The Comparative Statics of Sorting*

Axel Anderson†
Georgetown

Lones Smith‡
Wisconsin

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Abstract

We create a general and tractable theory of increasing sorting in pairwise matching models with transferable utility — one that subsumes Becker (1973) as the extreme cases with most and least sorting.

We first prove that the positive quadrant dependence order smartly measures increasing sorting, e.g. it rises in the correlation of matched partners.

Our theory centers on synergy — the cross partial difference or derivative of match production. Notably, if synergy everywhere increases, sorting need not rise, but cannot fall. To rescue increasing sorting, we posit often met cross-sectional restrictions on match synergy. Our theory illuminates top economics sorting papers, affording quick proofs of their results and new insights.

Synergy reflects basic economic forces, such as diminishing returns, moral hazard, insurance, and learning dynamics.

Our proof develops and exploits new monotone comparative statics methods. The main proof proceeds by induction with finitely many types, and secures the continuum type results by taking limits.

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†Email: aza@georgetown.edu; web page: http://faculty.georgetown.edu/aza/

‡Email: lones.smith@wisc.edu; web page: www.lonessmith.com
1 Introduction

Assortative matching is the allocational theme in the vast literature on decentralized matching. This finding has seen application in marriage, employment, partnerships, optimal assignment, and pairwise trade. Becker (1973) showed that it emerges when individual types are complementary. The power of this conclusion is also its weakness — higher “men” match with higher “women,” without exception. Since it is an ideal, how should we understand deviations from assortative matching? For instance, Shimer and Smith (2000) asked if these can be seen as evidence of search frictions. They found that sorting only holds under even stronger complementarity assumptions. Also, their matching set must be centered about Becker’s frictionless sorting partner.

While search and information frictions impact who matches with whom, surely match productivity is the main driving force. And currently, our only tractable and general matching theory is Becker’s that is so restrictive it allows but two conclusions: positive and negative assortative matching. His driving premises are also very restrictive: a globally positive or globally negative cross partial difference or derivative. Chade, Eeckhout, and Smith (2017) explore many natural and some well-cited economic matching settings where both assumptions fail. The lack of a predictive general theory of who matches with whom has greatly limited the analytic reach of the matching literature in economics. Barring a few closed form solvable cases, it has focused excessive attention on the extreme cases of perfect positive or negative sorting.

This paper fills this void: We develop a tractable general theory of sorting changes in the frictionless pairwise matching model with either finitely many or a continuum of types and transferable utility (TU). By using old and new methods for monotone comparative statics, we succeed without ever solving the planner’s problem, offering predictions for the matching papers that have influenced economics since Becker (1973). We hope to increase the reach of matching models to capture more economic settings.

We first search for a partial order that best corresponds to an economic meaning of “more assortative”. We prove that the positive quadrant dependence (PQD) does the trick. This stochastic order ranks matching measures by the mass in the southwest quadrant. Increases in the PQD order reduce the average distance between matched types, increase the correlation of matched types, and raise the regression coefficients of women on their partners’ types rise in PQD (Lemma 1). This ensures us that our sorting comparative statics conclusions are of direct empirical relevance in economics.

To illustrate the PQD order, consider the six possible complete matchings among three men and three women (Figure 1). Each man matches with a weakly closer partner in PAM than in NAM1 or NAM3, in turn each closer than in PAM2 or PAM4, and finally than in NAM. Meanwhile, the matchings NAM1 and NAM3, as well as PAM2
Figure 1: Pure Matchings with 3 Types. The possibilities are: negative and positive assortative matching (NAM and PAM), negative sorting in quadrants 1 and 3 (NAM1 and NAM3), and positive sorting in quadrants 2 and 4 (PAM2 and PAM4).

and PAM4, are incomparable. We have thus a partial order:

\[ \text{PAM} \succ_{PQD} \text{[NAM1, NAM3]} \succ_{PQD} \text{[PAM2, PAM4]} \succ_{PQD} \text{NAM} \]  

Next, our assumptions about production functions are a local version of Becker’s complementarity. Synergy is the cross partial difference of production with finitely many types, and with continuous types, the cross partial derivative. Becker (1973) finds that globally positive synergy induces positive sorting, and globally negative synergy induces negative sorting. To account for intermediate cases, where synergy sometimes changes sign, we uncover a new formula for total match output \( [4] \), rewriting it as a weighted average of match synergy. Our formula yields Becker’s Result at once by corollary, and shows how production only impacts sorting via synergy.

Becker’s theory implies that globally positive synergy gives us assortative matching. So then is sorting greater with more synergistic production? We show by a simple three-type example that this intuitive conjecture fails (Figure 3) — the optimal matching oscillates between NAM1 and NAM3 as synergy rises (Figure 1). The reason is that standard monotone comparative statics theory does not apply because the choice set — namely, all matching measures — is a partially ordered set in the PQD order, but not a lattice. This highlights the difficulty of extending Becker (1973). Our first major result, Proposition \( \text{I} \), says that sorting cannot fall if synergy everywhere increases.

While greater synergy need not yield more sorting, it does if production obeys an extra cross-sectional assumption. Corollary \( \text{I} \) is the simplest such result: If the production function changes, thereby globally increasing synergy, and if synergy in each case is monotone in match partner types, then sorting increases. But these assumptions are stronger than we need, and not met in many important matching models. So instead let us assume that total synergy changes its sign only from negative to positive on all unions of rectangular partner sets. With this premise, Proposition \( \text{2} \) (and Proposition \( \text{3} \)) show that sorting increases provided the sign of total synergy on any rectangle (or line segment) of types changes sign just once as it shifts north and/or east.

Next, in pursuit of a purely local theory amenable to the popular continuum type matching models, we then relax the monotonicity assumptions of Corollary \( \text{I} \) to single-
crossing assumptions. We show in Figure 5 that sorting can fall after a single-crossing rise. We then rescue increasing sorting by adding in a third assumption. We assume that the production function obeys a proportional upcrossing inequality that ensures that positive synergy increases proportionately more than absolute negative synergy. Proposition 4 concludes that sorting increases in this case. Finally, we show that monotone functions are proportional upcrossing, and thus Corollary 1 follows.

Our last result, Corollary 3 repurposes our increasing sorting theory for a new goal: type distribution shifts. We argue that upward shifts in type distributions emulate the effect on synergy of changing production functions, and thus increase sorting.

Our theory greatly expands the predictive reach of matching theory. For instance, with 100 men and 100 women, Becker (1973) makes predictions for just two possible synergy sign combinations. Our cross-sectional single crossing synergy encompasses a total of $2 \cdot 99^2$ sign combinations — and ones that specifically arise in applications.

Economic Applications of Our Theory. Becker’s work sparked a truly vast literature on the transferable utility matching paradigm. For he offered a quick way to check whether matching was perfectly assortative. Since global complementarity is restrictive, it was inevitable that models would arise without this property. We offer comparative statics for these papers, and numerically illustrate the optimal matchings; these plots reflect subtle and surprising global optimality considerations.

Firstly, there may be inherent economic reasons for the marriage model without complements. Kremer and Maskin (1996) proposed a partnership model with defined roles. Match output was thus the maximum of two supermodular functions — one for each role assignment. This inspired paper defines the outer limits of our theory — for we can conclude when sorting nowhere falls in parameters, but not where it increases.

Secondly, informational issues may undermine sorting — either adverse selection or moral hazard. Legros and Newman (2002) showed that supermodular production does not induce supermodular match payoff functions with imperfect credit constraints. Our nowhere decreasing theory subsumes their production function. But we instead focus on Guttman’s (2008) dynamic extension of Ghatak’s (1999) model of group lending with adverse selection — for which our stronger increasing sorting theory applies.

Next, Serfes (2005) investigated a pairwise matching model of principals and agents. He showed that negative sorting — or, more risk averse agents with safer projects — arises with a low disutility of effort, but positive sorting emerges for high disutility of effort. We complete his picture, and find that sorting rises in the disutility of effort.

Thirdly, dynamic models of matching naturally blunt complementarity. For even

1 We prove in a multi-dimensional extension of Karlin and Rubin’s 1956 upcrossing preservation.

2 For our upcrossing assumption, a sign change can occur after any of 99 men and 99 women.
with supermodular static payoffs, [Anderson and Smith (2010)] show that dynamic models with Bayesian updating need not inherit supermodularity. In our subsequent work with evolving human capital (Anderson and Smith, 2012), we show that preservation of supermodularity is highly exceptional. For general transition functions of old types into new types, the dynamic match values are rarely supermodular.

Finally, we address our proof logic. Our nowhere decreasing theory owes to a comparative static theory for non-lattice partially ordered sets that we develop. But our increasing sorting results evade all optimization theory we know of, and so owe to a unique induction argument on the number of types in the Appendix.

Longer proofs and new monotone comparative statics results are in the Appendix.

### 2 Becker’s Marriage Model and Planner’s Result

Assume pairwise matching by individuals either from two groups (men and women, firms and workers, buyers and sellers) or the same set (partnerships). In the general matching model, a unit mass of “women” and “men” have respective types $x, y \in [0, 1]$ with cdfs $G$ and $H$. To capture the literature, we allow for two cases: absolutely continuous type distributions $G$ and $H$, and finitely many types, when $G$ and $H$ are discrete measures with equal weights on female types $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$ and male types $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$. In the finite types case, we relabel women and men as $i, j \in \{1, 2, \ldots, n\}$, respectively.

We assume a $C^2$ production function $\phi > 0$, so that types $x$ and $y$ jointly produce $\phi(x, y)$. In the finite type model, the output for match $(i, j)$ is $f_{ij} = \phi(x_i, y_j) \in \mathbb{R}$. Production is supermodular or submodular (SPM or SBM) for all $x' < x''$ and $y' < y''$ if:

$$\phi(x', y') + \phi(x'', y'') \geq (\leq) \phi(x', y'') + \phi(x'', y') \quad (2)$$

Strict supermodularity (respectively, strict SBM) asserts strict inequality in (2). And production is modular when (2) always holds with equality.

Like Becker’s, our theory does not explore an extensive margin (whether to match). A matching is a bivariate cdf $M \in \mathcal{M}(G, H)$ on $[0, 1]^2$ with marginals $G$ and $H$. In the finite type case, $G$ and $H$ put equal unit weight on $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. A finite matching is a nonnegative matrix $[m_{ij}]$, with cdf $M_{i_0j_0} = \sum_{1 \leq i \leq i_0, 1 \leq j \leq j_0} m_{ij}$, and unit marginals $\sum_i m_{ij_0} = 1 = \sum_j m_{i_0j}$ for all men $i_0$ and women $j_0$. In a pure matching, $[m_{ij}]$ is a matrix of 0’s and 1’s, with everyone matched to a unique partner.

There are two perfect sorting flavors. In positive assortative matching (PAM), any woman type of $x$ at quantile $G(x)$ pairs with a man of type $y$ at the same quantile $H(y)$, and thus the match cdf is $M(x, y) = \min(G(x), H(y))$. In negative assortative matching...
(NAM), complementary quantiles match, and so \( M(x, y) = \max(G(x) + H(y) - 1, 0) \). Matched types are uncorrelated given uniform matching, and so \( M(x, y) = G(x)H(y) \).

The partnership (or unisex) model is a special case where types \( x \) and \( y \) share a common distribution, \( G = H \), the production function \( \phi \) is symmetric (\( \phi(x, y) = \phi(y, x) \)), and so too is the optimal matching distribution \( M(x, y) \equiv M(y, x) \). In this case, PAM simply reduces to the matching \( y = x \), or match with the same type.

A social planner maximizes total match output, namely \( \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(\theta)m_{ij} \) with finite types, or more generally \( \int_{[0,1]^2} \phi(x, y|\theta)M(dx, dy) \), where we index output \( \phi(x, y|\theta) \) by a (often suppressed) state \( \theta \in \Theta \), a partially ordered set. Solving for optimal matchings:

\[
\mathcal{M}^*(\theta) = \arg \max_{M \in \mathcal{M}(G,H)} \int_{[0,1]^2} \phi(x, y|\theta)M(dx, dy) \tag{3}
\]

Gretsky, Ostroy, and Zame (1992) deduce existence of \( \mathcal{M}^* \), and decentralize it as a competitive equilibrium — hence the planner’s problem is important. Maximization (3) has been solved in three general cases: Trivially, every feasible matching is optimal with modular (or additive) production, while Becker solved the PAM and NAM extremes.

**Becker’s Sorting Result.** Given SPM (SBM) production \( \phi \), PAM (NAM) is an optimal matching. Given strict SPM (SBM), these pairings are uniquely optimal.

