previous literature. We also characterize consumer’s demand for the platform and show how it relates to the simple one-sided monopoly case. This model is extended to include same side network effects and we derive similar results. We then extend the model to include competition between two homogeneous platforms. We determine the Nash Equilibrium for two competing platforms and show how these relate to several current two-sided markets such as those for video game consoles and smartphones. Lastly, we compare welfare between a monopoly platform and two competing platforms and show their are cases where a monopoly platform is welfare improving.

1 Introduction

Many transactions among multiple parties are conducted through an intermediary or platform. Transactions need not involve the simple cash payment for a particular good at a consignment store. For many consumers, purchases are made online through platforms like eBay or Amazon. Some buyers actually consume their purchases on the platform or network: Google users obtain searches on Google, Facebook users visit Facebook pages of their friends, smartphone users use apps on their phones, and video game users play games on their gaming console. Today, platforms or networks provide many commodities in our society and their market structures vary considerably. For example, Google and Facebook are near monopolists who provide their services for free; however smartphones and video game consoles are generally priced well above zero. Given their relevance and the variety of pricing strategies used, it is important for these markets to be thoroughly understood.
The early development of this literature began with markets where network externalities were prominent. Katz and Shapiro (1985) and Economides (1996a). Caillaud and Jullien (2003) investigate competition further but assume that coordination favors the incumbent platform; otherwise platforms may fail to launch. There has been a considerable amount of focus on two-sided markets: Evans (2003); Rochet and Tirole (2003); Armstrong (2006); Rochet and Tirole (2006). Many have attempted to resolve the issues surrounding the failure to launch and the multiple equilibrium aspects which occur in multi-sided market models, Ellison and Fudenberg (2003) and Hagiu (2006).

Weyl (2010) generalizes many of the previous pricing models including those of Armstrong and Rochet and Tirole for the monopoly case using what he calls insulating tariffs. With an insulating tariff, the price that a platform charges on one side of the market depends on the number of agents on the other side. An insulating tariff makes price a function of the number of users on the other side and in doing so resolves the failure to launch and multiple equilibrium issues which occur in other models. Weyl and White (2013) extend this model with insulating tariffs to the case where there exist multiple competing platforms. Both models are very general, allowing for rich heterogeneity and same side network effects. Same side network effects occur when the utility generated on one side of the platform depends on the number of participants on its own side; these can be positive. For example, the utility of video game users is increasing in the number of potential users with whom they can play. These same side network effects can also be negative, such as on eBay where bidders have to pay higher prices when there are more bidders present. A platform such as this has been investigated by Deltas and Jeitschko (2007).

Lee investigates how a contracting game between firms, such as game suppliers, and the platform determines equilibrium outcomes. In this sense firms are considered to be large players which have an effect on the pricing equilibrium; in prior literature all sides are considered price takers. Lee also gives insight as to how “tipping” in a platform setting can occur. Tipping takes place when a platform gains complete access to one side of the market and thus claims monopoly power of the entire market. Tipping remains one of the main topics of future research in the network and platform field. Another main area of platform research is entry and exit. Some work has been done on entry and exit when network effects exist: Economides (1996b); Kim et al. (2013); Goldfarb and Xiao (2011); Goolsbee and Syverson (2008); Prieger and Hu (2012); Fan and Xiao (2010). However, most of these papers are empirical. One of the main empirical contributions is Lee (2013). He investigates the video game market when Xbox entered the market from 2000-2005. He finds that through exclusive contracts, what we will consider as a form of platform differentiation, with game developers allowed Microsoft to enter the video game market with Xbox. We hope to develop a theoretical model that connects that connects the traditional models with some of these more recent findings.

Unfortunately, seminal models do not naturally extend to the study of platform entry and exit. For this reason, we propose a new model for two-sided markets that, in keeping with Weyl and White, allows for rich heterogeneity and incorporates entry and exit. We show that this model allows for applications which relate nicely to traditional one-sided market models,
provide new insights regarding multiple equilibrium, and allows for the inclusion of same side network effects in a well behaved manner. This model is a variation of many of the two-sided model frameworks and it will allow for more general forms of platform differentiation with richer insights into platform competition.

The rest of the paper is organized as follows. Section 2 describes the general model for a monopoly platform. Section 3 develops our base model of a two-sided market of consumers and firms. We thoroughly analyze this model and determine consumers demand for the platform. Section 4 adds consumer same side network effects to the base model and analyzes this addition. In section 5 we extend the base model to allow for multiple platforms and determine the Nash Equilibrium when two homogeneous platforms compete. We also determine how consumers and firms will allocate themselves, by either single-homing or multi-homing, on the platforms. Section 6 concludes.

2 General Model

Suppose there exists a market with two groups who would benefit from interaction but are unable to do so without an intermediary. We call this intermediary a platform. The platform charges agents in each group a price to participation on the platform. The utility of one group depends on the number of agents of the other group who join the platform. We start with a simple monopoly platform model and later extend the model to include consumer same side network effects, multiple differentiated platforms, platform competition, and entry.

Agents on both sides of the platform are described by continuous variables. Agents on side 1 are indexed by $\tau \in [0, \hat{N}_1]$ and agents on side 2 are indexed by $\vartheta \in [0, \hat{N}_2]$. The number of agents on side 1 who join the platform is denoted by $n_1 \in [0, \hat{N}_1]$ and the number of agents on side 2 who join the platform is denoted by $n_2 \in [0, \hat{N}_2]$ where $\hat{N}_i$ are the maximum number of agents on side $i$. The utility for an agent on side 1 of type $\tau$ is given by:

$$u_1(\tau) = v_1(\tau) + \alpha_1(\tau) \cdot n_2 - p_1,$$

where $v_1(\tau)$ is the membership value agent $\tau$ receives from joining the platform. $p_1$ is the price an agent on side 1 pays to join the platform, and $\alpha_1(\tau) \in [0, \infty)$ is the network effect, or marginal benefit, of an additional agent of side 2 on the platform for an agent of type $\tau$. Thus agents are heterogeneous in both the membership benefit and their marginal benefit. If $\alpha_1(\tau) = \alpha_1$ and $v_1(\tau) = v_1$ for all $\tau$ then consumers are homogeneous. We assume the platform knows $v_1(\tau)$ and $\alpha_1(\tau)$ but cannot distinguish the individual values for each consumer $\tau$. Thus it cannot price discriminate between consumers.

The utility for an agent on side 2 of type $\vartheta$ is given by:

$$u_2(\vartheta) = v_2(\vartheta) + \alpha_2(\vartheta) \cdot n_1 - p_2,$$

where $v_2(\vartheta)$ is the membership value agent $\vartheta$ receives from joining the platform. $p_2$ is the price an agent on side 2 pays to join the platform, and $\alpha_2(\vartheta)$ is the network effect, or marginal benefit, of an additional side 1 agent on the platform for an agent of type $\vartheta$. We again assume...
the platform cannot price discriminate between side 2 agents. Furthermore, we assume the
functions \( v_i(\cdot) \) and \( \alpha_i(\cdot) \) for \( i = 1, 2 \) are continuous, non-increasing, and twice continuously
differentiable.

### 2.1 General Pricing with a Monopoly Platform

In this section, the platform is a price setter. Platform profits are given by:

\[
\Pi^M = n_1(p_1 - f_1) + n_2(p_2 - f_2), \tag{3}
\]

where \( f_i \) is the marginal cost the platform pays for an agent on side \( i \) to participate on
the platform. The last agent on each side to join the platform will have utility equal to
zero. These agents \( \tau \) and \( \vartheta \) will give the total mass of agents on side 1 and 2 that join the
platform. Thus the number of agents on side 1 who join the platform is determined from:

\[
u_1(\tau = n_1) = v_1(\tau) + \alpha_1(\tau) \cdot n_2 - p_1 = 0.
\]

Solving for \( p_1 \) gives side 1 price as a function of the number agents on sides 1 and 2. Thus we have:

\[
p_1(n_1, n_2) = v_1(n_1) + \alpha_1(n_1) \cdot n_2. \tag{4}
\]

Similarly, on side 2: \( u_2(\vartheta = n_2) = v_2(n_2) + \alpha_2(n_2) \cdot n_1 - p_2 = 0 \) gives the number of side 2
agents that will participate on the platform. This implies

\[
p_2(n_1, n_2) = v_2(n_2) + \alpha_2(n_2) \cdot n_1. \tag{5}
\]

Substituting (4) and (5) into the profit function, (3), gives platform profits as a function of
\( n_1 \) and \( n_2 \):

\[
\Pi^M(n_1, n_2) = n_1[v_1(n_1) + \alpha_1(n_1)n_2 - f_1] + n_2[v_2(n_2) + \alpha_2(n_2)n_1 - f_2].
\]

Thus choosing the optimal number of agents on side 1 and 2 will determine optimal prices.
Maximizing platform profit with respect to \( n_1 \) and \( n_2 \) gives two first-order conditions and
two second-order conditions which must be satisfied:

\[
[v_1(n_1) + \alpha_1(n_1)n_2 - f_1] + n_1[v_1'(n_1) + \alpha_1'(n_1)n_2] + n_2\alpha_2(n_2) = 0, \tag{6}
\]

\[
n_1\alpha_1(n_1) + [v_2(n_2) + \alpha_2(n_2)n_1 - f_2] + n_2[v_2'(n_2) + \alpha_2'(n_2)n_1] = 0, \tag{7}
\]

\[
S_1 = 2[v_1'(n_1) + \alpha_1'(n_1)n_2] + n_1[v_1''(n_1) + \alpha_1''(n_1)n_2] < 0, \tag{8}
\]

\[
S_2 = 2[v_2'(n_2) + \alpha_2'(n_2)n_1] + n_2[v_2''(n_2) + \alpha_2''(n_2)n_1] < 0, \tag{9}
\]

\[
S_1 \cdot S_2 > [\alpha_1(n_1) + \alpha_1'(n_1)n_1 + \alpha_2(n_2) + \alpha_2'(n_2)n_2]^2. \tag{10}
\]
The second-order conditions, equations (8), (9), and (10), impose restrictions on the functional forms of the membership and network benefits. This restricts the amount of heterogeneity we can allow in our model. In what follows we assume these conditions hold. Later we will provide more structure and insight toward these functions. As noted by Weyl and White (2013), implementation of dominant strategies by the platform is infeasible with rich heterogeneity of the agents, (see page 12 of Weyl and White (2013)).