**Proof:** Assume finitely many types. A maximum of (3) exists. To see uniqueness, assume women \( x' < x'' \) and men \( y' < y'' \) are negatively sorted into matches \( (x', y') \) and \( (x'', y') \). Then output is not maximal, since SPM production (2) implies a higher payoff to the matches \( (x', y') < (x'', y'') \). A general proof of Becker’s Result is in §3.

We do not solve the general social planner’s problem (3) — a hard open question. Rather, we ask how the optimal matching set \( \mathcal{M}^*(\theta) \) changes as the state \( \theta \) increases. This paper derives comparative statics in \( \theta \) when output \( \phi(x, y|\theta) \) is neither SPM or SBM, and thus the optimal matching is neither PAM nor NAM. Throughout, a time series property relates to changes in the state \( \theta \) while a cross-sectional property relates to the behavior of production over the type space. We then apply our finding in several matching models throughout economics, without SPM or SBM output.

Throughout, we present finite type and continuum type results together. We draw intuition from the former, since the continuum type results follow by taking limits.

Koopmans and Beckmann (1957) decentralize the solution as a competitive equilibrium assuming TU. Legros and Newman (2007) show that some NTU models can be mapped into the TU paradigm. The term time-series is used to distinguish variation across matching markets from changes across types within a market. The state could also represent geographic differentiation in matching markets.
3 Synergy and Sorting Measurement

A. Synergy. In finite type models, (match) synergy is the cross partial difference of output:

\[ s_{ij}(\theta) = f_{i+1,j+1}(\theta) + f_{ij}(\theta) - f_{i+1,j}(\theta) - f_{ij+1}(\theta) \]

With continuum types, synergy is the cross partial \( \phi_{xy}(\theta) \). By Topkis (1998), production is SPM when synergy is everywhere nonnegative. To highlight the importance of synergy, let’s doubly sum match output by parts (the continuum analogue is in §A):

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} [f_{nj+1} - f_{nj}] j + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij} M_{ij} \tag{4}
\]

Equation (4) highlights that the matching distribution only impacts total match output via synergy. Any two production functions with the same synergies have the same matching, if everyone matches. For instance, synergy vanishes if production is linear in types. In this case, all match distributions yield the same output. As we will see this re-expression of match output helps build intuition for our measure of sorting.

B. Positive Quadrant Dependence (PQD). PQD is a partial order on bivariate probability distributions \( M_1, M_2 \in \mathcal{M}(G,H) \). Say that matching measure \( M_2 \) is PQD higher than \( M_1 \), or \( M_2 \succeq_{PQD} M_1 \), if \( M_2(x,y) \geq M_1(x,y) \) for all types \( x, y \). So \( M_2 \) puts more weight than \( M_1 \) on all lower (southwest) orthants. As \( M_1 \) and \( M_2 \) share marginals, \( M_2 \) puts more weight than \( M_1 \) on all upper (northeast) orthants too (Figure 2).

As §1 notes, PQD only partially orders the six possible pure matchings on three types. It helps to note that all match cdf’s are sandwiched above NAM and below PAM:

\[
\max(G(x) + H(y) - 1, 0) \leq M(x,y) \leq \min(G(x), H(y)) \tag{5}
\]
The second inequality says that the mass of matched men and women in \([0, x] \times [0, y]\) is at most the supply of men or women. The first inequality is more subtle — or \(1 - M(x, y) \leq \min(1 - G(x) + 1 - H(y), 1)\), says the mass of men and women in \([0, x] \times [0, y]\) not matched is at most the supply of men plus the supply of women.

Observe that Becker's Result follows immediately from (4) and (5). For SPM output implies nonnegative synergy, \(s_{ij} \geq 0\); and thus, by (4) output is highest when the match cdf \(M(x, y)\) is maximal — namely, PAM, as it dominates all other matchings in the PQD order by (5). Similarly, SBM implies globally nonpositive synergy, \(s_{ij} \leq 0\), and thus output is highest when the match cdf \(M(x, y)\) is minimal, namely, for NAM. More generally, the PQD and SPM orders coincide in \(\mathbb{R}^2\), i.e. increases in the PQD order increase (reduce) the total output for any SPM (SBM) function \(\phi\).

\[
M_2 \succeq_{PQD} M_1 \iff \int \phi(x, y)M_2(dx, dy) \geq \int \phi(x, y)M_1(dx, dy) \quad \forall \phi \text{ SPM} \quad (6)
\]

The PQD sorting measure shows up in some economically relevant measures:

**Lemma 1.** Fix increasing functions \(u\) and \(v\). Given a PQD order upward shift:
(a) the average geometric distance \(E[|u(X) - v(Y)|^\gamma]\) for matched types falls, if \(\gamma \geq 1\); (b) the covariance \(E_M[u(X)v(Y)] - E[u(X)]E[v(Y)]\) across matched pairs rises; (c) the coefficient in a linear regression of \(v(y)\) on \(u(x)\) across matched pairs rises.

Lemma 1 illustrates that the PQD order is scale invariant. To wit, if we claim that educational sorting rises in the PQD order, then it does so regardless of whether it is measured in highest degree attained, years of schooling, log years of schooling, etc.

**Proof of (a):** By inequality (6) it suffices that \(|u(x) - v(y)|^\gamma\) is SBM for all \(\gamma \geq 1\). Since \(-\psi(u - v)\) is SPM for all convex \(\psi\), by Lemma 2.6.2-(b) in Topkis (1998), we have \(-|u - v|^\gamma\) SPM for all \(\gamma \geq 1\). So, \(|u(x) - v(y)|^\gamma\) is SBM for all increasing \(u\) and \(v\).

**Proof of (b):** Since the marginal distributions on \(X\) and \(Y\) is constant for all \(M \in \mathcal{M}(G, H)\), and \(u(x)v(y)\) is supermodular for all increasing \(u\) and \(v\), the covariance \(E_M[XY] - E[X]E[Y]\) between matched types increases in the PQD order by (6).

**Proof of (c):** The coefficient \(c_1 = \text{cov}(u(X)v(Y))/\text{var}(v(X))\) in the univariate match partner regression \(v(y) = c_0 + c_1u(x)\) increases in the PQD order, by part (b).

For a useful counterpoint, posit a uniform type distribution on \([0, 1]\). Assume that every \(x \leq 1/2\) matches with \(x + 1/2\). Since it is increasing on the domain of larger match partners, Legros and Newman (2002) call this matching “monotone”. Notice that this matching maximizes the average distance between partners. To wit, it minimizes total match output for the supermodular production function \(f(x, y) = 1 - |x - y|\).

4 What Happens When Synergy Rises?

Since Becker shows that globally negative synergy leads to NAM, and globally positive synergy leads to PAM, one might guess that sorting increases if synergy increases everywhere. But Figure 3 refutes this conjecture: For as synergy increases, the uniquely optimal matching oscillates back and forth between NAM1 and NAM3, and these are not PQD comparable. Thus inspired, say that sorting is nowhere decreasing in θ if the matching never falls in the PQD order. So for all θ₂ ≥ θ₁, if M₁ ∊ 𝒜(θ₁) and M₂ ∊ 𝒜(θ₂) are ranked M₁ ≥_{PQD} M₂, then we have M₂ ∊ 𝒜(θ₁) and M₁ ∊ 𝒜(θ₂).

Proposition 1. Sorting is nowhere decreasing in θ if synergy is non-decreasing in θ.

Appendix B proves this by extending the theory of monotone comparative statics to our case with a single crossing condition, but not on a lattice domain. For notably, the space of matching cdf’s is not a lattice (Müller and Scarsini, 2006): To see this, consider (1). There, NAM1 and NAM3 are both upper bounds for PAM2 and PAM4, but there is no pure least upper bound. More strongly, PQD does not induce a lattice, as there is no least mixed least upper bound, M for PAM2 and PAM4.

A corollary of the theory to come, will address Figure 3. We will add some missing cross-sectional discipline: Say that synergy is (strictly) monotone in types if synergy is either non-decreasing (increasing) or non-increasing (decreasing) in (x, y). Say sorting increases in θ if M₂ ≥_{PQD} M₁ for all M₁ ∊ 𝒜(θ₁) and M₂ ∊ 𝒜(θ₂) and θ₂ ≥ θ₁.

Corollary 1. Assume synergy is non-decreasing in θ. Sorting is increasing in θ for: (a) generic finite type models if synergy is monotone in types and (b) continuum types model if synergy is strictly monotone in types.
5 Increasing Sorting

5.1 Rectangular Synergy

Index type space rectangles $[i_1, i_2] \times [j_1, j_2]$ by opposite corners, $r \equiv (i_1, j_1, i_2, j_2)$. Rectangular synergy is then

$$S(r|\theta) \equiv \sum_{i=11}^{i_2-1} \sum_{j=j_1}^{j_2-1} s_{ij}(\theta)$$

This coincides with the economic notion of a “sorting premium”, or the surplus from positively sorting couples $(i_1, j_1) < (i_2, j_2)$ vs. negatively sorting them as $(i_1, j_2)$ and $(i_2, j_1)$:

$$S(r|\theta) = f_{i_1j_1}(\theta) + f_{i_2j_2}(\theta) - f_{i_1j_2}(\theta) - f_{i_2j_1}(\theta)$$

For continuum types, $S(R|\theta) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi_{12}(x, y|\theta) dxdy$ for $R = (x_1, y_1, x_2, y_2)$.

Summed rectangular synergy adds the rectangular synergy on a finite set of non-overlapping rectangles: $\{r_k\}$ with finite types, or $\{R_k\}$ with a continuum of types. So summed rectangular synergy is upcrossing in $\theta$ if synergy is non-decreasing in $\theta$.

Our first cross-sectional assumption uses the northeast partial order on rectangles: $r \succeq_{NE} r'$, if diagonally opposite corners of $r$ are weakly higher than $r'$. Rectangular synergy is one-crossing in types if $S(r|\theta)$ is upcrossing (downcrossing) in $r$, for all $\theta$.

**Proposition 2.** Assume (A1) summed rectangular synergy is upcrossing in $\theta$ and (B1) rectangular synergy is one-crossing in types. If there is a unique optimal matching at $\theta_2 \succ \theta_1$, then sorting is PQD higher at $\theta_2$ than $\theta_1$.

For finite type models, the optimal matching is generically unique by Koopmans and Beckmann [1957]. We discuss uniqueness in continuum type models in §5.2.

We prove Proposition 2 in §E.2 by induction on the number of types. Lacking its time series assumption, Proposition 1 does not apply. But we can still argue that the optimal matching cannot fall in the PQD order. For instance, consider PAM4 and NAM. The difference is that (2, 1) and (3, 2) are matched pairs under PAM4, while NAM sorts them as (2, 2) and (3, 1). Thus, the PAM4 payoff exceeds the NAM payoff by rectangular synergy for women 2, 3 and men 1, 2, i.e. by $s_{21}(\theta)$. If NAM and PAM4 are uniquely optimal respectively at states $\theta'' \succ \theta'$, then $s_{21}(\theta'') < 0 < s_{21}(\theta')$, violating

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6A function $\Upsilon$ is upcrossing in $t$ on a partially ordered set $T$ if $\Upsilon(t) \geq (>)0 \Rightarrow \Upsilon(t') \geq (>)0$ for all $t' \succeq t$, downcrossing in $t$ if $\Upsilon(t') \geq (>)0 \Rightarrow \Upsilon(t) \geq (>)0$ for all $t' \preceq t$, one-crossing in $t$ if it is upcrossing or downcrossing. Strict versions of these conditions require that weak inequalities imply strict inequalities. For example, $\Upsilon$ is strictly upcrossing if $\Upsilon(x) \geq 0 \Rightarrow \Upsilon(x') > 0$, for all $x' > x$. 

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9
Step 1
\(N\) • − • − • •

Step 2
\(N\) • − • • − •

Step 3
\(N\) • + • − • − •

\(S(\cdot|\theta')\)

Step 1
\(P\) • • − • •

Step 2
\(P\) • • + • •

Step 3
\(P\) • • + • − •

\(S(\cdot|\theta'')\)

NAM1 vs. NAM3 Shifts

PAM2 vs. PAM4 Shifts

Figure 4: Precluding Unranked Shifts with \(n = 3\) and Nonzero Synergies. NAM1 at \(\theta'\) and NAM3 at \(\theta''\) is impossible, as is PAM2 at \(\theta'\) and PAM4 at \(\theta''\). The synergy signs in Steps 1\(_N\) and 1\(_P\) reflect local optimality. Step 2\(_N\) deduces \(s_{11}(\theta') < 0\) via upcrossing synergy from \(\theta''\) to \(\theta'\). Given PAM on rectangles \(r = (1, 1, 2, 3), (1, 1, 3, 2)\) at \(\theta'\), local optimality implies \(S(r|\theta') > 0\). As rectangular synergy is the sum of synergies, the synergy signs in Step 3\(_N\) follow — ruling out \(S(r|\theta')\) one-crossing in \(r\), a contradiction. Next, Step 2\(_P\) deduces \(s_{12}(\theta'') > 0\) via upcrossing synergy from \(\theta'\) to \(\theta''\). Given NAM on rectangles \(r = (1, 1, 2, 3), (1, 1, 3, 2)\) at \(\theta'\), local optimality implies \(S(r|\theta') < 0\). Since rectangular synergy is the sum of synergies, we can fully sign \(s_{ij}\). This sign pattern in Step 3\(_P\) violates \(S(r|\theta'')\) one-crossing in \(r\), a contradiction.

upcrossing summed rectangular synergy. The payoff difference between any two PQD ranked 3-type matchings \([\square]\) is the summed rectangular synergy for some set of couples.

Next, we use both time series and cross sectional reasoning to preclude non-PQD comparable shifts. Figure 4 traces the logic for the two relevant transitions: NAM1 to NAM3, and PAM2 to PAM4, or vice versa. That rectangular synergy is not one-crossing in types in our earlier Figure 3 is key to its sorting monotonicity failure.

5.2 Marginal Rectangular Synergy

We now provide a stronger, easier to check, cross-sectional assumption for increasing sorting. Specifically, the \(x\)-marginal rectangular synergy \(\Delta_i(i|j_1, j_2)\) is synergy summed over men in the interval \([j_1, j_2 - 1]\) and the \(y\)-marginal rectangular synergy \(\Delta_j(j|i_1, i_2)\) is synergy summed over women in the interval \([i_1, i_2 - 1]\), i.e.:

\[
\Delta_i(i|j_1, j_2, \theta) \equiv \sum_{j=j_1}^{j_2-1} s_{ij}(\theta) \quad \text{and} \quad \Delta_j(j|i_1, i_2, \theta) \equiv \sum_{i=i_1}^{i_2-1} s_{ij}(\theta)
\]

For a type continuum, the marginal rectangular synergy is an integral \(\Delta_x(x|y_1, y_2, \theta) \equiv \int_{y_1}^{y_2} \phi_{12}(x, y) dy\) or \(\Delta_y(y|x_1, x_2, \theta) \equiv \int_{x_1}^{x_2} \phi_{12}(x, y) dx\). These sums and integrals are one-crossing if they are respectively both upcrossing or both downcrossing in \(x\) and \(y\).

Proposition 3. Assume (A1) summed rectangular synergy is upcrossing in \(\theta\) and (B2)
marginal rectangular synergy is one-crossing. **Sorting rises in \( \theta \) in generic finite type, or continuum, types models with a strictly one-crossing marginal rectangular synergy.**

In §E.4 we integrate one-crossing marginal rectangular synergy (B2) to deduce one-crossing rectangular synergy (B1). Hence, Proposition 2 implies Proposition 3.

Next, we apply optimal transport theory to establish that the continuum optimal matching is unique when marginal rectangular synergy is strictly one-crossing.

### 5.3 Local Assumptions on Synergy

We now develop a fully local approach to synergy aggregation that simultaneously secures the needed cross-sectional and time series conditions — rectangular synergy one-crossing in types, and summed rectangular synergy upcrossing. Notice that for averages on very small sets, upcrossing synergy is necessary for upcrossing summed rectangular synergy. We now seek a condition that renders it sufficient.

We exploit our new upcrossing aggregation result in the Appendix. Denote by \( f^+ \equiv \max(f, 0) \) and \( f^- \equiv -\min(f, 0) \) the positive and negative parts of a function \( f \).

**Synergy is proportionately upcrossing** if for all \( z = (x, y), z' = (x', y') \), and \( \theta' \succeq \theta \), we have:

\[
\phi_{12}^+(z \wedge z', \theta)\phi_{12}^+(z \vee z', \theta') \geq \phi_{12}^-(z, \theta')\phi_{12}^+(z', \theta)
\] (7)

where the meet and join is defined with the usual (northeast) vector order on \( \mathbb{R}^2 \), or the reverse (southwest) vector order. The analogous definition applies to the finite type case when synergy \( s_{ij}(\theta) \) obeys (7) for all \( z = (i, j) \) and \( z' = (i', j') \) and \( \theta' \succeq \theta \).

While proportionately upcrossing synergy may be hard to appreciate, trivially note that it holds when synergy is increasing in \( \theta \) and monotone in types. But proportionately upcrossing synergy need not be monotone; if not, positive synergy must absolutely increase in \( \theta \) more than negative synergy does. Finally, for an easily checked condition in the continuum type case, note that synergy is proportionately upcrossing if it is smoothly log-supermodular (LSPM), namely when \( \sigma = \phi_{12} \) obeys \( \sigma_{ij} \sigma \geq \sigma_i \sigma_j \) (see §C.2).

**Proposition 4.** Assume (A2) synergy is upcrossing in \( \theta \), (B3) synergy is one-crossing in types, and (C1) proportionately upcrossing synergy. **Sorting increases in \( \theta \) in generic finite type models, or with continuum types if synergy strictly one-crosses in types.**

The Appendix proof derives Proposition 4 from Proposition 3. Specifically, We show that if synergy is upcrossing in \( \theta \) and proportionately upcrossing then summed

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7 \((z \vee z', \theta') \succeq (z', \theta) \Rightarrow \phi_{12}^+(z \vee z', \theta') \geq \phi_{12}^+(z', \theta), \) and \((z, \theta') \succeq (z \wedge z', \theta) \Rightarrow \phi_{12}^-(z \wedge z', \theta) \geq \phi_{12}^-(z, \theta').\)

8Assume negative synergy at couple \( z \), and positive at a higher couple \( z' = z \vee z' \succeq z \vee z' = z. \) Then (7) says that \( \phi_{12}^+(z', \theta')/\phi_{12}^+(z', \theta) \geq \phi_{12}^-(z, \theta')/\phi_{12}^-(z, \theta). \)
Match Payoffs: $f_{ij}(\theta') \rightarrow f_{ij}(\theta'')$

Synergy: $s_{ij}(\theta') \rightarrow s_{ij}(\theta'')$

Figure 5: Proportionately Upcrossing Failure. The unique efficient matching falls from NAM3 to PAM2 as $\theta'$ shifts up to $\theta''$. Note that synergy is upcrossing in $\theta$ and in types. But synergy is not proportionately upcrossing and sorting falls in $\theta$.

rectangular synergy is upcrossing in $\theta$, while if synergy is one-crossing in types and proportionately upcrossing then marginal rectangular synergy is one-crossing in types.

Also, Proposition 4 implies Corollary 1—namely, our easiest to state sorting result that we began with. For if synergy is increasing in $\theta$ and monotone in types, then it is upcrossing in $\theta$, one-crossing in types, and proportionately upcrossing.

Figure 5 presents an example in which synergy is both upcrossing in $\theta$ and in types, but in which sorting falls in $\theta$. To verify that synergy is not proportionately upcrossing, let $z = (2, 1)$, $z' = (2, 2)$, $t = \theta'$, and $t' = \theta''$, then we have:

$$\phi_{12}^- (z \wedge z', t) \phi_{12}^+ (z \vee z', t') = 2 \cdot 4 = 8 < 20 = 5 \cdot 4 = \phi_{12}^- (z, t') \phi_{12}^+ (z', t)$$

Notice that in this example both the time-series condition and cross-sectional conditions in Proposition 2 fail to hold. In particular, rectangular synergy is not upcrossing in types at $\theta''$, since $2 + (-1) > 0 > 4 + (-5)$. And rectangular synergy is not upcrossing in $\theta$, since $4 + (-2) > 0 > 4 + (-5)$. Notice that this latter failure is precisely what causes the optimal matching to fall from NAM3 to PAM2.

Corollary 2. Assume a continuum of types, with synergy upcrossing in $\theta$ (A2), strictly one-crossing in types (B3), and smoothly LSPM (C2). Then sorting is increasing in $\theta$.

6 Increasing Sorting and Type Distribution Shifts

Distributional shifts can be formally embedded in production functions, and thus allow us to use our comparative statics theory to deduce sorting predictions for changes in the type distributions $G(\cdot|\theta)$ and $H(\cdot|\theta)$. We say that $X$ types shift up (down) in $\theta$ if $G(\cdot|\theta)$ stochastically increases (decreases) in $\theta$, i.e. $G(\cdot|\theta') \leq G(\cdot|\theta)$ if $\theta' \succeq \theta$. Similarly, $Y$ types shift up (down) in $\theta$ if $H(\cdot|\theta)$ stochastically increases (decreases) in $\theta$.

The PQD order introduced in §3 only ranks matching distributions with the same marginals $G$ and $H$. To overcome this, we consider sorting in quantile space. Label every type by its quantile in the distribution, so $p = G(X(p, \theta)|\theta)$ and $q = H(Y(q, \theta)|\theta)$. 

12
Figure 6: Increasing Sorting with Type Shifts for Quadratic Production. These graphs depict optimally matched quantile pairs (blue dots) given an exponential distribution on types $G(x|\theta) = 1 - e^{-x/\theta}$ and $H(y|\theta) = 1 - e^{-y/\theta}$, and quadratic production $xy - (xy)^2$. By Corollary 3 since synergy is decreasing in types, quantile sorting increases as $\theta$ falls. The plots depict $\theta = 1, 2/3, 1/3$ at left, middle, and right.

Next for any matching distribution consider the associated bivariate copula which defines the sorting by quantiles, namely $C(p, q) = M(X(p, \theta), Y(q|\theta))$. The copula is the matching distribution defined on quantiles $(p, q)$ rather than types $(x, y)$. We say that quantile sorting is higher at $M''$ than $M'$ when the associated copulas are ranked $C'' \succeq_{PQD} C'$; namely, when $C''$ has more mass than $C'$ in all lower and upper orthants in $(p, q)$ space. The quantile sorting order generalizes the PQD order. For if $M''$ and $M'$ share the same marginals, then $C'' \succeq_{PQD} C'$ if and only if $M'' \succeq_{PQD} M'$. And since all copulas have uniform marginals by definition, we can compare two copulas in the PQD order even if the associated matching distributions do not share marginals.

By Lemma 1, greater quantile sorting order reduces the average geometric distance between matched quantiles, and raises the covariance across matched quantile pairs, and the coefficient in linear regression of male on female match partner quantiles.

**Corollary 3.** Quantile sorting increases if types shift up (down):
(a) generically with finite types, if synergy is non-decreasing (non-increasing) in types;
(b) given $G$ and $H$ absolutely continuous, if synergy is increasing (decreasing) in types.

For some insight into the proof in §E.6 consider the quantile production function $\varphi(p, q|\theta) \equiv \phi(X(p, \theta), Y(q, \theta))$ with quantile synergy:

$$\varphi_{12}(p, q|\theta) = \phi_{12}(X(p, \theta), Y(q, \theta))X_p(p, \theta)Y_q(q, \theta)$$

(8)

For concreteness assume productive synergy $\phi$ is increasing in types and that types shift up in $\theta$. Thus, implying that the composite function $\phi_{12}(X(p, \theta), Y(q, \theta))$ is increasing in quantiles and $\theta$. But we cannot conclude that quantile synergy is increasing in quantiles and $\theta$ since the right hand side of (8) includes the derivatives $X_p$ and $Y_q$. 
which need not be monotone in quantiles or \( \theta \). However, since these derivatives are positive, we can conclude that quantile synergy is upcrossing in types and \( \theta \). In fact, we verify in \( \S.E.6 \) that the assumptions in Corollary \( \S.3 \) imply the premise of Proposition \( \S.3 \). Figure \( \S.6 \) illustrates this result with quadratic production.

### 7 Economic Applications

We now explore economic applications. All matching plots depict optimally matched pairs (blue dots) for a discrete uniform distribution on 100 types of “women” and “men.”

#### 7.1 Diminishing Returns

Production complementarity is a key driving force behind match synergy. But because convex transformations preserve SPM, such complementarity is amplified by increasing returns, and therefore, undermined by diminishing returns: While widget production is complementary, there may be diminishing returns to more widgets.

For example, assume bi-quadratic production \( \phi(x, y) = \alpha xy + \beta(xy)^2 \). Then the match synergy function \( \phi_{12} = \alpha + 4\beta xy \) is strictly increasing in \( \alpha \) and \( \beta \). Becker gives PAM if \( \alpha, \beta > 0 \), and NAM if \( \alpha, \beta < 0 \). But if \( \alpha > 0 > \beta \), then synergy is strictly monotone in types, and Corollary \( \S.1 \) predicts that sorting increases in \( (\alpha, \beta) \).