The first-order conditions, equations (6) and (7), provide an intuitive description of optimal prices which are similar to the optimal prices found in previous literature on monopoly platform pricing. Using equations (4) and (6) gives

$$p_1 = f_1 - n_2 \alpha_2(n_2) + n_1 \cdot [-v'_1(n_1) - \alpha'_1(n_1)n_2].$$  

(11)

This says that the price side 1 agents pay equals the marginal cost of consumer participation, $f_1$, minus the external benefit to side 2, $n_2 \alpha_2(n_2)$, plus the gain to the marginal side 1 agent times the number of side 1 agents on the platform, $n_1 \cdot [-v'_1(n_1) - \alpha'_1(n_1)n_2]$. Using equations (5) and (7) gives

$$p_2 = f_2 - n_1 \alpha_1(n_1) + n_2 \cdot [-v'_2(n_2) - \alpha'_2(n_2)n_1].$$  

(12)

The side 2 price equals the marginal cost of participation, $f_2$, minus the external benefit to side 1, $n_1 \alpha_1(n_1)$, plus the gain to the marginal agent on side 2 who joins the platform times the number of side 2 agents who join the platform, $n_2 \cdot [-v'_2(n_2) - \alpha'_2(n_2)n_1]$. The socially optimal price is $p_1 = f_1 - \text{avg}_{\vartheta}(\alpha_2(\vartheta)) \cdot n_2$, were $\text{avg}_{\vartheta}(\alpha_2(\vartheta))$ is the average of $\alpha_2(\vartheta)$ over the $\vartheta$ who participate on the platform. The added term in equation (11) is the monopoly mark up. Thus the monopoly platform distorts the price from the socially optimal one in two ways. The first is the monopoly mark up due to market power. The second derives from the externality mark down which is taken with respect to the marginal side 2 agent by the monopoly platform instead of an average over all side 2 agents who participate on the platform. This is known as the Spence distortion.\footnote{This was first noticed in the platform models by Weyl (2010).}

The marginal externality is less than the average, hence $p_1$ is marked up further.

In the following section we introduce a base model were we apply this general framework to a model of a two-sided market with consumers and firms who interact through a monopoly platform.

### 3 Base Model

In this model we investigate a two-sided market with consumers and firms who interact through a monopoly platform. Many examples come to mind: video game platforms with consumers and game developers, smartphones with consumers and app producers, and Netflix

\footnote{For this reason we develop a base model in the next section that reduces the amount of heterogeneity but allows to investigate many aspects of two-sided markets.}
with consumers and video distributors. To gain insight into these kinds of two-sided markets we make assumptions on the functional forms of the consumer and firm utility functions.

On the consumer side, we assume that $v_1(\tau) = v$ for all $\tau$; that is, consumers are homogenous in their membership benefit for joining the platform. We also assume $\alpha_1(\tau)$ is decreasing in $\tau$ so that consumers are heterogeneous in their marginal benefit for an additional firm. The argument here is that consumers place a different marginal value for the other side of the platform. For example, I have many apps on my iPhone and look for new apps weekly; my mother has a handful of apps on her iPhone and never looks for new ones. Thus the marginal utility from the number of apps available differs. Since $\alpha_1(\tau)$ is decreasing it orders consumers by their marginal benefits. Consumers whose type is $\tau$ close to zero have marginal benefits that are high relative to those consumers whose type is located far from zero. Consumer utility is then given by:

$$u_1(\tau) = v + \alpha_1(\tau) \cdot n_2 - p_1.$$  
(13)

On the firm’s side we make different functional form assumptions which better describe the firm’s problem. We assume the marginal gain from an additional consumer is constant for all firms, $\alpha_2(\vartheta) = \alpha_2 > 0$. The intuition here is that an additional consumer joining the platform shifts the firms demand curve out by the same amount for all firms. Each firm being homogeneous to consumers implies this. Firms are heterogeneous in their cost to synchronize their product to the platform. This is the cost of configuring its product for use on the platform. We assume this cost is determined by the firms location $\vartheta \in [0, n_2]$ so that firm $\vartheta$ has a synchronization cost of $c \cdot \vartheta$. This implies $v_2(\vartheta) = -c\vartheta$. Thus profits for firm of type $\vartheta$ is given by:

$$u_2(\vartheta) = \pi_2(\vartheta) = -c \cdot \vartheta \alpha_2 \cdot n_1 - p_2,$$  
(14)

We next develop the platforms problem in the base model.

### 3.1 Pricing with a Monopoly Platform

The platform profit function is the same as in the general model. So platform profits are given by equation [3]. Again, the last agent on each side to join the platform will have utility equal to zero so that $u_1(\tau = n_1) = 0$ and $\pi_2(\vartheta = n_2) = 0$. Thus we can again get prices as a function of the number of consumers and firms who participate on the platform, $p_1(n_1, n_2)$ and $p_2(n_1, n_2)$ which follow equations [4] and [5]. Substituting these into the platforms profit function and maximizing profits with respect to $n_1$ and $n_2$ we have two first-order conditions and three second-order conditions that must be satisfied:

$$[v + \alpha_1(n_1) n_2 - f_1] + n_1[\alpha_1'(n_1) n_2] + \alpha_2 n_2 = 0,$$  
(15)

$$n_1 \alpha_1(n_1) + [\alpha_2 n_1 - c n_2 - f_2] - c n_2 = 0,$$  
(16)
\[2n_2\alpha'_1(n_1) + n_1n_2\alpha''(n_1) < 0, \quad (17)\]
\[-2c < 0, \quad (18)\]
\[2c[2n_2(-\alpha'_1(n_1)) - n_1n_2\alpha''(n_1)] - [\alpha_1(n_1) + n_1\alpha'_1(n_1) + \alpha_2]^2 > 0. \quad (19)\]

The second-order conditions, equations (17), (18), and (19), essentially imply \(\alpha''_1(\cdot)\) cannot be positive. Note, the second-order conditions hold for all linear \(\alpha_1(\cdot)\). The first-order conditions, equations (15) and (16), give the optimal prices the platform charges:

\[p_1 = f_1 - n_2\alpha_2 + n_1 \cdot [-\alpha'_1(n_1)n_2], \quad (20)\]
\[p_2 = f_2 - n_1\alpha_1(n_1) + c \cdot n_2. \quad (21)\]

For the price consumers face \(f_1\) is the marginal cost to the platform for an additional consumer, \(n_2\alpha_2\) is the benefit to the firm side for an additional consumer, and \(n_1 \cdot [-\alpha'_1(n_1)n_2] > 0\) is the monopoly markup. For the firm price \(f_2\) is the marginal cost to the platform for an additional firm, \(n_1\alpha_1(n_1)\) is the benefit to the marginal consumer for an additional firm, and \(c \cdot n_2\) is the monopoly markup.

Using this base model we can derive closed form solutions when we specify the function \(\alpha_1(\tau)\). With this additional structure we can investigate how how the two-sided market problem relates to the classic one-sided price/quantity model. We do this by holding the firm side of the market fixed, both \(p_2\) and \(n_2\), and derive a conditional consumer demand for the platform. Furthermore, by then allowing \(n_2\) to vary with \(n_1\), the shape of the consumer inverse demand curve for the platform is determined and the existence of two effects from a change in consumer price, \(p_1\) are realized. First, there is a direct effect on \(n_1\) from the price change, then there is a feedback effect on \(n_1\) through the change in \(n_2\). In the following subsection we derive closed form solutions and investigate these concepts further. Note, the framework used in this base model and the functional form specifications used to solve for closed form solutions are still general relative to much of the multi-sided market literature that investigates closed form fixed prices.

### 3.2 Closed Form Solutions and Welfare

In this subsection we assume \(\alpha_1(\cdot)\) is linear, \(\alpha_1(\tau) = a - b\tau\). This occurs when consumer \(\alpha_1\)'s are distributed uniformly over \([0, a]\) and the number of potential consumers present on the platform is \(\tilde{N}_1 = \frac{a}{b}\). Consumer utility is now given by

\[u_1(\tau) = v + [a - b\tau] \cdot n_2 - p_1. \quad (22)\]

The firm’s profit function is the same as in the base model:
\[ \pi_2(\vartheta) = \alpha_2 \cdot n_1 - c \cdot \vartheta - p_2, \] (23)

Since \(\alpha_1(\tau) = a - b\tau\), we have \(\alpha_1'(\tau) = -b\) for all \(\tau \in [0, \frac{\alpha}{b}]\). Using the first order conditions, (15) and (16), we have\(^3\)

\[ n_1 = \frac{1}{2b} [a + \alpha_2] + \frac{v - f_1}{2bn_2}, \] (24)

\[ n_2 = \frac{n_1}{2c} [a - bn_1 + \alpha_2] - \frac{f_2}{2c}. \] (25)

To make analysis simpler we assume \(v = f_1 \geq 0\) and \(f_2 = 0\). These assumptions are not extremely critical in the analysis and they make computations simple. The assumption that \(v \geq f_1\) is the only part that has an affect. This case with \(v = f_1 \geq 0\) and \(f_2 = 0\) fits with the smartphone and video game examples we have and will discuss through out this paper. The marginal cost of firms, or app producers, is zero. Firms pay synchronization costs that drive marginal cost to zero for the platform. Consumers receive membership benefits from have a smartphone or a video game console, they can make phone calls or watch DVDs, and producing each piece of hardware for an individual consumer has a positive marginal cost. So long as this membership value is close to and/or greater than this marginal cost we expect this analysis to be accurate.

Since the number of potential consumers who can join the platform is finite, \(\hat{N}_1 = \frac{a}{b}\), there exists a corner solution. We give the interior solution first, then we give the corner solution. The interior solution will never occur when \(\alpha_2 \geq a\), this will be clear when we present the interior solution. This means the marginal gain from the other side of the platform is greater for firms; firms have a higher marginal gain then every consumer. To extract the most surplus, the platform will want to capture as many consumers as they can with a low consumer price; this will generate a much greater surplus on the firm side which the monopoly platform will extract from. When \(\alpha_2 < a\) we will compare the platform profits between the corner solution and interior solution and determine which parameter values result in each type of solution.

For the interior solution, the assumptions \(\alpha_2 < a\), \(v = f_1 \geq 0\), and \(f_2 = 0\) and equations (20), (21), (24), and (25) imply:

\[ n_1^* = \frac{1}{2b} [a + \alpha_2], \] (26)

\[ p_1^* = v + \frac{1}{16bc} [a + \alpha_2]^2 (a - \alpha_2) > 0, \] (27)

\[ n_2^* = \frac{1}{8bc} [a + \alpha_2]^2, \] (28)

\(^3\)The second-order conditions hold for this example. This can be seen by substituting the linear \(\alpha_1(\cdot)\) into the second-order conditions, equations (17), (18), and (19).
\[ p_2^* = \frac{1}{8b} [a + \alpha_2](3\alpha_2 - a). \] (29)

There are two key things to notice in this equilibrium. First, recall the usual monopoly problem with inverse demand, \( p = a - bQ \) and marginal cost equal to zero. The equilibrium number of consumers who purchase from the monopoly is given by \( Q^* = \frac{a}{2b} < n_1^* \). So in equilibrium, a monopoly platform will have more consumers than a traditional monopolist. This is since additional consumers generate additional surplus on the platform. Second, firm price can be negative. Firms are subsidized to join the platform when \( 3\alpha_2 < a \). Intuitively this means that if adding firms generates a significantly larger amount of surplus for consumers than consumers generate for firms, then the total surplus on the firm side is less important. The platform will subsidize firms allowing for a greater extraction of surplus on the consumer side. In what follows we focus on the interior solutions so \( \alpha_2 < a \).

Given equilibrium prices, the number consumers, and the number of firms, we can calculate platform profits. The superscript \( i \) denotes the interior solution.

\[ \Pi^i = n_1(p_1 - f_1) + n_2(p_2 - f_2) = \frac{1}{64b^2c}[a + \alpha_2]^4. \] (30)

Looking at the comparative statics we see that \( \frac{\partial \Pi}{\partial c}, \frac{\partial \Pi}{\partial b} < 0 \). This implies that an increase in \( c \), the synchronizing cost firms face, will decrease platform profits, and an increase in \( b \), the rate at which \( \alpha_1 \) decreases across consumers, will also decrease platform profits. We also have \( \frac{\partial \Pi}{\partial a}, \frac{\partial \Pi}{\partial \alpha_2} > 0 \). This implies an increase in \( a \), or an upward shift in \( \alpha_1 \) for all consumers, will increase platform profits, and an increase in the gain firms receive from additional consumers, \( \alpha_2 \), will increase platform profits. These are the comparative static results one would expect and they give insight as to how the platform would like to affect these parameters if they were made endogenous through some form of technological improvement.

We can also derive consumer, firm, and total surplus given the platform’s optimal pricing scheme:

\[ CS^i \equiv \int_0^{n_1^*} [v + \alpha_1(\tau)n_2^* - p_1^*] d\tau = \frac{(a + \alpha_2)^4}{64b^2c} = \Pi^M, \] (31)

\[ FS^i \equiv \int_0^{n_2^*} [\alpha_2 n_1^* - c\theta - p_2^*] d\theta = \frac{(a + \alpha_2)^4}{128b^2c} = (1/2)\Pi^M, \] (32)

\[ W^i = TS^i = \frac{5(a + \alpha_2)^4}{128b^2c}. \] (33)

Notice, consumer surplus is equal to platform profits which are twice as large as firm surplus. Thus comparative statics for the types of surplus are the same as those for the platforms profit. This is not surprising since the comparative statics are all in relation to generating or reducing some kind of surplus which affects all agents through the platform in this model.

We now investigate the corner solution which always occurs when \( \alpha_2 \geq a \), and once this is complete we investigate for which cases it occurs when \( \alpha_2 < a \). In the corner solution we
know all consumers participate on the platform so for the consumer side of the market we have:

\[ n_1^{**} = \frac{a}{b} = \hat{N}_1, \]  
(34)

\[ p_1^{**} = v = f_1 \geq 0. \]  
(35)

This is the highest consumer price the platform can charge that will induce all consumers to join the platform when \( \alpha_2 > a \). Given this, the platform maximizes profits with respect to \( n_2 \) with \( p_2 = p_2(n_1 = a/b, n_2) = \frac{\alpha_2 a}{b} - cn_2 \). This gives the firm side equilibrium for the corner solution:

\[ n_2^{**} = \frac{a\alpha_2}{2bc}, \]  
(36)

\[ p_2^{**} = \frac{a\alpha_2}{2b}. \]  
(37)

Given the corner solution, we calculate welfare. The superscript \( c \) denotes the corner solution results.

\[ \Pi^c = n_1(p_1 - f_1) + n_2(p_2 - f_2) = n_1(0) + n_2p_2 = \frac{\alpha_2^2a^2}{4b^2c}, \]  
(38)

\[ CS^c \equiv \int_0^{n_1^{**}} [v + \alpha_1(\tau)n_2^{**} - p_1^{**}]d\tau = \frac{a^3\alpha_2}{4b^2c}, \]  
(39)

\[ FS^c \equiv \int_0^{n_2^{**}} [\alpha_2n_1^{**} - c\vartheta - p_2^{**}]d\vartheta = \frac{\alpha_2^2a^2}{8b^2c} = (1/2)\Pi^c, \]  
(40)

\[ W^c = TS^c = \frac{\alpha_2^2a^2}{8b^2c}[3\alpha_2 + 2a]. \]  
(41)

The comparative statics are what we expect. Surpluses, firm price and firm participation are increasing in \( a \) and \( \alpha_2 \), which are the parameters that generate surplus, and are decreasing in \( b \) and \( c \), which are the parameters that extract surplus. By comparing platform profits for when \( \alpha_2 < a \), \( \Pi^c > \Pi^i \) if and only if \( a^4 + 4a^3\alpha_2 - 10a^2\alpha_2^2 + 4\alpha^3_2 + \alpha_2^4 < 0 \). This inequality never holds true for \( \alpha_2, a \geq 0 \); so when \( \alpha_2 < a \) we have the interior solution and when \( \alpha_2 \geq a \) we have the corner solution. In the next subsection we focus on the case when \( \alpha_2 > a \) and we determine consumer inverse demand.

### 3.2.1 Consumer Inverse Demand for the Platform.

As discussed above we can derive consumer inverse demand for the platform, the price effect, and the feedback effect for consumers when the price for the firm side is fixed. In this section we investigate the case for \( v = f_1 \geq 0 \) and \( f_2 = 0 \); see the proof of Theorem 1 in the appendix
for the general case. For the simplicity of illustrating these concepts, we assume $3\alpha_2 = a$, this means the total number of consumers present is $\hat{N}_1 = \frac{a}{b} = \frac{3\alpha_2}{b}$. This condition and equations (25) and (29) implies $n_2^* = \frac{n_1}{2c}[3\alpha_2 - bn_1 + \alpha_2] = \frac{p}{2c}[4\alpha_2 - bn_1]n_1$ and firm price is zero, $p_2^* = 0$. First we characterize consumer inverse demand when both firm price and participation are fixed. This implies:

$$D(n_1|n_2, p_2^* = 0) = p_1 = v + (a - bn_1)n_2$$

(42)

We know $n_2^*$ is a function of $n_1$, $n_2^* = n_2^*(n_1) = \frac{1}{2c}[4\alpha_2 - bn_1]n_1$, so keeping $p_2^*$ fixed we allow the equilibrium number of firms to vary and this gives consumer’s inverse demand for the platform:

$$D(n_1|p_2^* = 0) = p_1 = v + (a - bn_1)n_2^*(n_1) = v + \frac{1}{2c}(n_1)(3\alpha_2 - bn_1)(4\alpha_2 - bn_1).$$

(43)

This means on the set $n_1 \in [0, \hat{N}_1]$ the inverse demand is hill shaped. When $n_1 > \hat{N}_1$ the function will eventually begin to increase again. However, the set of potential consumers is restricted hence the inverse demand is only the hill shaped part of this function. See Figure 1. This purely hill shaped inverse demand is true even when we generalize to the case when $v, f_1, f_2 \geq 0$.

**Figure 1: Consumer Inverse Demand**

![Figure 1: Consumer Inverse Demand](image)

Thus, for low $n_1$, $n_1$ is increasing in price. This is true because a low $n_1$ implies $n_2$ is also low, hence the utility consumers receive is low. A hill shaped demand implies that

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4For further analysis of hill shaped demand see (Becker 1991).
there exists a maximum price, $\overline{p}_1$, and a corresponding $\overline{n}_1$, such that for prices above $\overline{p}_1$ the market collapses. The hill shaped demand means that for every price below $\overline{p}_1$ there exists two potential outcome quantities of consumers and firms, a low consumer/firm pair and a high pair. This can be seen in Figure 1. However, the outcome for the low pair will not be “stable” in equilibrium since at this point the platform can increase profits by raising price $p_1$. In this sense the pair is unstable:

**Definition 1.** An outcome $(p'_1, n'_1)$, is **unstable** when $\frac{\partial p'_1}{\partial n'_1} > 0$. Otherwise we say the outcome is a **stable equilibrium**.

In the monopoly analysis, we will only focus on stable equilibria since at an unstable outcome the platform will just increase its price. This analysis does give insight towards a potential resolution to the multiplicity of equilibrium issue many multi-sided market papers face. By increasing price, platforms can choose the outcome which maximizes their profit. Using equations (26) and (43) we can identify the stable equilibrium and the unstable outcome.

Stable : $(n_1^s = \frac{2\alpha_2}{b}, p_1^* = \frac{2\alpha_2^3}{bc})$,

Unstable : $(n_1^u, p_1^* = \frac{2\alpha_2^3}{bc})$.

$n_1^u$ is the smallest root that solves $\frac{2\alpha_2^3}{bc} = \frac{1}{2c}(n_1)(3\alpha_2 - bn_1)(4\alpha_2 - bn_1)$. Note that the stable equilibrium is consistent with the profit maximizing solution, (26) and (27), assuming $3\alpha_2 = a$. These are displayed in Figure 2 along with consumer inverse demand, $D_1(n_1|p_2^* = 0)$, and three examples of the $p_1(n_1|n_2, p_2^* = 0)$ used to generate the consumer inverse demand.
In the above analysis we assumed $3\alpha_2 = a$, $v = f_1$, and $f_2 = 0$ to simplify the computations. However, this hill shaped demand exists more generally as noted in our first theorem; all proofs are in the appendix.

**Theorem 1.** If $\alpha_1(\tau) = a - b\tau$ and $v, f_1, f_2 \geq 0$ but restricted to such values where the market exists, see proof for details, then consumer inverse demand for the platform is hill shaped and there exist a stable equilibrium.

### 3.2.2 Direct Price and Feedback Effects

Lastly, using the $p_1(n_1|n_2, p_2^*)$ functions and continuing to assume that $3\alpha_2 = a$ so that $p_2^* = 0$ we can determine the price and feedback effects on consumer participation from a change in consumer price, $p_1$. The **price effect** is simply the change in $n_1$ given a change in $p_1$ when $n_2 = \bar{n}_2$ is fixed. This is given by:

$$
\left. \frac{dn_1}{dp_1} \right|_{n_2=\text{constant}} = -b\bar{n}_2 < 0 \quad (46)
$$

This allows us to break the total effect down into the price effect plus the feedback effect:

$$
\frac{dn_1}{dp_1} = \frac{\partial n_1}{\partial p_1} \bigg|_{n_2=\text{constant}} + \frac{\partial n_1}{\partial n_2} \frac{\partial n_2}{\partial p_1} \quad (47)
$$

Figure 3, displays the price and feedback effects.
This simple example using a uniform distribution of $\alpha_1$ has allowed us to investigate welfare, one-sided demand for the platform, and the price and feedback effects associated with a change in price. In the following section we extend the general model to include same side consumer network effects.