With bi-cubic production \( \phi(x, y) = \alpha xy + \beta(xy)^2 + \gamma(xy)^3 \), synergy falls in types when \( \beta, \gamma < 0 \), and rises when \( \beta, \gamma > 0 \). But if \( \beta\gamma < 0 \), synergy may not be monotone in types; we can then only conclude that sorting is nowhere decreasing, by Proposition \( \S.4 \).

#### 7.2 Task Assignment

Asymmetric roles is a source of negative match synergies. \cite{kremer1996} assume that agents are assigned to the manager or deputy roles, where \( x^\theta y^{1-\theta} \) is output if \( x \) is the manager and \( y \) the deputy, and \( \theta \in [0, 1/2] \). As a unisex model, match output is then the maximum of two SPM functions \( \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta \} \). But this function is neither SPM nor SBM, since minimization preserves SPM, and maximization preserves SBM. We can consider more general production that allows a smooth role transition:

\[
\phi(x, y|\theta) = x^\theta y^\theta (x^\rho + y^\rho)^{1-2\theta} \rightarrow \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta \} \quad \text{as} \quad \rho \rightarrow \infty \quad (9)
\]

\( ^9 \)More generally, let \( \phi(x, y|\theta) = \psi(xy|\theta) \) for \( \psi \) increasing and concave with “relative risk aversion” \(-z\psi''(z|\theta)/\psi'(z|\theta) \) decreasing in \( z \) and \( \theta \). Then we show in \( \S.F \) that synergy is upcrossing in types and \( \theta \) and proportionately upcrossing: sorting increases in \( \theta \) (as \( \psi \) becomes less concave) by Proposition \( \S.4 \). Symmetrically, sorting increases for convex transformations of a SBM function.
Figure 7: Kremer-Maskin Payoffs. We plot the payoffs for three type \((x, y) \in \{1, 2, 6.5\}\) matchings for \((\theta)\) (NAM is dominated by PAM2=PAM4). On the left, matching is nowhere decreasing, from PAM2=PAM4 (brown), to NAM1 (blue), to NAM3 (green), to PAM (red) as \(\theta\) rises \((\rho = 100)\). On the right, matching is nowhere increasing, from PAM (red), to NAM3 (green), to NAM1 (blue), as \(\varrho\) rises \((\theta = 0.32)\).

Figure 7 explores the optimal matching correspondence with three types. Sorting is not monotone in either parameter, as the optimal matching switches between NAM1 and NAM3. To see why increasing sorting fails in this case, note that \(x^\theta y^{1-\theta}\) and \(x^{1-\theta} y^\theta\) are supermodular. So rectangular synergy is positive for rectangles not straddling the diagonal. But we can show that it is negative for small rectangles straddling the diagonal \(10\) and so is not one-crossing in types, as Proposition 2 requires.

Sorting is not monotone, but is nowhere decreasing in \((\theta, -\varrho)\), as proven in §B.3.

7.3 Information Frictions

A. ADVERSE SELECTION. We consider the dynamic extension of Ghatak’s (1999) model of group lending with adverse selection by Guttman’s (2008). Borrowers vary by their project success chance \(x\); a success pays \(\pi > 0\) and a failure nothing. Pairs of borrowers sign lending contracts, and project outcomes are independent. Borrowers observe each other’s type in the matching market, but banks do not observe types.

After the project outcome, a borrower either repays the loan, or defaults. Each pays the debt \(d > 1\) if both repay. If only one defaults, the other repays collateral plus debt \(c + d > d\). In the paper, a borrower repays when his project succeeds if \(\pi \geq c + d\). If both default, each loses access to credit markets. Borrowers observe each other’s type in the matching market, but banks do not observe types.

To model the dynamics, we let \(\delta \leq 1\) defaulting if both projects fail. A project pair \((x, y)\) has discounted value:

\[
\phi(x, y) = x((\pi - d) - (1-y)c) + y((\pi - d) - (1-x)c) + \delta(1 - (1-x)(1-y))\phi(x, y)
\]  

The static payoff has the weakest link (SPM) flavor, since the collateral \(c\) is paid when

\[10\]If \(y > x\) and \(0 < \theta < 1/2\), rectangular synergy is \(\phi(x, x|\theta) + \phi(y, y|\theta) - 2\phi(x, y|\theta) = x+y-2xy^{1-\theta} < 0\) for \(1 < y/x < 1 + \varepsilon\).

\[11\]The discounted value is well defined for \(\delta = 1\), since both projects eventually fail.
Figure 8: **Increasing Sorting in the Group Lending Model.** If parameters obey \( \delta < c/(c+\pi-d) \), the optimal is PAM; otherwise, the optimal matching is mixed. These graphs depict optimally matched man-woman pairs (blue dots) assuming a uniform distribution on 100 types for \( \delta = 0.8 \). The left matching is drawn for \((\pi, c, d) = (10, 0, 2)\), the middle for \((\pi, c, d) = (10, 2, 2)\), and the right for \((\pi, c, d) = (4, 2, 2)\).

precisely one of the two projects fails. But since the pair only loses access to credit markets when *both* projects fail, the continuation chance has a strongest link (SBM) flavor\(^{12}\). Indeed, synergy \( \phi_{12} \) is globally positive if \( \delta \leq \delta^* \equiv c/[c + (\pi - d)] < 1 \). But with more patience, synergy is positive for low types and negative for high types.

In §\( \text{F} \) we apply Proposition 3 to show that sorting rises in the collateral \( c \), and falls in the net payoff \( \pi - d \). For these respectively amplify and lessen the static weakest link force (Figure 8). While PAM is not optimal for \( \delta \in (\delta^*, 1) \),\(^{13}\) sorting is not monotone in the discount factor \( \delta \), as synergy is globally non-negative when \( \delta \leq \delta^* \) and at \( \delta = 1 \).

**B. Moral Hazard with Endogenous Contracts.** [Serfes (2005)](https://example.com) explores a pairwise matching principal-agent model. The output for any project is the sum of the agent’s unobservable effort and a mean zero Gaussian error. Project variances \( y \in [y, \overline{y}] \) vary across principals, while agents differ by risk aversion parameter \( x \in [x, \overline{x}] \), and share a scalar dis-utility of effort \( \theta > 0 \). Contracts are signed after matching takes place; they specify the agent’s wage as a function of realized output. Serfes derives (his equation (2)) the equilibrium expected output and cross-partial of an \((x, y)\) match:

\[
\phi(x, y | \theta) = \frac{1}{2\theta (1 + \theta x y)} \quad \Rightarrow \quad \phi_{12}(x, y | \theta) = \frac{\theta x y - 1}{2(1 + \theta x y)^3}
\]  

\(^{14}\)Legros and Newman (2002) consider matching with imperfect credit markets. Conditional on receiving financing, output is SPM \( x y \) minus upfront fixed cost \( q \). But creditors only finance pairs with \( x y \geq \kappa > q \), which has a weakest link flavor. Altogether, output is \( \phi = (x y - q)\mathbb{1}_{x y \geq \kappa} \), which is neither globally SPM nor globally SBM. Synergy is not one-crossing in types, and sorting is not monotone in \( \kappa \) or \( q \). But our theory yields sorting nowhere decreasing in \( \theta = q/\kappa \).

\(^{13}\)When \( \delta \in (c/(c + \pi - d), 1) \), the symmetric synergy function \( \phi_{12}(x, x) \) is strictly negative for \( x \) close to 1. Thus, cross matching types \( x \) and \( x + \varepsilon \) beats sorting them, for high \( x \) and low \( \varepsilon \).
Figure 9: Increasing Sorting in the Principal-Agent Model. NAM is optimal for low dis-utility of effort $\theta$, PAM for high $\theta$, and the optimal matching is mixed for intermediate $\theta$. These graphs depict optimally matched pairs (blue dots) for a discrete uniform distribution on 100 types of principals and agents. The left plot is drawn for $\theta = 0.65$, the middle for $\theta = 0.72$, and the right for $\theta = 0.82$. In all plots, the matching obeys local optimality — if the matching slopes up, then synergy is positive (shaded). In all plots, the reverse implication fails due to subtle global optimality considerations.

Serfes observes that synergy is globally negative for $\theta \bar{x} \bar{y} < 1$ and globally positive for $\theta \bar{x} \bar{y} > 1$. Thus by Becker’s Sorting Result, NAM obtains for $\theta < (\bar{x} \bar{y})^{-1}$ and PAM obtains for $\theta > (\bar{x} \bar{y})^{-1}$. This result reflects two counterveiling forces for sorting. First, if all contracts were the same, then efficient insurance across principal-agent pairs favors NAM: less risk averse agents work on higher variance projects. But the slope of the equilibrium wage contract is $(1 + \theta \bar{x} \bar{y})^{-1}$; and thus, the incentives to provide effort are SPM for high types. The sign of synergy (11) implies that the insurance effect dominates for low types, and the incentive effect dominates for high types.

We claim that sorting is increasing in $\theta$ when extremal types obey $\bar{x} \bar{y} \leq 2 \bar{x} \bar{y}$ (14). Assume $\theta' > \theta$. In both cases, sorting is weakly higher at $\theta'$ than $\theta$. Assume $\theta' \bar{x} \bar{y} \leq 1 < \theta \bar{x} \bar{y}$. Then by (14) we have $\theta' \bar{x} \bar{y} \leq 2$, and since the function $(t - 1)/(1 + t)^3$ is increasing for $t \in (0, 2]$, synergy (11) is increasing in $\theta$, $x$, and $y$. Altogether, NAM obtains for $\theta \leq (\bar{x} \bar{y})^{-1}$, PAM for $\theta \geq (\bar{x} \bar{y})^{-1}$, and sorting is increases in $\theta$ between these two extremes, by Corollary 3 as in Figure 9. Also, since synergy increases in types when PAM is suboptimal, quantile sorting increases when types shift up, by Corollary 3.

\footnote{Ackerberg and Botticini (2002) investigate matching between landowners (principals) and tenants (agents) in 15th century Tuscany. Using data on crop types (parameterizing project variance $y$) and tenant wealth (a proxy for risk aversion $x$), they find significant positive covariance in $y$ and $x$ across matched crop-tenant pairs. But since the matching is not perfect sorting (PAM), our theory provides a framework for analyzing changes in crop-tenant matching across markets.}
7.4 Learning Dynamics

Assume pairwise matching in periods one and two. Production is the symmetric, increasing and SPM function $\phi^0(x, y)$. But agents may learn over time: If types $x$ and $y$ match in period one, they enter period two as type $x' = \tau(x, y)$ and $y' = \tau(y, x)$ with $\tau_1 > 0$. Given SPM output, PAM is statically optimal in period two. But in period one, the social planner weights output by $(1 - \delta, \delta)$, so that the payoff function is:

$$\phi(x, y) = (1 - \delta)\phi^0(x, y) + \frac{\delta}{2} [\phi^0(\tau(x, y), \tau(x, y)) + \phi^0(\tau(y, x), \tau(y, x))]$$  \hspace{1cm} (12)

Synergy $\phi_{12}$ will be a $(1 - \delta, \delta)$ weighted average of static synergy $\phi^0_{12} > 0$ and dynamic synergy. To understand the sign of dynamic synergy, notice that:

$$[\phi^0(\tau(x, y), \tau(x, y))]_{12} = (\phi^0_{11} + 2\phi^0_{12} + \phi^0_{22}) \tau_1 + (\phi^0_1 + \phi^0_2) \tau_{12}$$  \hspace{1cm} (13)

As seen in §7.1, diminishing returns induces negative synergy — the first term is negative when $\phi^0(x, x)$ is concave. But negative synergy may also reflect a submodular transition function $\tau$. This naturally arises in learning environments, where the lower type learns from the higher, as protege from the mentor. At the extreme, $\tau(x, y) = \max\{x, y\}$ is SBM and $\tau(x, y) = \min\{x, y\}$ is SPM. In a marriage market interpretation of the model, married couple with types $(x, y)$ producing offspring with types $(x', y')$.

Then $\tau = \max$ is a reduced form model for dominant (genetic or social) inheritance of productivity, while $\tau = \min$ captures recessive type transmission.

Now consider comparative statics in the discount rate, assuming dynamic synergy is negative, consistent with the empirical results in Herkenhoff, Lise, Menzio, and Phillips (2018). Synergy is then increasing in $1 - \delta$; and thus, sorting will be nowhere decreasing in $1 - \delta$ by Proposition. Since synergy is increasing in $1 - \delta$, the time-series premise of each of our increasing sorting results is met. But, stronger assumptions are required to satisfy the cross-sectional assumptions. The most transparent case is when static synergy and dynamic synergy are both monotone in types in the same direction. Then sorting will be decreasing in $\delta$ by Corollary.