4 A General Model With Consumer Same Side Network Effects

Here we extend the general model to include same side network effects for consumers. This means for a given number of consumers $n_1$ present on the platform, consumer $\tau$ receives an additional $\beta(n_1, \tau)$ utility; were this marginal utility for the same side of the market can depend on individual consumers type $\tau$ and the number of consumers present, $n_1$. If we have decreasing marginal returns in the number of consumers on the consumer side then $\frac{\partial \beta}{\partial n_1} \equiv \beta_1 = 0$ If the consumer same side effect is homogeneous for consumers then $\frac{\partial \beta}{\partial \tau} \equiv \beta_2 = 0$ for all $\tau$ and $\beta$ is simply a function of the number of consumers present. Note, $\beta$ can be positive or negative. For example, $\beta > 0$ for video game platforms, Xbox 360 and Playstation 3, since playing video games with friends increases utility and this can only be done if friends are using the same platform. eBay is an example where consumers, or
bebidders, dislike having other bidders present since this causes prices to increase, hence β < 0. Consumer utility is given by:

\[ u_1(\tau) = v_1(\tau) + \alpha_1(\tau) \cdot n_2 + \beta(n_1, \tau) - p_1, \]  
(48)

Firm’s utility function is the same as in the general model, equation (2).

Given this extension, it is important to compare the differences in this model from our original general model where β(n_1, τ) = 0 for all n_1 and τ. Thus we again solve the monopoly platform’s problem of maximizing profits. As before we derive platform profits as a function of n_1 and n_2. This gives a the platform profit function:

\[ \Pi^M(n_1, n_2) = n_1[v_1(n_1) + \alpha_1(n_1)n_2 + \beta(n_1, n_1) - f_1] + n_2[v_2(n_2) + \alpha_2(n_2)n_1 - f_1]. \]  
(49)

Maximizing with respect to n_1 and n_2 gives two first-order conditions and three second-order conditions which must be satisfied:

\[ [v_1(n_1) + \alpha_1(n_1)n_2 + \beta(n_1, n_1) - f_1] + n_1[\alpha'_1(n_1)n_2 + \beta_1(n_1, n_1) + \beta_2(n_1, n_1)] + n_2\alpha_2(n_2) = 0, \]  
(50)

\[ n_1\alpha_1(n_1) + [v_2(n_2) + \alpha_2(n_2)n_1 - f_2] + n_2[v_2'(n_2) + \alpha_2'(n_2)n_1] = 0, \]  
(51)

\[ S'_1 = n_1[v_1''(n_1) + \alpha_1''(n_1)n_2 + \beta_1(n_1, n_1) + 2\beta_11(n_1, n_1) + \beta_22(n_1, n_1)] + 2[v_1'(n_1) + \alpha_1'(n_1)n_2 + \beta_1(n_1, n_1) + \beta_2(n_1, n_1)] < 0, \]  
(52)

\[ S'_2 = 2[v_2'(n_2) + \alpha_2'(n_2)n_1] + n_2[v_2''(n_2) + \alpha_2''(n_2)n_1] < 0, \]  
(53)

\[ S'_1 \cdot S'_2 > [\alpha_1(n_1) + \alpha_1'(n_1)n_1 + \alpha_2(n_2) + \alpha_2'(n_2)n_2]^2. \]  
(54)

where \( \beta_{11}(n_1, n_1) = \frac{\partial^2 \beta}{\partial n_1^2}, \) \( \beta_{12}(n_1, n_1) = \frac{\partial^2 \beta}{\partial n_1 \partial n_2 | n_2 = n_1}, \) and \( \beta_{22}(n_1, n_1) = \frac{\partial^2 \beta}{\partial n_2^2}. \)

These conditions are similar to equations (15), (16), (17), (18) and (19) from the base model. The second-order conditions put restrictions on the functional forms of \( \beta(\cdot), \) the \( \alpha(\cdot)s, \) and the \( v(\cdot)s. \) The first-order conditions, equations (50) and (51), give us the optimal pricing scheme:

\[ p_1 = f_1 - n_2\alpha_2(n_2) + n_1 \cdot [-v_1'(n_1) - \alpha'_1(n_1)n_2 - \beta_1(n_1, n_1) - \beta_2(n_1, n_1)], \]  
(55)

\[ p_2 = f_2 - n_1\alpha_1(n_1) + n_2 \cdot [-v_2'(n_2) - \alpha'_2(n_2)n_1]. \]  
(56)

These equations imply consumer price, \( p_1 \) is directly affected by the added surplus from the consumer same side effect for the marginal consumer \( \tau = n_1, -n_1 \cdot [\beta_1(n_1, n_1) + \beta_2(n_1, n_1)]. \)

However, firm price, \( p_2, \) is not directly affected by the additional \( \beta. \) It is indirectly affected by the consumer same side effect through the change in \( n_1 \) which will occur when \( p_1 \) is adjusted to include the consumer same side effects. In the following subsection we add same side effects to the base model we previously investigated.
4.1 Base Model with Consumer Same Side Effects

We now extend our base model to include consumer same side network effects. Firm’s profit function remains the same as in the base model, it is given by equation (14). Consumer utility is the same as in the base model with equation (13) except we add the same side effect. We assume the same side effect function \( \beta \) is linear with respect to \( n_1 \) and homogeneous with respect to \( \tau \), i.e. \( \beta = \beta n_1 \). Thus consumer utility is given by:

\[
    u_1(\tau) = v + \alpha_1(\tau) \cdot n_2 + \beta n_1 - p_1.
\]  

Using this specifications, equations (50), (51), (52), (53), and (54) imply:

\[
    [v + \alpha_1(n_1)n_2 + \beta n_1 - f_1] + n_1[\alpha'_1(n_1)n_2 + \beta] + \alpha_2 n_2 = 0, \tag{58}
\]

\[
    n_1 \alpha_1(n_1) + [\alpha_2 n_1 - cn_2 - f_2] - cn_2 = 0, \tag{59}
\]

\[
    2n_2 \alpha'_1(n_1) + 2\beta + n_1 n_2 \alpha''_1(n_1) < 0, \tag{60}
\]

\[
    -2c < 0, \tag{61}
\]

\[
    2c[2n_2(-\alpha'_1(n_1)) - 2\beta - n_1 n_2 \alpha''_1(n_1)] > [\alpha_1(n_1) + n_1 \alpha'_1(n_1) + \alpha_2]^2. \tag{62}
\]

The first-order conditions, equations (58) and (59), give the optimal pricing scheme for this example:

\[
    p_1 = f_1 - \alpha_2 n_2 - \beta n_1 + [\alpha'_1(n_1)n_2] \cdot n_1 \tag{63}
\]

\[
    p_2 = f_2 - \alpha_1(n_1) \cdot n_1 + cn_2 \tag{64}
\]

Compared to the price in the original base model, equation (20), \( p_1 \) has an additional term, \(-\beta n_1\). That is, the external benefit/cost to consumers is subtracted from the price they face. Although the firm price is not directly affected, it is indirectly affected through \( n_1 \) and \( n_2 \). To ensure maximizing behavior, we must restrict \( \beta \). If \( \beta \) is too large then \( p_1(n_1, n_2) \) will be u-shaped and the second-order conditions will fail. Thus for equations (60) and (62) to hold, the following condition must be met.

\[
    \beta \leq \min\{-\alpha'_1(n_1)n_2 - (1/2)n_1 n_2 \alpha''_1(n_1) - (1/4c)[\alpha_1(n_1) + n_1 \alpha'_1(n_1) + \alpha_2]^2, -\alpha'_1(n_1)n_2 - (1/2)n_1 n_2 \alpha''_1(n_1)\}. \tag{65}
\]

Given that \( \beta \) satisfies (65), we can investigate how changes in \( \beta \) affect the model.

**Proposition 1.** For all \( \alpha_1(\tau) \) continuous, decreasing and twice differentiable and for all \( v, f_1, f_2 \geq 0 \) we have \( \frac{\partial n_2}{\partial \beta} > 0 \).
Proofs are provided in the appendix. This result confirms our intuition; if the marginal benefit consumers receive from the number of consumers increases, then the number of consumers will increase even though some of the surplus is extracted by the monopoly platform.

**Proposition 2.** For all \( \alpha_1(\tau) \) continuous, decreasing and twice differentiable and for all \( v, f_1, f_2 \geq 0 \) we have \( \frac{\partial \Pi^M}{\partial \beta} > 0 \).

The intuition here is clear. If the platform holds prices fixed with an increase in \( \beta \), then \( n_1 \) and \( n_2 \) will increase. This will increase profits, hence optimal prices must increase profits. When there exists more surplus to extract the platforms profits will increase. Unfortunately the other comparative statics with respect to \( \beta \) are less clear.

**Proposition 3.** In general, \( \frac{\partial n_2}{\partial \beta} \), \( \frac{\partial p_1}{\partial \beta} \), and \( \frac{\partial p_2}{\partial \beta} \) are ambiguous. Furthermore, \( \frac{\partial CS}{\partial \beta} \), \( \frac{\partial FS}{\partial \beta} \), and \( \frac{\partial W}{\partial \beta} \) are ambiguous without further parameter restrictions.

Using this framework, the natural extension is to allow for entry and platform competition.

### 5 Base Model of Static Competition

In this section, we assume that there exists two platforms, \( A \) and \( B \), that play a static pricing game. For simplicity we assume consumer and firm utilities structures that are similar to those defined in the base model without consumer same side effects. Thus firms face heterogeneous synchronization costs and homogeneous marginal benefits for an additional consumer while consumers receive a homogeneous membership benefit for joining the platform and are heterogeneous in their marginal benefit for an additional firm. With multiple platforms we assume consumers and firms can either join a single platform (single-home) or join multiple platforms (multi-home). We assume the synchronization cost for a firm to join the platform is the same for platforms \( A \) and \( B \). So the cost to joining one platform for firm \( \vartheta \) is \( c_{\vartheta} \) and \( 2c_{\vartheta} \) to join both platforms. This means synchronization to one platform has no affect on the cost to synchronize to the other platform.

On the consumer side, if a consumer participates on two platforms his benefit from membership to the second platform diminishes by \( \delta \in [0, 1] \), so that his total membership benefit from the two platforms is \( (1 + \delta)v \). If \( \delta = 0 \), then there is no additional membership benefit from joining the second platform. If \( \delta = 1 \), then the membership benefit is unaffected by being a member of another platform. We assume that each platform has its own network effect function denoted by \( \alpha_{1,A}(\cdot) \) and \( \alpha_{1,B}(\cdot) \). The differences in these two functions describe

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5 For a closer look at these restrictions see the proof in the appendix.

6 Note, having both \( \alpha_{1,A}(\cdot) \) and \( \alpha_{1,B}(\cdot) \) be continuous implies some kind of correlation of marginal utilities for consumers across platforms. Although this means the competition model is not completely general, it is still considerably more general then the usual Hotelling or Salop models. As noted by Weyl and White (2013), implementation of dominant strategies by the platform is infeasible with rich consumer and firm heterogeneity, (see page 12 of Weyl and White (2013)).
the amount of platform differentiation that exists in the market. This differentiation is both vertical, higher $\alpha_1$ functions, and horizontal, different slopes of the $\alpha_1$ functions.

**Definition 2.** Platforms $A$ and $B$ are **homogeneous** if $\alpha_{1,A}(\cdot) = \alpha_{1,B}(\cdot) \equiv \alpha_1(\cdot)$ for all $\tau$.

Let $N_1$ ($N_2$) denote the total number of consumers (firms) who participate in at least one platform and let $n_{1A}^A$ ($n_{1B}^B$) denote the total number of consumers who participate on platform $A$ ($B$). Finally, let $n_{1m}$ be the number of consumers who multi-home; hence, $N_1 = n_{1A}^A + n_{1B}^B - n_{1m}$. Similarly, we have $n_{2A}^A$, $n_{2B}^B$, and $n_{2m}$ for the firm side. If firm $\vartheta$ single-homes on platform $i$, its profit is given by $\pi^i_2(\vartheta)$; if it multi-homes its profit is given by $\pi^{AB}_2(\vartheta)$. These functions are given by:

$$\pi^i_2(\vartheta) = \alpha_2 n_{1i} - c_2 - p^i_2,$$  \hspace{1cm} (66)  

$$\pi^{AB}_2(\vartheta) = \alpha_2 (n_{1A}^A + n_{1B}^B - n_{1m}) - 2c_2 - p^A_2 - p^B_2.$$  \hspace{1cm} (67)

If consumer $\tau$ single-homes on platform $i$ his utility is given by $u^i_1(\tau)$; if he multi-homes his utility is given by $u^{AB}_1(\tau)$ where,

$$u^i_1(\tau) = v + \alpha_{1,i}(\tau)n_{2i} - p^i_1,$$  \hspace{1cm} (68)

$$u^{AB}_1(\tau) = (1+\delta) v + \alpha_{1,A}(\tau)[n_{2A}^A - n_{2m}] + \alpha_{1,B}(\tau)[n_{2B}^B - n_{2m}] + \max\{\alpha_{1,A}(\tau), \alpha_{1,B}(\tau)\}n_{2m} - p^A_1 - p^B_1.$$  \hspace{1cm} (69)

For a consumer who multi-homes, the products which he can consume on either platform, $n_{2m}$, he consumes on the platform which provides a higher utility. This characterizes the forth term, $\max\{\alpha_{1,A}(\tau), \alpha_{1,B}(\tau)\}n_{2m}$, in equation (69).

The sequence of play is as follows: first the platforms simultaneously choose consumer and firm prices, $p^i_1$ and $p^i_2$ for $i = A, B$; then consumers and firms simultaneously choose which platforms to join.

This form of differentiation describes many types of two-sided markets better than much of the literature which uses the Hotelling line on both sides of the market to differentiate two competing platforms. For example, the synchronization costs app producers incur rarely varies across platforms; this cannot be modelled with the Hotelling line. Furthermore, the Hotelling line only differentiates platforms through their membership benefit. In our setting we allow differentiation to occur through the marginal benefits consumers receive. This is exactly the differentiation that exists between the Nintendo Wii, which has physically interactive gaming, and the Xbox 360, which has a high graphics based traditional controller gaming experience. Thus in our model of competition, differentiation occurs through the consumer side of the market; then based on the unrestricted allocation decisions allowed of the consumers and firms, firms must decide whether it is worth joining a platform based on the additional fixed cost and the additional price they would incur. To the best of our knowledge, this is the first paper to introduce platform differentiation in this way.
In the following subsection we determine the equilibrium when the two platforms are homogenous. We determine exactly which consumers and firms single-home or multi-home in equilibrium and the equilibrium allocations we get closely resemble what is seen in several two-sided markets. To the best of our knowledge, this is the first paper that allows both sides of the market to determine their homing choice in a static game.

5.1 Static Competition with Homogeneous Platforms

In this subsection we investigate the equilibrium that occur when there exists two homogeneous platforms, A and B, who compete in prices. This implies \( \alpha_1, A(\cdot) = \alpha_1, A(\cdot) = \alpha_1(\cdot) \). Furthermore, if platforms charge the same consumer prices, \( p_1^A = p_1^B \), then consumers will join the platform with more firms. If consumer prices and the number of firms are equal across platforms then consumers are indifferent between the two platforms; in this case, consumers will either not participate, multi-home, or single-home on platform A or B with equal probability. This implies that if prices are equal on both sides of the platform then in expectation the number of consumers will be the same on each platform and the number of firms will be the same on each platform, \( E(n_1^A) = E(n_1^B) \) and \( E(n_2^A) = E(n_2^B) \).

Once prices are set, consumers and firms simultaneously choose which platforms to join. This can lead failure to launch issues. Specifically, how will consumers and firms allocate if \( p_1^A > p_1^B \) and \( p_2^B > p_2^A \). When platforms price both sides equally, then, in expectation, the number of consumers and firms on each platform are equal. However, when prices on each side are not equal and neither platform sets both the low prices then how consumers and firms allocate themselves will depend on the beliefs and the beliefs of beliefs and so on. To resolve this we assume the market fails to launch when prices are non-equal and one platform does not set both the low prices and both \( p_1^A, p_1^B > \delta v \). The additional requirement, \( p_1^A, p_1^B > \delta v \), will becomes clear in the analysis below; but notice that when consumer prices fall below \( \delta v \) then all consumers will join regardless of what the firms do. So this requirement is important.

As we will see, in many cases the homogeneous platforms will set their prices equal to each other for both the consumer and firm prices so that \( p_1^A = p_1^B = p_1 \) and \( p_2^A = p_2^B = p_2 \). Before investigating how, and for which parameter sets this occurs, we first investigate how firms and consumers allocate on the platforms when they face symmetric prices. Given the timing of the game and the lack of coordination between and within both sides of the platform, if a consumer or firm decides to single-home on a platform then it must be indifferent between which platform it chooses. We assume it then joins platform A or B with equal probability; hence, it must be true that in expectation \( n_1^A = n_1^B \) and \( n_2^A = n_2^B \). In the following subsections we investigate how firms and consumers allocate by either not participating, single-homing, or multi-homing on the platforms.
5.1.1 Allocation of Firms with Symmetric Pricing

Firms who join the platform can be separated into two disjoint groups: those who single-home and those who multi-home:

Case 1: The \( \vartheta \) such that \( \pi^A_2(\vartheta) > \pi^B_2(\vartheta) \) and \( \pi^A_2(\vartheta) > 0 \) are the multi-homing firms.

Case 2: The \( \vartheta \) such that \( \pi^B_2(\vartheta) > \pi^B_2(\vartheta) \) and \( \pi^B_2(\vartheta) > 0 \) are the single-homing firms.

Case 1 implies that \( 0 < \alpha_2(n_1^B - n_1^m) - c\vartheta - p_2 \) and \( 0 < \alpha_2(n_1^A + n_1^B - n_1^m) - 2c\vartheta - 2p_2 \). However, for all \( \vartheta \geq 0 \) and since in expectation \( n_1^A = n_1^B \), we have \( \alpha_2(n_1^B - n_1^m) - c\vartheta - p_2 \leq \alpha_2(n_1^A + n_1^B - n_1^m) - 2c\vartheta - 2p_2 \), i.e. the first equation implies the second. Thus the firms who multi-home are the \( \vartheta \) such that \( \vartheta \geq 0 \) and \( \vartheta \leq \frac{\alpha_2(n_1^B - n_1^m) - p_2}{c} \). Note, if the firm price, \( p_2 \), is large enough, then this set is empty and no firms multi-home. Alternatively, if \( n_1^m \), the number of consumers who multi-home, is large enough, then the set is again empty and no firm will multi-home. Thus for multi-homing firms to exist we must have firm price and the number of multi-homing consumers to be sufficiently low. Assuming this is true, i.e. the inequality \( 0 < \alpha_2(n_1^B - n_1^m) - c\vartheta - p_2 \) holds, then the set of firms who multi-home are:

\[
\vartheta \in \left[ 0, \frac{\alpha_2(n_1^B - n_1^m) - p_2}{c} \right) = \left[ 0, n_2^m \right).
\] (70)

Case 2 implies that \( 0 > \alpha_2(n_1^B - n_1^m) - c\vartheta - p_2 \) and \( 0 < \alpha_2 n_1^A - c\vartheta - p_2 \). For this set to be nonempty we need \( \alpha_2 n_1^A - c\vartheta - p_2 > \alpha_2(n_1^B - n_1^m) - c\vartheta - p_2 \) which is in fact true since \( n_1^m \geq 0 \). Thus, so long as prices are sufficiently low for the market to exist, the set of firms who single-home is always non-empty when the competing platforms set prices equal. By solving for \( \vartheta \) we can determine the total number of firms that single-home on platforms \( A \) or \( B \):

\[
\vartheta \in \left[ \frac{\alpha_2(n_1^B - n_1^m) - p_2}{c}, \frac{\alpha_2 n_1^A - p_2}{c} \right) = \left[ n_2^m, N_2 \right).
\] (71)

Assuming prices are such that both sets are nonempty there are two key points worth making here. First, we can determine the number of firms who will join each platform in expectation: \( E(n_1^A) = E(n_1^B) = n_2^m + (1/2)(N_2 - n_2^m) = (1/2)(N_2 + n_2^m) \) where \( (1/2)(N_2 - n_2^m) \) is the expected number of single-homing firms to join each platform. Second, we see that the firms that choose to multi-home instead of single-home are the firms with the sufficiently low synchronization costs. For firms with higher synchronization costs it becomes to costly to join more than one platform. Thus for given prices a firm is more likely to multi-home if it faces a lower synchronization cost to join a platform.

5.1.2 Allocation of Consumers with Symmetric Pricing

In a similar fashion we can separate the consumers into those who multi-home and those who single-home.

Case 1: The \( \tau \) such that \( u^{AB}_2(\tau) > u^A_2(\tau) \) and \( u^{AB}_2(\tau) > 0 \) are the multi-homing consumers.
Case 2: The \( \tau \) such that \( u_2^A(\tau) > u_2^{AB}(\tau) \) and \( u_2^A(\tau) > 0 \) are the single-homing consumers.