7.5 Pairwise Trading with Multidimensional Characteristics

Our last pairwise matching application reaches outside the traditional realm of Becker sorting results in economics, and inquires about a classic unit trade model. We build

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[15] HLMP add search frictions to the matching model in this subsection, and estimate the model on US labor market data. They find that dynamic synergy is significantly negative.
on Shapley and Shubik (1971), assuming home buying couples indexed by income $x$ and home sellers indexed by home size $y$ with increasing selling costs $c(y)$. Home buyer values are $x + \alpha xy + \beta zy$, where $z$ is home buyer family size and $\alpha, \beta \geq 0$. We assume that $z$ is the realization of a random variable with conditional expectation $Z(x)$ and that matching takes place before $z$ is realized. Altogether the match value is:

$$
\phi(x, y|\theta) = \max(x + \alpha xy + \beta Z(x)y - c(y), 0)
$$

The homogeneous double auction corresponds to $\alpha = \beta = 0$, for which the match payoffs (14) are SBM — so that Becker’s Matching Theorem predicts that low cost sellers trade to high value buyers. But let us now focus on just those market participants who do trade. Match synergy is $\alpha + \beta Z'(x)$ for all positive surplus trades.

If family size does not matter ($\beta = 0$) or positively covaries with income ($Z'(x) \geq 0$), then $\phi$ is SPM for all trading pairs and thus PAM is optimal among those that trade. Also, if $\alpha = 0$ and family size negatively covaries with income $Z'(x) < 0$ then NAM is optimal among those that trade. Finally, assume $\alpha, \beta > 0$ and $Z'(x) < 0$. If $Z'$ is also monotone, then synergy is monotone in $x$, increasing in $\alpha$, and decreasing in $\beta$, and Corollary predicts that sorting rises in $\alpha$ and falls in $\beta$.

8 Conclusion

Becker’s 1973 insight that supermodularity yields positive sorting launched the pairwise matching literature. But an impassable wall of mathematical complexity has prevented any general theory for non-assortative matching. The large number of economic models without perfect sorting invites a more general approach. Bypassing the solution of the optimal matching, we provide the missing general theory for comparative statics. Using an economically-motivated notion of increasing sorting, we answer when the match sorting increases given shifts in productivity or type distributions.

Our primary hurdle is that more synergistic matching need not lead to more sorting. Rather, we can only conclude that sorting does not fall. Stronger conclusions require cross-sectional discipline. We show that if synergy one-crosses as types increase, then sorting increases if either (i) the summed rectangular synergy on all sets of couples

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17 If family size were known to the buyers at the time of purchase, then buyer types would be a vector $(x, z)$ and this would be a multidimensional type matching model as in Lindenlaub (2017). The comparative statics of sorting with multidimensional types is beyond the scope of the current work.

18 This is consistent with the well documented negative relationship between income and family size across a wide range of countries.

19 Our comparative static echoes the finding in Lindenlaub (2017) that sorting increases on the cognitive dimension with an increase in the relative weight on cognitive (vs. manual) complementarities.
increases, or \( \text{(ii)} \) synergy linearly or monotonely changes, or \( \text{(iii)} \) synergy upcrosses through zero, and proportionately so, in a sense we define.

We revisit the matching literature since 1990, quickly deriving and strengthening their findings. Our paper offers a tractable foundation for future theoretical and empirical analysis of matching. A subtle and valuable direction for future work is a multidimensional extension of our theory (see Lindenlaub (2017)).

### A Match Output Reformulation

**Finitely Many Types.** Summing \( \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} f_{ij} m_{ij} \right] \) by parts in \( j \) and then \( i \) yields:

\[
\sum_{i=1}^{n} \left[ f_{in} \sum_{j=1}^{n} m_{ij} - \sum_{j=1}^{n-1} \left[ f_{i,j+1} - f_{ij} \right] \sum_{k=1}^{j} m_{ik} \right]
\]

\[
= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} \sum_{i=1}^{n} \left[ f_{i,j+1} - f_{ij} \right] \sum_{k=1}^{j} m_{ik}
\]

\[
= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} \left( \left[ f_{n,j+1} - f_{n,j} \right] \sum_{\ell=1}^{j} \sum_{k=1}^{\ell} m_{\ell k} - \sum_{i=1}^{n-1} s_{ij} \sum_{\ell=1}^{i} \sum_{k=1}^{j} m_{\ell k} \right)
\]

\[
= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} \left( \left[ f_{n,j+1} - f_{n,j} \right] j - \sum_{i=1}^{n-1} s_{ij} M_{ij} \right)
\]

**Continuum of Types.** Given type sets \( \mathcal{I} \equiv [0, 1] \) and \( \mathcal{J} \equiv (0, 1] \), then:

\[
\int_{\mathcal{I} \times \mathcal{J}} \phi(x, y) M(dx, dy) = \int_{\mathcal{I}} \phi(x, 1) G(dx) - \int_{\mathcal{J}} \phi_2(1, y) H(y) dy + \int_{\mathcal{J} \times \mathcal{I}} \phi_1(x, y) M(x, y) dx dy
\]

**Proof:** If \( \psi \) is \( C^1 \) on \([0, 1]\) and \( \Gamma \) is a cdf on \([0, 1]\), integration by parts yields:

\[
\int_{[0,1]} \psi(z) \Gamma(dz) = \psi(1) \Gamma(1) - \int_{[0,1]} \psi'(z) \Gamma(z) dz
\]

where the interval \((0, 1]\) accounts for the possibility that \( \Gamma \) may have a mass point at 0.

Since \( M(dx, y) \equiv M(y|x) G(dx) \) for a conditional matching cdf \( M(y|x) \), we have:

\[
M(x, y) \equiv \int_{[0,x]} M(y|x') G(dx')
\]

By Theorem 34.5 in [Billingsley (1995)] and then in sequence (15), (16) and Fubini’s
Theorem, (15), the objective function \( \int_{[0,1]} \int_{[0,1]} \phi(x,y)M(dy|x)G(dx) \) in (3) equals:

\[
\int_{[0,1]} \int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]} \int_{[0,1]} \phi_2(x,y)M(y|x)dyG(dx)
\]

\[
= \int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]} \left[ \phi_2(1,y)M(1,y) - \int_{[0,1]} \phi_{12}(x,y)M(x,y)dx \right] dy
\]

which easily reduces to the desired expression, using \( M(1,y) = H(y) \).

B Nowhere Decreasing Optimizers

The space of matching cdf’s is not a lattice, since the meet and the join are not defined for arbitrary matchings.\(^{20}\) The matching problem (3) does not have a lattice constraint or an objective function that is quasi-supermodular in the control: standard monotone comparative static results (e.g. Milgrom and Shannon (1994)) do not apply. The next section presents a general comparative result static for single-crossing functions on partially ordered sets (posets) without assuming a well-defined meet or join.\(^{21}\) We then apply this result to our sorting model to get a nowhere decreasing sorting result.

B.1 Nowhere Decreasing Optimizers for Arbitrary Posets

Let \( Z \) and \( \Theta \) be posets. The correspondence \( \zeta : \Theta \to Z \) is nowhere decreasing if \( z_1 \in \zeta(\theta_1) \) and \( z_2 \in \zeta(\theta_2) \) with \( z_1 \succeq z_2 \) and \( \theta_2 \succeq \theta_1 \) imply \( z_2 \in \zeta(\theta_1) \) and \( z_1 \in \zeta(\theta_2) \).

**Theorem 1 (Nowhere Decreasing Optimizers).** Let \( F : Z \times \Theta \to \mathbb{R} \), where \( Z \) and \( \Theta \) are posets, and let \( Z' \subseteq Z \). If \( \max_{z \in Z'} F(z,\theta) \) exists for all \( \theta \) and \( F \) is single crossing in \((z,\theta)\), then \( \mathcal{Z}(\theta|Z') \equiv \arg \max_{z \in Z'} F(z,\theta) \) is nowhere decreasing in \( \theta \) for all \( Z' \). If \( \mathcal{Z}(\theta|Z') \) is nowhere decreasing in \( \theta \) for all \( Z' \subseteq Z \), then \( F(z,\theta) \) is single crossing.

\((\Rightarrow)\): If \( \theta_2 \succeq \theta_1 \), \( z_1 \in \mathcal{Z}(\theta_1) \), \( z_2 \in \mathcal{Z}(\theta_2) \), and \( z_1 \succeq z_2 \), optimality and single crossing give:

\[
F(z_1,\theta_1) \geq F(z_2,\theta_1) \Rightarrow F(z_1,\theta_2) \geq F(z_2,\theta_2) \Rightarrow z_1 \in \mathcal{Z}(\theta_2)
\]

\(^{20}\)As shown in Proposition 4.12 in Müller and Scarsini (2006): If \( M \) dominates PAM2 and PAM4, then \( M(2,1) \geq 1/3 \) and \( M(1,2) \geq 1/3 \), but \( M(1,1) = 0 \) if NAM1 and NAM3 dominate \( M \). So then \( M(2,2) = 2/3 \), but then NAM1 cannot PQD dominate \( M \).

\(^{21}\)This may be a known result. We include it for completeness, and as we cannot find any reference.
Now assume \( z_2 \notin \mathcal{Z}(\theta_1) \). By optimality and single crossing, we get the contradiction:

\[
F(z_1, \theta_1) > F(z_2, \theta_1) \Rightarrow F(z_1, \theta_2) > F(z_2, \theta_2) \Rightarrow z_2 \notin \mathcal{Z}(\theta_2)
\]

(\( \Leftarrow \)): If \( F \) is not single crossing, then for some \( z_2 \geq z_1 \) and \( \theta_2 \geq \theta_1 \), either: (i) \( F(z_2, \theta_1) \geq F(z_1, \theta_1) \) and \( F(z_2, \theta_2) < F(z_1, \theta_2) \); or, (ii) \( F(z_2, \theta_1) > F(z_1, \theta_1) \) and \( F(z_2, \theta_2) \leq F(z_1, \theta_2) \). Let \( Z' = \{z_1, z_2\} \). In case (i), \( z_2 \in \mathcal{Z}(\theta_1|Z') \) and \( z_1 = \mathcal{Z}(\theta_2|Z') \) precludes \( \mathcal{Z}(\theta|Z') \) nowhere decreasing in \( \theta \), since \( z_2 \notin \mathcal{Z}(\theta_2|Z') \). In case (ii), \( z_2 = \mathcal{Z}(\theta_1|Z') \) and \( z_1 \in \mathcal{Z}(\theta_2|Z') \) precludes \( \mathcal{Z}(\theta|Z') \) nowhere decreasing in \( \theta \), since \( z_1 \notin \mathcal{Z}(\theta_1|Z') \). \( \square \)

### B.2 Nowhere Decreasing Sorting

We say that **weighted synergy is upcrossing**\(^{23}\) in \( \theta \) if the following is upcrossing in \( \theta \):

- \( \int \phi_{12}(x, y|\theta)\lambda(x, y)dxdy \) for all nonnegative (measurable)\(^{23}\) functions \( \lambda \) on \([0, 1]^2\)
- \( \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)\lambda_{ij} \) for all positive weights \( \lambda \in \mathbb{R}_+^{(n-1)^2} \)

**Theorem 2.** Sorting is nowhere decreasing in \( \theta \) if weighted synergy is upcrossing in \( \theta \), and thus if synergy is nondecreasing in \( \theta \). Also, if sorting is nowhere decreasing in \( \theta \) for all type distributions \( G, H \), then any rectangular synergy is upcrossing in \( \theta \).

**Proof of (a):** First, \( M' \geq_{PQD} \bar{M} \) iff \( \lambda \equiv \bar{M}' - \bar{M} \geq 0 \). As weighted synergy upcrosses:

\[
\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij} - M_{ij}) \geq (>) 0 \Rightarrow \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M_{ij}) \geq (>) 0
\]

\[
\int_{[0,1]^2} \phi_{12}(\cdot|\theta')(M' - \bar{M}) \geq (>) 0 \Rightarrow \int_{[0,1]^2} \phi_{12}(\cdot|\theta')(M' - \bar{M}) \geq (>) 0
\] \hspace{1cm} (17)

Thus, match output is single crossing in \( (M, \theta) \) by (4) (for finite types) and Appendix A for continuum types. Then the optimal matching \( \mathcal{M}^*(\theta) \) (in the space of feasible matchings \( \mathcal{M}(G, H) \)) is nowhere decreasing in the state \( \theta \), by Theorem 1.

**Proof of (b):** Assume a two type model with women \((x_1, x_2)\) and men \((y_1, y_2)\). Assume that \( S(R|\theta) \) is not upcrossing in \( \theta \), i.e. for some \( \theta'' \geq \theta' \) and rectangle \( R = (x_1, y_1, x_2, y_2) \), we have \( S(R|\theta'') \leq 0 \leq S(R|\theta') \) with one inequality strict. These inequalities respectively imply that NAM is optimal at \( \theta'' \) and PAM is optimal at \( \theta' \). Since one inequality is strict, either NAM is uniquely optimal at \( \theta'' \) or PAM is uniquely optimal at \( \theta' \). Either case precludes nowhere decreasing sorting. \( \square \)

---

\(^{22}\) Let \( Z \) be a partially ordered set. The function \( \sigma : Z \rightarrow \mathbb{R} \) is upcrossing if \( \sigma(z) \geq (>0) \) implies \( \sigma(z') \geq (>0) \) for \( z' \geq z \), downcrossing if \( -\sigma \) is upcrossing. Similarly, \( \sigma \) is strictly upcrossing if \( \sigma(z) \geq 0 \) implies \( \sigma(z') > 0 \) for all \( z' > z \), with strictly downcrossing defined analogously.

\(^{23}\) To save space, we henceforth assume measurable sets for integrals whenever needed.
B.3 Nowhere Decreasing Sorting in Kremer and Maskin (1996)

We now establish that sorting is nowhere decreasing in $\theta$ and nowhere increasing in $\varrho$ in the Kremer-Maskin model.

**Step 1.** PAM is not optimal if $\varrho > (1-2\theta)^{-1}$, and is uniquely optimal for $\varrho < (1-2\theta)^{-1}$.

*Proof:* In a unisex model, PAM is optimal iff the symmetric rectangular synergy $S(x,x,y,y)$ is globally positive. Its sign is constant along any ray $y = kx$, and proportional to:

$$s(k) = 2^{1-2\varrho} (1 + k) - 2k^\theta (1 + k^\varrho)^{1-2\varrho}$$  \hfill (18)

Since $s(1) = s'(1) = 0$, $s''(1) \propto (1 + \varrho(2\theta - 1))$, and $\theta \in [0, 1/2]$, we have $s(k) < 0$ close to $k = 1$ precisely when $\varrho > (1-2\theta)^{-1} \geq 1$. In this case, the symmetric rectangular synergy is negative in a cone around the diagonal, and PAM fails.

Conversely, posit $\varrho < (1-2\theta)^{-1}$. Then $s(k) > 0$ for all $k \in [0, 1]$. Since $S(x,x,y,y)$ is symmetric about $y = x$, it is globally positive and PAM is uniquely optimal. \hfill $\square$

**Step 2.** If $\varrho \geq (1-2\theta)^{-1}$ then weighted synergy is upcrossing in $\theta$, downcrossing in $\varrho$.

*Proof:* Change variables $y = kx$. If $\Delta(k) = \int_0^1 \lambda(x,kx)dx$, weighted synergy is

$$\int \int \phi_{12}(x,y)\lambda(x,y)dydx = 2 \int_0^1 \int_0^1 x\phi_{12}(x,kx)\lambda(x,kx)dkdx = \int_0^1 \sigma(k,\theta,\varrho)\Delta(k)dk$$

where $\sigma = \sigma_A\sigma_B$ for $\sigma_A \equiv 2k^{\theta-1}(1 + k^\varrho)^{1-2\varrho} - \varrho$ and $\sigma_B \equiv \varrho(1 - \theta)(1 + k^{2\varrho}) + (1 - \varrho + 2\theta(\theta - 1 + \varrho))k^\varrho$. As $\varrho \geq (1-2\theta)^{-1}$, $\sigma_A > 0$ is LSPM in $(k,\theta,\varrho)$, $\sigma_B$ is increasing in $(\theta, -k, -\varrho)$ for $k \in [0, 1]$. So $\sigma = \sigma_A\sigma_B$ is proportionately downcrossing in $(k,\theta)$ and $(k, -\varrho)$. Weighted synergy is upcrossing in $\theta$, downcrossing in $\varrho$, by Theorem 3. \hfill $\square$

**Step 3.** Sorting is nowhere decreasing in $\theta$ and nowhere increasing in $\varrho$.

*Proof:* Pick $\theta'' > \theta'$. If $\varrho < (1-2\theta'')^{-1}$, then PAM is uniquely optimal at $\theta''$ (Step 1) and sorting increases from $\theta'$ to $\theta''$. If $\varrho \geq (1-2\theta'')^{-1}$, then $\varrho > (1-2\theta')^{-1}$ and weighted synergy is upcrossing on $[\theta', \theta'']$ (Step 2) and sorting is non-decreasing (Proposition 2).

Now pick any $\theta$ and $\varrho'' > \varrho'$. If $\varrho' < (1-2\theta)^{-1}$, then PAM is uniquely optimal at $\varrho'$ (Step 1) and sorting is decreasing from $\varrho'$ to $\varrho''$. If, instead, $\varrho' \geq (1-2\theta)^{-1}$, then, necessarily, $\varrho'' > (1-2\theta)^{-1}$, weighted synergy is downcrossing from $\varrho'$ to $\varrho''$ (Step 2) and sorting is non-increasing in $\varrho$, by Proposition 2. \hfill $\square$
C Integral Preservation of Upcrossing Properties

C.1 Integral Preservation of Upcrossing Functions on Lattices

Given a real or integer lattice $\mathbb{Z} \subseteq \mathbb{R}^N$ and poset $(\mathcal{T}, \succeq)$, the function $\sigma : \mathbb{Z} \times \mathcal{T} \to \mathbb{R}$ is proportionately upcrossing if $\forall z, z' \in \mathbb{Z}$ and $t' \geq t$:

$$\sigma^-(z \wedge z', t)\sigma^+(z \vee z', t') \geq \sigma^-(z, t')\sigma^+(z', t) \quad (19)$$

**Theorem 3.** Let $\sigma(z, t)$ be proportionately upcrossing. Then $\Sigma(t) \equiv \int_{z} \sigma(z, t)d\lambda(z)$ is weakly upcrossing in $t$ and upcrossing in $t$ if $\sigma(z, t)$ is upcrossing in $t$.

This generalizes a key information economics result by [Karlin and Rubin (1956)]: If $\sigma_0(z)$ is upcrossing in $z \in \mathbb{R}$, and $\sigma_1 \geq 0$ is LSPM, then $\int \sigma_0(z)\sigma_1(z, t)d\lambda(z)$ is upcrossing. Our result subsumes theirs when $n = 1$ and $\sigma = \sigma_0\sigma_1$ is proportionally upcrossing.

**Proof:** [Karlin and Rinott (1980)] prove the following: If functions $\xi_1, \xi_2, \xi_3, \xi_4 \geq 0$ obey $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for $z \in \mathbb{Z} \subseteq \mathbb{R}^N$, then for all positive measures $\lambda$:

$$\int \xi_3(z)d\lambda(z) \int \xi_4(z)d\lambda(z) \geq \int \xi_1(z)d\lambda(z) \int \xi_2(z)d\lambda(z) \quad (20)$$

Now, if $t' \geq t$, then (19) reduces to $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for the functions:

$$\xi_1(z) \equiv \sigma^+(z, t), \quad \xi_2(z) \equiv \sigma^-(z, t'), \quad \xi_3(z) \equiv \sigma^+(z, t'), \quad \xi_4(z) \equiv \sigma^-(z, t)$$

Thus, by (20):

$$\int \sigma^+(z, t')d\lambda(z) \int \sigma^-(z, t)d\lambda(z) \geq \int \sigma^+(z, t)d\lambda(z) \int \sigma^-(z, t')d\lambda(z) \quad (21)$$

This precludes $\int \sigma^+(z, t)d\lambda(z) > \int \sigma^-(z, t)d\lambda(z)$ and $\int \sigma^+(z, t')d\lambda(z) < \int \sigma^-(z, t')d\lambda(z)$, simultaneously. And thus, $\Sigma(t) > 0$ implies $\Sigma(t') \geq 0$, proving weakly upcrossing.

We now argue $\Sigma$ upcrossing. First assume $\Sigma(t) > 0$. Then $\int \sigma^+(z, t)d\lambda(z) > \int \sigma^-(z, t)d\lambda(z)$. By (21), either $\int \sigma^+(z, t')d\lambda(z) > \int \sigma^-(z, t')d\lambda(z)$, or $\int \sigma^+(z, t')d\lambda(z) = \int \sigma^-(z, t')d\lambda(z)$.

---

24 We prove a stronger than needed result, as it applies to general lattices; we just need it for $\mathbb{R}^2$.

25 This result is related to Theorem 2 in [Quah and Strulovici (2012)]. They do not assume (19).

26 Proportionately upcrossing implies weakly upcrossing; namely, $\sigma(z, t) \geq 0$ implies $\sigma(z', t') \geq 0$ for all $(z', t') \succeq (z, t)$. Proof: Fix $t = t'$ and suppress $t$. If $z' \geq z$, inequality (19) is an identity. If $z \geq z'$, inequality (19) becomes $\sigma^-(z')\sigma^+(z) \geq \sigma^-(z)\sigma^+(z')$, which precludes $\sigma(z) < 0 < \sigma(z')$.

27 The proof for the integer lattice requires that $\lambda$ be a counting measure. Also true: if $\lambda$ does not place all mass on zeros of $\sigma$, then $\Sigma(t) \equiv \int_{z} \sigma(z, t)d\lambda(z)$ is upcrossing in $t$. 24
\( \int \sigma^{-}(z, t')d\lambda(z) = 0 \). But the latter is impossible, since \( \int \sigma^{+}(z, t')d\lambda(z) = 0 \) implies \( \int \sigma^{+}(z, t)d\lambda(z) = 0 \), as \( \sigma(z, t) \) is upcrossing in \( t \) — contradicting \( \Sigma(t) > 0 \). So \( \Sigma(t') > 0 \).

Next, posit \( \Sigma(t) = 0 \), then \( \int \sigma^{+}(z, t)d\lambda(z) = \int \sigma^{-}(z, t)d\lambda(z) \). By (21), either \( \int \sigma^{+}(z, t')d\lambda(z) \geq \int \sigma^{-}(z, t')d\lambda(z) \), and so \( \Sigma(t') \geq 0 \). Or, we have \( \int \sigma^{+}(z, t)d\lambda(z) = \int \sigma^{-}(z, t)d\lambda(z) = 0 \) — as \( \sigma(z, t) \) is upcrossing in \( t \), and so \( \sigma^{-}(z, t) \) is downcrossing. Thus, \( \int \sigma^{+}(z, t')d\lambda(z) \geq \int \sigma^{-}(z, t')d\lambda(z) \), or \( \Sigma(t') \geq 0 \). □

### C.2 Proportionately Upcrossing and Log-supermodularity

We now introduce a sufficient condition for (19) that emphasizes the link between log-complementarity and proportional upcrossing. Let \( \theta \in \mathbb{R} \), and call \( \sigma(z, \theta) \) smoothly signed log-supermodular (LSPM) if its derivatives obey the inequality \( \sigma_{ij}\sigma \geq \sigma_{i}\sigma_{j} \).

**Theorem 4.** If \( \sigma(z, \theta) \) is upcrossing and smoothly signed LSPM, then \( \sigma \) obeys (19).

**Step 1: Ratio Ordering.** Abbreviate \( w = (z, \theta) \in \mathbb{R}^{N+1} \). Assume \( \hat{w} \geq w \), sharing the \( i \) coordinate \( w_i = \hat{w}_i \), with \( \sigma(\hat{x}, \hat{w}_{-i}) < 0 < \sigma(\hat{w}) \) for some \( \hat{x} > w_i \). Then we prove:

\[
\sigma_i(x, w_{-i})\sigma(x, \hat{w}_{-i}) \geq \sigma_i(x, w_{-i})\sigma(x, w_{-i}) \quad \forall \ x \in [w_i, \hat{x}]
\]

(22)

Since \( \sigma \) is upcrossing, \( \sigma(x, w_{-i}) < 0 < \sigma(x, \hat{w}_{-i}) \) for all \( x \in [w_i, \hat{x}] \). If (22) fails, then for some \( x' \in [w_i, \hat{x}] \):

\[
\frac{\sigma_i(x', w_{-i})}{\sigma(x', w_{-i})} > \frac{\sigma_i(x', \hat{w}_{-i})}{\sigma(x', \hat{w}_{-i})}
\]

This contradicts smoothly LSPM, as \( (\sigma_i/\sigma_j)_{ij} \geq 0 \) for all \( \sigma \neq 0 \) and \( i \neq j \). So (22) holds.