Case 1 implies that \( 0 < \delta v + \alpha_1(\tau)(n_2^B - n_2^m) - p_1 \) and \( 0 < v + \alpha_1(\tau)n_2^A - p_1 \). However, for all \( \tau \geq 0 \) and since in expectation \( n_2^A = n_2^B \) we have \( \delta v + \alpha_1(\tau)(n_2^B - n_2^m) - p_1 \leq v + \alpha_1(\tau)n_2^A - p_1 \). This means that the first equation implies the second. Thus the consumers who multi-home are those with \( \tau \geq 0 \) and \( 0 < \delta v + \alpha_1(\tau)(n_2^B - n_2^m) - p_1 \). Note, if the consumer price, \( p_1 \), is large enough, then this set is empty and no consumer will multi-home. If \( n_2^m \), the number of consumers who multi-home, is large enough then the set may again be empty and no consumer will multi-home. For consumers, unlike firms, there always exists an additional membership benefit, \( \delta v \), to join the additional platform. Thus if \( \delta v \geq p_1 \), then all consumers will multi-home regardless of the firm’s decisions. Assuming \( 0 < \delta v + \alpha_1(\tau)(n_2^B - n_2^m) - p_1 \) holds, then the set of consumers who multi-home is given by:

\[
\tau \in [0, \alpha_1^{-1}\left(\frac{p_1 - \delta v}{n_2^B - n_2^m}\right)] = [0, n_1^m].
\]

(72)

Case 2 implies that \( 0 > \delta v + \alpha_1(\tau)(n_2^B - n_2^m) - p_1 \) and \( 0 < v + \alpha_1(\tau)n_2^A - p_1 \). For both of these to hold we need \( v + \alpha_1(\tau)n_2^A - p_1 \geq \delta v + \alpha_1(\tau)(n_2^B - n_2^m) - p_1 \) to hold for all \( \tau \), which it does. Since \( \alpha_1(\cdot) \) is decreasing in \( \tau \), this set of single-homing consumers are the consumers with larger \( \tau \) relative to those \( \tau \) who multi-home. Furthermore, the above inequalities imply that if \( p_1 > \delta v \) then for all \( \delta > 1 \) the set of single-homing consumers is nonempty since \( v + \alpha_1(\tau)n_2^A > \delta v + \alpha_1(\tau)(n_2^B - n_2^m) \). It is reasonable to assume that \( p_1 > \delta v \) will hold in most practical cases. For most products the membership benefit will depreciate almost to zero when a consumer multi-homes. In the video game case, the membership benefit would be the ability to play music and watch movies from the video game console. Once I have one console I can do this but adding an additional console is of almost no added benefit. This would imply a \( \delta \) close to zero. Thus for \( p_1 > \delta v \), the set of single-homing consumers is given by:

\[
\tau \in [\alpha_1^{-1}\left(\frac{p_1 - \delta v}{n_2^B - n_2^m}\right), \alpha_1^{-1}\left(\frac{p_1 - v}{n_2^A}\right)] = [n_1^m, N_1].
\]

(73)

So consumers allocate in a fashion that is similar to the firms. Assuming the sets of multi-homing and single-homing consumers are nonempty, the consumers who multi-home are those with small \( \tau \), i.e. those with the higher marginal utility for firms. This makes sense. In the case for video games, this implies that consumers who purchase multiple gaming consoles are those with the highest marginal utility for the number of games available on these consoles. The consumers who single-home are those who have lower marginal utilities for the number of firms available and those with the lowest marginal utilities do not enter the market. Furthermore, we see that in expectation we have \( E(n_1^A) = E(n_1^B) = n_1^m + (1/2)(N_1 - n_1^m) = (1/2)(N_1 + n_1^m) \) where \( (1/2)(N_1 - n_1^m) \) is the expected number of single-homing consumers to join each platform. Figure 4, displays how both consumers and firms allocate by their respective types.
Figure 4: Allocation by Individual Consumer or Firm Type

Investigating the allocation decisions of the consumers and firms provides solid insight into two-sided markets where both sides can either single-home or multi-home if it is optimal to do so. In reality this is how most two-sided markets work and to the best of our knowledge this is the first paper to investigate this setting. Given the consumer and firm allocation rules for equal prices across platforms, we can determine the Nash Equilibrium that occur in this static model.

5.1.3 Nash Equilibria in the Static Game

In determining the Nash Equilibria for the static game, the main parameter inequality that affects both how consumers and firms allocate and how platforms choose to price is \( f_1 \geq \delta v \). The argument for \( f_1 \geq \delta v \) is that for many products the membership benefit will depreciate almost to zero when a consumer multi-homes. This would imply the marginal cost for an additional consumer on the platform is greater than the additional membership benefit from joining an additional homogeneous platform, \( f_1 \geq \delta v \). In the video game case, the membership benefit would be the ability to play music and watch movies from the video game console. Once I have one console I can do this, but adding an additional console is of almost no added benefit since I can already watch movies and play music on my first console. This would imply a \( \delta \) close to zero and the additional benefit would not overcome the cost of producing the additional console.

Thus we first investigate the case when \( f_1 \geq \delta v \). The alternative case will be investigated once this case is complete. In this case where \( f_1 \geq \delta v \), competition between the two homogeneous platforms results in the Bertrand Paradox:

**Proposition 4.** If \( f_1 \geq \delta v \) then \( p_1^A = p_1^B = f_1 \) and \( p_2^A = p_2^B = f_2 \) are the unique symmetric prices in equilibrium and \( \Pi^A = \Pi^B = 0 \).

Given these prices, by using the allocation rules above we can determine the Nash Equilibrium of the stage game for the case when \( p_1 = f_1 \geq \delta v \). We first investigate the case when there exists both consumers who single-home and consumers who multi-home.
Theorem 2. When \( \delta v \leq f_1 \) we have unique Nash Equilibrium prices charged by the two platforms, \( p_1^A = p_1^B = f_1 \) and \( p_2^A = p_2^B = f_2 \), and at least one, potentially four, allocations of firms and consumers that can be present in Nash Equilibrium.

1. All consumers single-home and all firms multi-home. This is always a Nash Equilibrium.

2. There exists consumers that multi-home and single-home and firms that multi-home and single-home; here, existence depends on model parameters.

3. All firms single-home and many, potentially all, consumers multi-home. This has strong restrictions when \( p_2 > 0 \) and when \( p_2 = 0 \), it requires \( v = 0 \).

4. All consumers multi-home and all firms single-home. This requires \( v = 0 \).

In case 1, which always exists, is an important equilibria because it very closely resembles the two-sided market for smartphones where almost all consumers single-home, they own one phone, and almost all firms multi-home, apps are available on all types of smartphones. In case 2, allocations resemble current allocations we see in many two-sided markets. For video game platforms, there exists consumers who multi-home and single-home and there exists game designers who multi-home and single-home. In cases 2 and 4, the restrictions must be much stronger for existence to exist and there does not exist known examples of actual two-sided markets where consumers and firms allocate in these ways with all firms single-homing.

We now turn to the case when \( \delta v > f_1 \). In this case a platform can charge a consumer price of \( p_i^1 = \delta v \) and guarantee itself profit since consumers will either single-home on platform \( i \) or if a consumer is already on platform \( j \neq i \) then they will be indifferent to multi-homing. Hence, if \( p_i^1 = \delta v \) then consumers will join platform \( i \) regardless of the firms’ decisions. Thus for this case with \( \delta v > f_1 \), both platforms are guaranteed profits and all consumers \( \tau \in [0, \tilde{N}_1] \) are guaranteed to join at least one platform. Furthermore, in this case we do not have failure to launch issues since both platforms are completely able to establish themselves on the consumer side of the market. This implies we can investigate and determine the Nash Equilibrium allocations for the cases when \( p_i^1 > p_1^1 \) and \( p_2^1 > p_2^1 \). This also leads to potentially uncountably many Nash Equilibrium.

Theorem 3. When \( \delta v > f_1 \), there exists at least one and potentially uncountably many Nash Equilibria.

1. There exists a unique symmetric Nash Equilibria where \( p_1^A = p_1^B = \delta v, p_2^A = p_2^B = f_2 \), all consumers \( \tau \in [0, \tilde{N}_1] \) multi-home, and the firms who join a platform do so by single-homing on platform A or B with equal probability; this Nash Equilibrium always exists.

2. The uncountably many asymmetric Nash Equilibrium that can potentially exist are characterized by \( p_2^1 = f_2, p_2^1(\epsilon) = f_2 - \epsilon, p_1^1 = \delta v, p_1^1 = p_1^1(\epsilon) \geq \delta v, \) and \( p_2^1 = f_2 \).
for $\epsilon \geq 0$ with $p_1^1(\epsilon = 0) = \delta v$. If there exists an $\epsilon > 0$ where certain conditions hold, see proof, then the resulting Nash Equilibrium allocations will have all consumers multi-homing and all firms who decide to participate do so by joining platform $j$.

Furthermore, in all NE platforms receive positive profits: $\Pi^A = \Pi^B = \hat{N}_1(\delta v - f_1) > 0$.

Thus, if a platform has significantly high retained membership benefit for consumers when they multi-homing, then platforms can avoid the Bertrand Paradox on the consumer side of the market and make positive profits when competing with a homogeneous platform. This completes our investigate of the Nash Equilibrium in the static game. The Nash Equilibrium outcomes in this model are realistic and provide us with insight into the decision of an agent to single-home or multi-home. We next investigate the theorems derived here by looking at closed form solutions.

### 5.2 Closed Form Solutions and Welfare for Static Nash Equilibrium

In this subsection we investigate the same closed form solution that we determined above in the case for a monopoly platform. We calculate welfare and compare it with the welfare that occurs with a monopoly platform. Note, there exists two effects on welfare between having a monopoly platform and two competing platforms. Competition will result in lower prices however it also can destroy network surplus or create more synchronization costs from firms who choose to multi-home. Depending on which affect dominates welfare may increase or decrease with increased competition.

As before we assume $\alpha_1(\tau) = a - b\tau$ so that the maximum number of consumers that can join a platform is $\hat{N}_1 = a/b$. Furthermore, we assume $v = f_1 \geq 0$ and $f_2 = 0$. This implies $f_1 = v \geq \delta v$ so Proposition 4 and Theorem 2 hold but Theorem 3 does not. Proposition 4 implies $p_1^A = p_1^B = f_1 = v \geq 0$ and $p_2^A = p_2^B = f_2 = 0$. Given this, we investigate allocation 1 in Theorem 2 where all consumers single-home and all firms multi-home. We know this equilibrium exists and is relevant for real world applications.