Given \( \sigma(x, \hat{w}_{-i}) \neq 0 \), the ratio \( \sigma(x, w_{-i})/\sigma(x, \hat{w}_{-i}) \) is non-decreasing in \( x \) on \([w_i, \hat{x}]\), so that:

\[
\frac{\sigma(w)}{\sigma(\hat{w})} = \frac{\sigma(\hat{x}, w_{-i})}{\sigma(\hat{x}, \hat{w}_{-i})}
\]

(23)

**Step 2: \( \sigma \) obeys (19).** By assumption \( \theta' \geq \theta \) (now a real). So if \((z, \theta') \leq (z \wedge z', \theta)\), we have \( z \leq z' \) and \( \theta' = \theta \), in which case (19) is an identity. If not \((z, \theta') \leq (z \wedge z', \theta)\), then let \( i_1 < \ldots < i_K \) be the indices with \((z, \theta'_{i_k}) > (z \wedge z', \theta)_{i_k} \) for \( k = 1, \ldots, K \). Let’s change \( w^0 \equiv (z \wedge z', \theta) \) into \( w^K \equiv (z, \theta') \) in \( K \) steps, \( w^0, \ldots, w^K \), one coordinate at a time, and likewise \( \hat{w}^0 \equiv (z', \theta) \) into \( \hat{w}^K \equiv (z \vee z', \theta') \), changing coordinates in the same order. Notice that \( w^{k-1}_{i_k} = \hat{w}^{k-1}_{i_k} = (z', \theta)_{i_k} < (z, \theta)_{i_k} \) and \( \hat{w}^k \geq w^k \) for all \( k \).

Now, inequality (19) holds if its RHS vanishes. Assume instead the RHS of (19) is positive for some \( \theta' \geq \theta \), so that \( \sigma(z, \theta') < 0 < \sigma(z', \theta) \); and so, replacing \( \hat{w}^0 = (z', \theta) \) and \( w^K = (z, \theta') \), we get \( \sigma(w^K) < 0 < \sigma(\hat{w}^0) \). But then since the sequences \( \{w^k\} \) and \( \{\hat{w}^k\} \) are increasing and \( \sigma \) is upcrossing, we have \( \sigma(w^k) < 0 < \sigma(\hat{w}^{k-1}) \) for all \( k \).
Figure 10: Non-pure Matching Example. We depict the matching support for the synergy function $\alpha - \beta \min\{x_i, x_j\}$. In each graph synergy is positive (negative) on the shaded (unshaded) regions. In the left most graph $\alpha = 0.4$ and $\beta = 1.3$. In the middle graph $\alpha = 0.4$ and $\beta = 1$, while $\alpha = 0.6$ and $\beta = 1.3$ on the right.

Altogether, we may repeatedly apply inequality (23) to get:

\[
\frac{\sigma(z \land z', \theta)}{\sigma_z(\theta)} = \frac{\sigma(w^0)}{\sigma(\hat{w}^0)} \leq \frac{\sigma(w^k)}{\sigma(\hat{w}^k)} \leq \cdots \leq \frac{\sigma(w^K)}{\sigma(\hat{w}^K)} = \frac{\sigma(z, \theta')}{\sigma(z \lor z', \theta')}
\]

So given $\sigma(z \land z', \theta), \sigma(z, \theta') < 0 < \sigma(z', \theta), \sigma(z \lor z', \theta')$, inequality (19) follows from:

\[
\frac{\sigma^-(z \land z', \theta)}{\sigma^+(z', \theta)} \geq \frac{\sigma^-(z, \theta')}{\sigma^+(z \lor z', \theta')}
\]

D An Example with a Complex Matching Pattern

We now show via numerical example that our increasing sorting theory applies to examples with elaborate matching patterns. Since synergy is all that matters for matching patterns (by (4)), we can solve for the optimal matching patterns in Figure 10 given synergy function $\alpha - \beta \min\{x, y\}$. Notice that these finite type plots suggest that the optimal matching is not pure (one-to-one) with continuum types, but fortunately, none of our continuum model sorting results require purity. This synergy function is non-decreasing in $\alpha$ and non-increasing in $\beta$. It is also monotone in types: non-decreasing in types when $\beta \leq 0$ and non-increasing in types when $\beta \geq 0$. Altogether, sorting will be increasing in $\alpha$ and falling in $\beta$ by Corollary 1.
E Omitted Proofs

E.1 One Crossing Weighted Synergy via Linear Synergy

Claim 1. If synergy is linear in a parameter \( \theta \), say \( \phi_{12}(x, y|\theta) = A(x, y) + \theta B(x, y) \) or \( s_{ij}(\theta) = A_{ij} + \theta B_{ij} \), and \( A \) is globally positive (negative), then weighted synergy is strictly downcrossing (upcrossing) in \( \theta \), and so also is summed rectangular synergy.

Proof: Assume the case with \( A > 0 \) globally. Then for all \( \theta'' > \theta' \geq 0 \) and \( \lambda \geq 0 \):

\[
\int A(x, y) \lambda(x, y) + \theta' \int B(x, y) \lambda(x, y) \leq 0 \Rightarrow \int A(x, y) \lambda(x, y) + \theta'' \int B(x, y) \lambda(x, y) < 0
\]

as \( A > 0 \) and \( \theta > 0 \), together imply \( \int B(x, y) \lambda(x, y) < 0 \). Symmetric logic establishes the finite type case and weighted synergy strictly upcrossing in \( \theta \) when \( A < 0 \). \( \square \)

E.2 Proof of Proposition 2: Increasing Sorting for Finite Types

Lemma 2. An optimal matching is generically unique and pure for finite types.

Proof: The optimal matching is generically unique, by Koopmans and Beckmann (1957). A non-pure matching \( M \) is a mixture \( M = \sum_{\ell=1}^{L} \lambda_{\ell} M_{\ell} \) over \( L \leq n + 1 \) pure matchings \( M_1, \ldots, M_n \), with \( \lambda_{\ell} > 0 \) and \( \sum_{\ell} \lambda_{\ell} = 1 \).

(a) Consider the generic case with unique optimal pure matchings \( \mu_i \), described by men partners \( (\mu_1, \ldots, \mu_n) \) of women, or women partners \( (\omega_1, \ldots, \omega_n) \) of men.

(b) To emphasize the dependence on the number of types \( n \), write rectangular synergy as \( S^n(r|\theta) \), and the summed rectangular synergy as \( S^n(K|\theta) = \sum_{k=1}^{n} S^n(r_k|\theta) \) for any finite set of non-overlapping rectangles \( K = \{r_k\} \).

(c) We consider the summed rectangular synergy dyad \( (S^n(K|\theta'), S^n(K|\theta'')) \) for generic \( \theta'' \geq \theta' \). Let domain \( D_n \) be the space of summed rectangular synergy dyads \( (S^n(K|\theta'), S^n(K|\theta'')) \) that are each upcrossing in \( K \) on rectangles \( R \) and upcrossing in \( \theta \) on \( \{\theta', \theta''\} \) for any \( K \in R \). The domain \( \hat{D}_n \subseteq D_n \) further insists that they be upcrossing in \( \theta \) for finite sets of non-overlapping rectangles \( K \). Proposition 2 assumes that summed rectangular synergy dyads are in \( \hat{D}_n \) for all \( n \).

---

28This follows from Carathéodory’s Theorem. It says that non-empty convex compact subset \( X \subseteq \mathbb{R}^n \) are weighted averages of extreme points of \( X \). The extreme points here are the pure matchings.
(d) Removing couple \((i, j)\) from an \(n\)-type market induces rectangular synergy \(S^n_{ij} \) among the remaining \(n-1\) types, satisfying the formula:

\[
S^n_{ij}(r|\theta) \equiv S^n(r + \mathcal{I}_{ij}(r)|\theta) \quad \text{for} \quad \mathcal{I}_{ij}(r) = (1_{r_1 \geq i}, 1_{r_2 \geq j}, 1_{r_3 \geq i}, 1_{r_4 \geq j})
\]  

(24)

where \(\mathcal{I}_{ij}(r)\) increments by one the index of the women \(i' \geq i\) and men \(j' \geq j\), where the type indices refer to the original model whenever removing types henceforth.

(e) To avoid ambiguity when changing the number \(n\) of types, we denote by \((i_n, j_n)\) the \(i\)th highest woman and the \(j\)th highest man. Now, consider the sequence models with \(\kappa = n + k, n + k - 1, \ldots, n\) indices induced by removing couple \((i'_n, j'_n)\) at \(\theta'\) and \((i''_n, j''_n)\) at \(\theta''\) from the \(\kappa\) type model. We say the sequence of couples has higher partners at \(\theta'\) than \(\theta''\) if \((i'_n, j'_n) \geq (i''_n, j''_n)\) and \(i'_n = i''_n\) or \(j'_n = j''_n\).

(f) Domain \(\hat{D}^*_n\) is the set of summed rectangular synergy dyads \((S^n(K|\theta'), S^n(K|\theta''))\) induced by sequentially removing \(k\) optimally matched couples with higher partners at \(\theta'\) than \(\theta''\) from dyads \((S^{n+k}(K|\theta'), S^{n+k}(K|\theta''))\) in \(\hat{D}^*_{n+k}\), for some \(k \in \{0, 1, \ldots\}\).

A. Key Properties of our Domains and Pure Matchings.

**Fact 1.** Given a summed rectangular synergy dyad in \(\mathcal{D}^*_{n+1}\), removing couple \((i', j')\) at \(\theta'\) and \((i'', j'')\) at \(\theta''\) induces a summed rectangular synergy dyad in \(\mathcal{D}^*_n\) if \((i', j') \geq (i'', j'')\) and \(i' = i''\) or \(j' = j''\).

**Fact 2.** Given a summed rectangular synergy dyad in \(\mathcal{D}^*_{n+1}\), removing couple \((i', j')\) at \(\theta'\) and \((i'', j'')\) at \(\theta''\) induces a summed rectangular synergy dyad in \(\mathcal{D}^*_n\) if \((i' = i''\) and \(j' \geq j'')\) or \((j' = j''\) and \(i' \geq i'')\).

**Proof:** We prove this for \(i' = i''\) and \(j' \geq j''\). For any \(\theta\), rectangular synergy \(S^n_{ij}(r|\theta)\) is upcrossing in \(r\), needing fewer inequalities. To see that summed rectangular synergy is upcrossing in \(\theta\) on rectangular sets in \(\mathcal{Z}^2_{n-1}\), assume \(S^n_{ij}(r|\theta') \geq (>0)\) for some \(r\). Then

\[
S^{n+1}(r + \mathcal{I}_{ij}(r)|\theta') \geq (>0) \quad \Rightarrow \quad S^{n+1}(r + \mathcal{I}_{ij'}(r)|\theta') \geq (>0)
\]

\[
S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta'') \geq (>0) \quad \Rightarrow \quad S^{n+1}_{ij''}(r|\theta'') \geq (>0)
\]

respectively, as \((i) S^{n+1}(r|\theta)\) is upcrossing for rectangles \(r\), non-increasing \(\mathcal{I}_{ij}\) in \(j\), and \(j'' \leq j'\), and \((ii) S^{n+1}(r|\theta)\) is upcrossing in \(\theta\) for rectangles \(r\), and \((iii)\) by (24). ☐

**Fact 3.** The domains are nested: \(\hat{D}_n \subseteq \mathcal{D}^*_n \subseteq \mathcal{D}_n\).

**Proof:** Trivially, \(\hat{D}_n \subseteq \mathcal{D}^*_n\), since we may set \(k = 0\) in the definition of \(\mathcal{D}^*_n\).
To get $\mathcal{D}_n^* \subseteq \mathcal{D}_n$, pick $(S^n(K|\theta'), S^n(K|\theta'')) \in \mathcal{D}_n^*$. This dyad is induced by removing $k$ optimally matched couples with higher partners at $\theta'$ than $\theta''$ from a dyad $(S^{n+k}(K|\theta'), S^{n+k}(K|\theta'')) \in \mathcal{D}_{n+k} \subseteq \mathcal{D}_{n+k}$, where $k \geq 0$. For $\ell = 1, \ldots, k$, induce dyads $(S^{n+k-\ell}(K|\theta'), S^{n+k-\ell}(K|\theta''))$, sequentially removing optimally matched couples. So $(S^{n+k-\ell}(K|\theta'), S^{n+k-\ell}(K|\theta'')) \in \mathcal{D}_{n+k-\ell}$ for $\ell = 1, \ldots, k$, as removed couples are ordered, as Fact 2 needs. So $(S^n(K|\theta'), S^n(K|\theta'')) \in \mathcal{D}_n$. □

**Fact 4.** If $M \neq \check{M}$ are pure $n$-type matchings, $\check{\mu}_i > \mu_i$ at some $i$ and $\check{\omega}_j > \omega_j$ at some $j$.