Given all consumers single-home and all firms multi-home, $n_1^m = 0$ and $n_2^A = n_2^B = n_2^m$. Furthermore, allocations (70) and (73) imply:

$$n_2^m = (1/c)(\alpha_2 n_1^A - f_2)$$

$$n_1^A = (1/2)[a/b]$$

These two equations and two unknowns determine the equilibrium: $n_1^A = n_1^B = a/2b$, $n_1^m = 0$, so $N_1 = a/b = \hat{N}_1$, $n_2^A = n_2^B = n_2^m = \alpha_2 a/2bc$, $p_1 = f_1 = v$, and $p_2 = f_2 = 0$. Notice, all consumers join a platform i.e. we have full participation. Given this equilibrium we calculate welfare. The superscript 1 represents allocation 1 in Theorem 2.

$$\Pi^1 = 0, \quad (74)$$
Recall from the monopoly model at the end of section 3.2 we showed how the corner solution occurs where the monopolist captures all consumers when $\alpha_2 \geq a$, otherwise the interior solution occurs. Notice with here how consumer surplus is the same as in the monopoly model when a corner solution occurs. This is since consumer prices are the same in both cases and all consumers are single-homing in this case with two platforms. Note, the difference in surplus between this case and the monopoly corner solution case will be the same for $v > f_1$ since this additional surplus will either be captured by the monopolist without any distortion or by the consumers in this competitive equilibrium. Firm surplus is greater here than in the monopoly corner solution.

The most striking result is that welfare with perfect competition is strictly less than welfare with a monopoly platform who implements the corner solution and captures all consumers. If we compare welfare between the case here and the case when the monopolists has an interior solution, $\alpha_2 < a$, we see that $W^1 < W^i$ occurs if and only if $0 < a_2^3 - a_2^2 a + a_2^2$. This inequality never holds for $\alpha_2, a \geq 0$. Thus we have the following theorem.

**Theorem 4.** For $v = f_1 \geq 0$, $f_2 = 0$, $\alpha_1(\tau) = a - b \tau$. When $\alpha_2 \geq a$ the monopoly corner solution that occurs has higher welfare than the perfectly competitive equilibrium between two homogeneous platforms when all consumers single-home and all firms multi-home. When $\alpha_2 < a$ the monopoly interior solution that occurs has lower welfare than the perfectly competitive equilibrium between two homogeneous platforms when all consumers single-home and all firms multi-home.

This means that when the two-sided market is driven by the firms, $\alpha_2 \geq a$, then the monopoly will capture all consumers and this will generate more welfare than the case where we have two homogeneous platforms who compete perfectly competitively with all consumers single-homing and all firms multi-homing. When the monopoly platform is already capturing all consumers than in an sense he is already pricing low enough for competition to not be enough to increase surplus. Otherwise, the tradeoff for lower prices is sufficient to increase welfare.

There are three more important points to be made here. First, we are assuming that there does not exist decreasing marginal returns for the other side of the platform. In reality consumers probably have a decreasing marginal return for the number of apps available on their smartphone. Overall this would decrease surplus for both levels of competition but it would hurt the monopolist more since now its prices would change and destroy more surplus. Second, we assumed $v = f_1 \geq 0$. For $v > f_1$ similar arguments follow and welfare results
will closely resemble what we have here. However, for \( v < f_1 \) it will be less likely for the
corner solution to occur since in that case it will be costly for the monopoly platform to
capture all consumers. This will make competition welfare improving in more cases. Finally,
we assumed homogeneous platforms. If platforms are quite different we can imagine more
cases with high price competition but strong network benefits. This would make competition
welfare enhancing.

Even with these pitfalls, this result is quite interesting and one can imagine a case where
a monopoly is welfare improving. For example, when the first iPhone came out it was
essentially a monopolist in terms of pure smartphones. At this time, if a homogeneous
competitor was also present, one can imagine welfare suffering from destroying and splitting
network surpluses between these two platforms even if we assume no coordination issues.

Allocation 2 is hard, depends on 6 parameters, add to appendix maybe. Since \( p_2 = 0 \)
allocations 3 and 4 in Theorem 2 only occur when \( v = 0 \), not relevant since then both prices
must be zero and MC of consumers must be zero, so no real examples.

6 Conclusion

This paper makes four contributions to the existing platform and two-sided market litera-
ture. First, it establishes a base model for two-sided markets, with consumers, firms, and
a monopoly platform, that is intuitive and general while allowing for multiple extensions
and well behaved closed form solutions. Second, it develops one-sided analysis, from the
consumer’s side, that characterizes consumer demand for the platform and connects the
two-sided market framework to the traditional markets seen in the industrial organization
literature. Third, it introduces a static competition game between two homogeneous plat-
forms where both sides of the market, consumers and firms, may either single-home or multi-
home. Lastly, in determining the Nash Equilibrium for the static competition game we now
have a model that allows us to naturally compare welfare across to levels of competition; a
monopoly platform and two competing homogeneous platforms.

Moving forward, the framework of this model allows for more extensions to be developed.
For future research, this model will be extended to a stage game with entry. The results of
the stage game will depend on the amount of differentiation between platforms. For example,
given a situation where an incumbent platform has established a presence in the market, a
homogeneous potential entrant will have no way of gaining momentum to enter the market.
This could lead to many interesting results about entry and exit in two-sided markets which
will be left for future research.
7 Appendix

7.1 Proofs

Proof of Theorem 1

Proof. First we show consumer inverse demand is hill shaped then we show the existence of a stable equilibrium. With \( v, f_1, f_2 > 0 \), (Without loss of generality we assume \( a = 3\alpha_2 \)), the two main equations to characterize consumer inverse demand are now:

\[
p_1 = D(n_1|n_2, p_2^*) = v + (a - b n_1) n_2 = v + (3\alpha_2 - b n_1) n_2
\]

\[
n_2(n_1) = \frac{1}{2c}(n_1)[a - b n_1 + \alpha_2] - \frac{f_2}{2c} = \frac{1}{2c}(n_1)[4\alpha_2 - b n_1] - \frac{f_2}{2c}
\]

By substitution and simplifying we have:

\[
p_1 = v - \frac{3\alpha_2 f_2}{2c} + \frac{1}{2c}(n_1)[b^2 n_1^2 - 7\alpha_2 b n_1 + (12\alpha_2^2 + b f_2)]
\]

So the added membership benefit, \( v \geq 0 \), only shifts the inverse demand curve up or down. It does not affect the hill shaped curvature. This is not true for \( f_2 \) which does affect the curvature. To ensure the inverse demand is hill shaped for \( n_1 \in [0, \hat{N}_1] \) we will take the derivative, \( \frac{dp_1}{dn_1} \), and evaluate it at \( n_1 = \hat{N}_1 = \frac{3\alpha_2}{b} \). If the sign is negative at this point then we know inverse demand is hill shaped. First, we take the derivative:

\[
\frac{dp_1}{dn_1} = \frac{1}{2c}[3b^2 n_1^2 - 14\alpha_2 b n_1 + (12\alpha_2^2 + b f_2)]
\]

Evaluating this at \( n_1 = \hat{N}_1 = \frac{3\alpha_2}{b} \) we see that \( \frac{dp_1}{dn_1} \bigg|_{n_1 = \frac{3\alpha_2}{b}} < 0 \) if and only if \( f_2 < \frac{3\alpha_2^2}{b} = \alpha_2 \hat{N}_1 \).

This means we have hill shaped inverse demand so long as the marginal cost for an additional firm is less than the maximum amount of surplus the consumers can generate for firms, \( \alpha_2 \hat{N}_1 \). However, when this inequality fails the market collapse since the marginal cost is greater than the maximum amount of possible surplus that can be generated. Thus, when marginal cost is low enough for the market to exist, consumers will have hill shaped inverse demand for the platform.

To show the existence of a stable equilibrium we need to investigate the relationship of two equations (24) and (25):

\[
n_1 = \frac{1}{2b}[4\alpha_2] + \frac{v - f_1}{2b n_2},
\]

\[
n_2 = \frac{1}{2c}(n_1)[4\alpha_2 - b n_1] - \frac{f_2}{2c}.
\]

We graph these equations below. Notice that an intersection always exists such that \( n_1, n_2 > 0 \) for ”reasonable” parameter values. If this intersection is not a stable equilibrium
then it cannot be an equilibrium since the platform would just increase price to increase profits. Thus there must always exist a stable equilibrium.

**Figure: Proof of Theorem 1**

Proof of Proposition 1

**Proof.** From (59) we have $n_2 = (1/2c)[\alpha_1(n_1)n_1+\alpha_2n_1-f_2]$ which implies $\frac{\partial n_2}{\partial n_1} = (1/2c)[\alpha'_1(n_1)n_1 + \alpha_1(n_1) + \alpha_2]$. Note this derivative is ambiguous. Using equation (58) we can solve for $\beta$. Thus

$$\beta = (-1/2n_1)[v + \alpha_1(n_1)n_2 - f_1 + \alpha'_1(n_1)n_1n_2 + \alpha_2n_2].$$

(78)

Given $\beta$ as a function of $n_1$ and $n_2$ we can determine $\frac{\partial \beta}{\partial n_1}$ using $\frac{\partial n_2}{\partial n_1}$ and this will have the same sign as $\frac{\partial n_1}{\partial \beta}$. After some algebra we have:

$$\frac{\partial \beta}{\partial n_1} = -\frac{2n_1\{2\alpha'_1(n_1)n_2 + \alpha''_1(n_1)n_1n_2 + (1/2c)[\alpha'_1(n_1)n_1 + \alpha_1(n_1) + \alpha_2]^2\} + 4n_1\beta}{4n_1^2}.$$

Thus $\frac{\partial n_1}{\partial \beta} > 0$ if and only if $-2n_1\{2\alpha'_1(n_1)n_2 + \alpha''_1(n_1)n_1n_2 + (1/2c)[\alpha'_1(n_1)n_1 + \alpha_1(n_1) + \alpha_2]^2\} + 4n_1\beta > 0$. This is in fact just the second-order condition (61). Thus $\frac{\partial n_1}{\partial \beta} > 0$. □

Proof of Proposition 2

**Proof.** If the platform holds prices fixed with an increase in $\beta$ then $n_1$ and $n_2$ will increase. This will increase profits, hence optimizing prices must increase profits. □
Proof of Proposition 3

Proof. Using the equation for \(n_2\) in the proof of Proposition 1 we see that \(\frac{\partial n_2}{\partial \beta} = \left(\frac{\partial n_2}{\partial \alpha_1}\right)\left(\frac{\partial n_1}{\partial \beta}\right)\) and we know from above that \(\frac{\partial n_2}{\partial \alpha_1}\) is ambiguous. Furthermore, it is ambiguous in a way that provides little meaning; \(\frac{\partial n_2}{\partial \alpha_1} > 0\) if \(\alpha_1(n_1) + \alpha_2 > -\alpha'_1(n_1)n_1 > 0\).

From (63), we have: \(\frac{\partial n_2}{\partial \alpha_1} = -\alpha_2 \frac{\partial n_2}{\partial \beta} - n_1 - \beta \frac{\partial n_2}{\partial \beta} - \alpha'_1(n_1)\frac{\partial n_2}{\partial \beta}n_1n_2 - \alpha'_1(n_1)\frac{\partial n_2}{\partial \beta} = \alpha_1(n_1)n_1n_2 - \alpha'_1(n_1)n_1\frac{\partial n_2}{\partial \beta}\). This depends on the value of certain parameters and on other ambiguous derivatives hence it is ambiguous. In numerical examples the sign varied over the ranges of \(\beta\).

From (64), we have: \(\frac{\partial p_1}{\partial \beta} = -\alpha'_1(n_1)\frac{\partial n_2}{\partial \beta}n_1 - \alpha_1(n_1)\frac{\partial n_2}{\partial \beta} + c\frac{\partial n_2}{\partial \beta} > 0\) we require \(\alpha_2 > -\alpha'_1(n_1) - \alpha_1(n_1) > 0\) which is a condition similar to that for \(\frac{\partial n_2}{\partial \alpha_1} > 0\) but this condition is a bit stronger. If the condition does not hold then \(\frac{\partial p_1}{\partial \beta}\) is ambiguous.

The sign of \(\frac{\partial CS}{\partial \beta}, \frac{\partial FS}{\partial \beta},\) and \(\frac{\partial W}{\partial \beta}\) depend on the derivatives above which are ambiguous. Hence these derivatives are also ambiguous. \(\square\)

Proof of Proposition 4

Proof. Clearly, symmetric prices that are less than their respective marginal costs is worse and both platforms would deviate. If the symmetric firm price is greater than \(f_2\) then both platforms have the incentive to undercut the price by \(\epsilon\) and capturing all firms and thus all consumers. Thus we must have \(p_2 = f_2\). If \(p_1 > f_1\) then both platforms have an incentive to undercut the price by \(\epsilon\) and capture the whole market. This undercutting only captures all consumers when \(p_1 \geq \delta v\). Otherwise all consumers will multi-home. Thus \(p_1 = f_1\). This implies \(\Pi^i = n^i_1(p_1 - f_1) + n^i_2(p_2 - f_2) = n^i_1(0) + n^i_2(0) = 0\) for \(i = A, B\). \(\square\)

Proof of Theorem 2

Proof. Here we show the equilibrium allocations for general prices \(p_1\) and \(p_2\). Equilibrium prices are then given by Proposition 4, this completes the Nash Equilibriums.

Allocation 1:

For all firms to multi-home, notice that allocation (71) implies all firms multi-home only when all consumers single-home. Furthermore, since \(n^m_2 = n^A_1 = n^B_1\), allocation (72) implies no consumer multi-homes. Hence, the allocation where all firms multi-home and all consumers single-home is a Nash Equilibrium when prices are equal and \(p_1 \geq \delta v\). This is an important equilibria because it very closely resembles the two-sided market for smartphones where almost all consumers single-home, they own one phone, and almost all firms multi-home, apps are available on all types of smartphones.

Allocation 2:

Since \(p_1 \geq \delta v\), allocation (72) implies the set of multi-homing consumers is non-empty only when the number of multi-homing firms is not to large. In what proceeds we determine a Nash Equilibria that occurs under certain parameter restrictions. Let \(x \in [0, 1]\) be the percent of consumers who multi-home so that in expectation \(n^m_i = xn^A_i = xn^B_i\). Given
this, using allocation (70) we can find the cutoff \( x = x^m \) such that above \( x^m \) no firm will multi-home. Allocation (70) implies \( 0 = \alpha_2(1-x^m)n_1^B - p_2 \). Thus,

\[
x^m = 1 - \frac{p_2}{\alpha_2 n_1^B}.
\] (79)

And for all \( x > x^m \) no firm multi-homes. Note, we are assuming \( p_2 < \alpha_2 n_1^B \) since otherwise the market collapses, hence \( x^m \in (0, 1) \).

Now if we let \( 0 < x < x^m \) then some firms will single-home and some firms will multi-home. Allocation (70) implies \( n_2^m = \frac{\alpha_2(1-x)}{\alpha_2(2-x)}n_1^B - p_2 \) and allocation (71) implies \( n_2^m = (1/2)(N_2 + n_2^m) = (1/2c)[\alpha_2(2-x)n_1^A - 2p_2] \). Similarly, allocation (72) defines the number of multi-homing consumers: \( \delta v = \alpha_1(n_1^m)(n_2^m - n_2^m) \); using this equation and the equations for \( n_2^m, n_2^B \), and \( n_1^m = xn_1^A = xn_1^B \) we can characterize \( x \) by:

\[
0 = \delta v + \alpha_1(n_1^m)(n_2^m - n_2^m) = \delta v + \alpha_1(xn_1^A)(1/2c)[\alpha_2 \cdot xn_1^A],
\] (80)

Furthermore, allocation (73) defines \( n_1^A \), the number of consumers on platform \( A \): \( 0 = v + \alpha_1(N_1)n_2^A - p_1 \). Thus we have:

\[
0 = v + \alpha_1(N_1)n_2^A - p_1 = v + \alpha_1(n_1^A)(1/2c)(\alpha_2 \cdot (2-x)n_1^A - 2p_2) - p_1.
\] (81)

Thus, we have two equations (80) and (81) and two unknowns, \( x \) and \( n_1^A \). If the resulting value of \( x \) is such that \( x \in (0, x^m) \) then we have a Nash Equilibrium for the whole static game where there is a complete mix of single-homing and multi-homing consumers and firms.

This outcome also resembles many two-sided markets. Video game platforms fit this equilibrium where there exist consumers who multi-home and single-home and there exists game designers who multi-home and single-home. Note this equilibrium may not exist for sets of parameters. If the \( x \) characterized in equations (80) and (81) is not in \( [0, x^m) \) then this equilibrium does not exist. There are four other allocation cases we need to check as Nash Equilibrium. These cases are: all firms multi-home, all consumers single-home, all firms single-home, and all consumers multi-home; in case 1 we showed the first two occur simultaneously.

**Allocation 3:**

For all firms to single-home, allocation (71) implies all firms single-home only if the number of multi-homing consumers is large relative to the number of single-homing consumers, \( n_1^B \leq n_1^m + p_2/\alpha_2 \). If \( p_2 = 0 \), then this holds only when all consumers multi-home. By allocation (73), this will only be an equilibrium when \( v = 0 \). If \( p_2 > 0 \), then allocation (73) implies there exists an equilibrium where all firms single-home and a large portion of consumers multi-home given prices such that \( \alpha_1^{-1}(\frac{m-v}{n_2^m}) - \alpha_1^{-1}(\frac{m-\delta v}{n_2^m}) \leq \frac{2p_2}{\alpha_2} \). This potential equilibrium requires relatively restrictive assumption. In reality we do not see equilibrium of this form where all firms single-home and many or all consumers multi-home.
Proof. We first prove the case for $\epsilon = 0$. I must show there does not exist a non-epsilon deviation that makes the platform worse off and the allocation stated above is indeed Nash. Given prices are $p_1^A = p_1^B = \delta v$ and $p_2^A = p_2^B = f_2$, consumers are indifferent between single-homing on either platform and multi-homing so we assume they multi-home. If $p_1$ lowered by $\epsilon$ then this would occur. Thus every consumer available in the market, $\hat{N}_1$, joins both platforms $A$ and $B$. Given all consumers multi-home and firms face equal prices between platforms $A$ and $B$, for all firms $\vartheta$ we have $\pi_2^A(\vartheta) = \pi_2^B(\vartheta) > \pi_2^{AB}(\vartheta)$. Thus the firms who join a platform will single-home to platform $A$ or $B$ with equal probability and the number of single-homing firms who join a platform is given by $\pi_2^A(\vartheta) = \pi_2^B(\vartheta) = N_2 = (1/c)[\alpha_2\hat{N}_1 - f_2]$. To see why the prices are NE we investigate deviations. If a platform were to increase its consumer price from $\delta v$, then it will lose consumers and thus lose all firms and thus lose all consumers driving profits to zero. If a platform lowers consumer price below $\delta v$, then the allocation is unchanged and they receive less payment; thus lowering profits. Thus $p_1^A = p_1^B = \delta v$ is a NE set of consumer prices. Given this, platforms compete for firms a la Bertrand and we have $p_2^A = p_2^B = f_2$. The resulting profits are $\Pi = n_1^A(p_1^A - f_1) + n_2^A(p_2^A - f_2) = \hat{N}_1(\delta v - f_1) > 0$. This completes the proof for $\epsilon = 0$.

Without loss of generality we assume platform $A$ plays the strategy $p_1^A = \delta v$ and $p_2^A = f_2$ and platform $B$ plays the $\epsilon$-strategy. For $\epsilon > 0$, if there exists an epsilon, there could be uncountably many equilibrium such that the following conditions hold:

1. The number of consumers on each platform are equal so that platform $B$ will have the opportunity to capture the firms. All consumers will join platform $A$ no matter what. So platform $B$ must set price so that all consumers, $\tau \in [0, \hat{N}_1]$, multi-home: $u_1^{AB}(\hat{N}_1) = (1 + \delta)v + \alpha_1(\hat{n}_1) \cdot n_2^B(\epsilon) - p_1^A - p_2^B \geq u_1^A(\hat{n}_1)|_{(n_2^B(\epsilon) > 0, n_2^A = 0)} = (1 - \delta)v$. This equation states that for all consumers it is optimal to multi-home instead of single-home on $A$ given platform $B$ captures $n_2^B(\epsilon)$ firms and platform $A$ does not capture any firms. The coordination argument here is that all firms know all consumers will multi-home so now they will single-home to the platform with the lowest price which is platform $B$. This implies $p_2^B$ must satisfy $\delta v < p_1^B \leq \delta v + \alpha_1(\hat{n}_1)n_2^B(\epsilon)$.
2. Profits between the two platforms are equal so that neither has an incentive to deviate, 
\[ \Pi^B = \hat{N}_1(p_1^B - f_1) + n_2^B(\epsilon)(-\epsilon) = \Pi^A = \hat{N}_1(\delta v - f_1). \] The flexibility of \( p_2^B \) makes this a reasonable possibility for many small \( \epsilon > 0 \).

Given these conditions hold, platform \( A \) has no incentive to deviate. Thus we have Nash Equilibria with different consumer prices and different firm prices where all consumers multi-home and firms single-home to the platform that charges them a lower price. This completes the proof.

Proof of Theorem 4

Proof. This follows directly from comparing equation (77) with equations (41) and (33) and from the last paragraph in section 3.2 where we showed the monopoly corner occurs when \( \alpha_2 \geq a \) and the interior solution occurs when \( \alpha_2 < a \).

References