**Proof:** Since $M \neq \check{M}$, there is a highest type man $j$ matched with woman $\check{\omega}_j > \omega_j$. Logically then, woman $i = \check{\omega}_j$ is matched to a lower man under $M$, i.e. $j = \check{\mu}_i > \mu_i$. □

Adding a couple $(i_0, j_0)$ to a matching $\mu$ creates a new matching $\check{\mu}$ with indices of women $i \geq i_0$ and men $j \geq j_0$ renamed $i + 1$ and $j + 1$, respectively. Equivalently, this means inserting a row $i$ and column $j$ into the matching matrix $m$ — with all 0’s except 1 at position $(i, j)$ — and shifting later rows and columns up one.

**Fact 5.** Adding respective couples $(1, \check{m}) \leq (1, m)$, or $(\check{w}, 1) \leq (w, 1)$, to the $n$-type matchings $\check{\mu} \succeq_{PQD} \mu$ preserves the PQD order for the resulting $n + 1$ type matchings.

**Proof:** We just consider adding couples $(1, \check{m}) \leq (1, m)$, as the analysis for $(\check{w}, 1) \leq (w, 1)$ is similar. For pure matchings $\mu$, let $C(\check{m}, i_0, j_0)$ count matches by women $i \leq i_0$ with men $j \leq j_0$, and so call $C(\check{m}, 0, j) = C(\mu, 0, j) = 0$. So $\check{\mu} \succeq_{PQD} \mu$ iff $C(\check{\mu}) \geq C(\mu)$.

By adding a couple $(1, m)$, the new count is:

$$C(\check{\mu})(i, j) \equiv C(\mu)(i - 1, j - 1) I_{j \geq m} + I_{j \geq m} \text{ for all } i, j \in \{1, 2, \ldots, n + 1\}$$

To prove the step, we must show that if $\check{\mu} \succeq_{PQD} \mu$, then $C(\check{\mu}) \geq C(\mu)$ for all $\check{m} \leq m$.

By assumption $\check{\mu} \succeq_{PQD} \mu$ and thus, $C(\check{\mu}) \geq C(\mu)$. So since $\check{m} \leq m$:

$$C(\check{\mu})(i, j) - C(\mu)(i, j) = \begin{cases} C(\check{\mu})(i - 1, j) - C(\mu)(i - 1, j) & \geq 0 \text{ for } j < \check{m} \\ C(\check{\mu})(i - 1, j - 1) + 1 - C(\mu)(i - 1, j) & \geq 0 \text{ for } \check{m} \leq j < m \\ C(\check{\mu})(i - 1, j - 1) - C(\mu)(i - 1, j - 1) & \geq 0 \text{ for } j \geq m \end{cases}$$

To understand the middle line, note that this match count can be written as

$$C(\check{\mu})(i - 1, j - 1) - C(\mu)(i - 1, j - 1) = [C(\mu)(i - 1, j) - C(\mu)(i - 1, j - 1)]$$

As $C(\mu)(i - 1, j) - C(\mu)(i - 1, j - 1) \leq 1$, this is at least $C(\check{\mu})(i - 1, j - 1) - C(\mu)(i - 1, j - 1) \geq 0$. □

**B. THE INDUCTION PROOF.** We use induction on the number of types. Let $M'_n$ and $M''_n$ be uniquely optimal $n$ type matchings at $\theta'$ and $\theta''$. Proposition 2 assumes summed rectangular synergy dyads in $\mathcal{D}_n$. We prove the result on the larger domain $\mathcal{D}_n^*$:
Premise $P_n$: Summed rectangular synergy dyad is in $D_n^* \Rightarrow M''_n \succeq_{PQD} M'_n$.

Step 1. Base Case $P_2$: Summed rectangular synergy dyad is in $D_2^* \Rightarrow M'_2 \succeq_{PQD} M'_2$.

Proof: If not, then NAM is uniquely optimal at $\theta''$ and PAM at $\theta'$. Since $D_2^* \subseteq D_2$ by Fact 3, rectangular synergy is upcrossing in $\theta$. This precludes negative rectangular synergy at $\theta''$ (NAM) and positive rectangular synergy at $\theta'$ (PAM).

- A pair refers to two couples, such as $(i_1, j_1)$ and $(i_2, j_2)$.
- A pair is a PAM pair if $(i_1, j_1) < (i_2, j_2)$, and a NAM pair if $i_1 < i_2$ and $j_1 > j_2$.

Step 2. If the summed rectangular synergy dyad is in $D_{n+1}^*$, then neither $M''_{n+1}$ nor $M''_{n+1}$ includes a matched NAM pair that exceeds a matched PAM pair.

Proof: By Fact 3, $D_{n+1}^* \subseteq D_{n+1}$. So $S^{n+1}(r|\theta)$ is upcrossing in rectangles $r$ for $\theta'$ and $\theta''$. Also, PAM (NAM) is optimal for a pair iff $S^{n+1}(r|\theta) \geq (\leq) 0$ on rectangle $r$. As the optimal matching is unique, $S^{n+1}(r|\theta) \neq 0$ for all optimally matched pairs.

Steps 3-8 impose premises $P_2, \ldots, P_n$, but not $P_{n+1}$, and arrive at a contradiction:

(†††): In a model with summed rectangular synergy dyads in $D_{n+1}^*$, the uniquely optimal matchings at $\theta'' > \theta'$ are not ranked $\mu'' \succeq_{PQD} \mu' \ (\omega'' \succeq_{PQD} \omega')$.

Step 3. At states $\theta'$ and $\theta''$, the matchings obey $\mu''_1 = \mu'_1 + 1 \geq 2$ and $\omega''_1 = \omega'_1 + 1 \geq 2$.

We establish the first relationship. Symmetric steps would prove the second.

Proof of $\mu''_1 > \mu'_1$: If not, then $\mu''_1 \leq \mu'_1$. In this case, remove couple $(1, \mu'_1)$ at $\theta'$, and couple $(1, \mu''_1)$ at $\theta''$. The remaining matching is PQD higher at $\theta''$, by Induction Premise $P_n$ and Fact 1. By Fact 3, if we add back the optimally matched pairs $(1, \mu'_1)$ and $(1, \mu''_1)$, then the PQD ranking still holds with $n + 1$ types, given $\mu''_1 \leq \mu'_1$, namely $\mu'' \succeq_{PQD} \mu'$. This contradiction to (†††) proves that $\mu''_1 > \mu'_1$. □

Proof of $\mu''_1 < \mu'_1 + 2$: If not, then $\mu''_1 \geq \mu'_1 + 2$. By Fact 4, choose a woman $i > 1$ with $\mu''_i < \mu'_i$. Remove couples $(i, \mu'_i)$ at $\theta'$, and $(i, \mu''_i)$ at $\theta''$. Since $\mu''_i < \mu'_i$, the resulting matching is PQD higher at $\theta''$ than $\theta'$, by Fact 1 and Premise $P_n$. In the resulting model, woman 1 is not matched to a higher man at $\theta''$ than $\theta'$. This is impossible if $\mu''_1 \geq \mu'_1 + 2$, as $\mu''_1 - \mu'_1$ falls by at most 1 when removing man $\mu_i$ at $\theta'$ and $\mu''_1$ at $\theta''$. □

Step 4. The couple $(\omega''_1, \mu''_1)$ is matched at $\theta'$, namely, $\mu'_{\omega''_1} = \mu''_1$ and $\omega'_{\mu''_1} = \omega''_1$. 

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The middle panel of Figure 11 depicts the takeout of Steps 3–4. We iteratively use this matching pattern to show how (††) greatly restricts the matching at \( \theta' \) and \( \theta'' \).

**Step 5.** \( \mu_1' \geq \mu_i' = \mu''_i - 1 \) for \( i = 1, \ldots, \omega_1' \) and \( \omega_i' \geq \omega_j'' = \omega_j'' - 1 \) for \( j = 1, \ldots, \mu_1' \).

**Proof:** We proved this for \( i = 1 \) and \( j = 1 \), and now prove the claimed ordering \( \mu_1' \geq \mu_i' = \mu''_i - 1 \) for \( i = 2, \ldots, \omega_1' \). By symmetry, \( \omega_1' \geq \omega_j'' = \omega_j'' - 1 \) for \( j = 2, \ldots, \omega_1' \).

**Part (a):** \( \mu_i' < \mu_1' \) for \( i = 2, \ldots, \omega_1' \). If not, then \( \mu_i' \geq \mu_1' \) for some \( 2 \leq i \leq \omega_1' \). And since \( \mu_i' = \mu_1' \) entails overmatching, we have \( \mu_i' > \mu_i' \) for \( i = 2, \ldots, \omega_1' \). Thus, \( \mu_i' \) involves a PAM pair \((1, \mu_1') < (i, \mu_i')\). We claim that \((i, \mu_i') \) and \((\omega_i'', \mu_i'')\) constitutes a higher NAM pair, violating the upcrossing of \( S(r|\theta) \) in \( r \), by Step 2. Indeed, \( i \leq \omega_i' < \omega_i'' \) (by the premise above, and Step 3 respectively). Also, \( \mu_i' > \mu_1' \), since we have assumed \( \mu_i' > \mu_1' \), and deduced \( \mu_i' = \mu_i' - 1 \) in Step 3 and, in Step 4 that \( \mu_i'' \) is matched to \( \omega_i'' \) at \( \theta' \), and we just showed \( \omega_i' > i \). (See the right panel of Figure 11.)

**Part (b):** \( \mu_i' < \mu_i'' \) for \( i = 2, \ldots, \omega_1' \). If not, then \( \mu_i' \geq \mu_i'' \) for some \( 2 \leq i \leq \omega_1' \). Since \( \mu_i' \geq \mu_i'' \), if we remove couple \((i, \mu_i')\) at \( \theta' \) and couple \((i, \mu_i'')\) at \( \theta'' \), then the resulting
matching is PQD higher at $\theta''$, by Fact 1 and $\mathcal{P}_n$. In the resulting matching, woman 1’s partner is thus not higher at $\theta''$ than $\theta'$. But $\mu''_i = \mu'_i + 1$ by Step 3 and $\mu'_i > \mu''_i$ by part (a) and the premise of (b). Both removed men $\mu'_i$ and $\mu''_i$ are then strictly below $\mu'_1$. So, woman 1’s partner is still 1 higher at $\theta''$ than $\theta'$. Contradiction. 

**Part (c):** $\mu'_i \geq \mu''_i - 1$ for $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i < \mu''_i - 1$ for some $2 \leq i^* \leq \omega'_1$. Remove couple $(\omega'_1, \mu''_i)$ at $\theta'$ (matched, by Step 4), and the couple $(\omega'_1, 1)$ at $\theta''$. By Fact 1 and Assumption $\mathcal{P}_n$, the resulting matching is PQD higher at $\theta''$.

But since $\omega''_1 > \omega'_1$ by Step 3 all women $i = 1, \ldots, \omega'_1$ remain. Each has a weakly lower partner at $\theta'$ than $\theta''$, since we started with $\mu'_i < \mu''_i$ for $i = 1, \ldots, \omega'_1$ by Step 3 for $i = 1$, and part (b) for $i > 1$. Also, woman $i^*$ has a strictly lower partner, as $\mu'_i < \mu''_i - 1$. The resulting matching cannot be PQD higher at $\theta''$. Contradiction. 

**Step 6.** The matching $\mu''$ is NAM among men and women at most $\omega'_1 = \mu''_1 \geq 2$.

**Proof of $\omega''_1 = \mu''_1$.** By Steps 3 and 5 we get $\mu''_i = \mu'_i + 1 \geq \mu''_i$ for $i = 1, \ldots, \omega'_1 = \omega''_1 - 1$ and $\mu''_i \geq 2 > 1 = \mu''_1$. So in matching $\mu''$, women $i \leq \omega''_1$ match with men $j \leq \mu''_i$. Hence, $\mu''_1 = \omega''_1$. Ditto, by Steps 3 and 5 $\omega''_j = \omega''_j$ for $j = 1, \ldots, \mu''_1$, and in matching $\omega''$, men $j \leq \mu''_1$ match with women $i \leq \omega''_1$. Hence, $\mu''_1 \leq \omega''_1$. Thus, $\mu''_1 = \omega''_1 \geq 2$. 

**Proof of $\mu'_i = \mu''_i - i + 1$ for $i = 1, \ldots, \omega''_1$.** This is an identity at $i = 1$ and true at $i = \omega''_1$ by $\omega''_1 = \mu''_1$ (just proven) and $\mu''_1 = 1$. So, henceforth assume $i \in \{2, \ldots, \omega''_1 - 1\}$. We claim that for all such $i$, $\mu'_i \geq \mu''_i$. Indeed, by Steps 3 and 5 $\mu''_i = \mu'_i + 1 \geq \mu''_i$; and since we do not over match, $\mu''_i \neq \mu''_i$ for $i \neq 1$. Since $\mu'_1 \geq \mu''_1$, Step 5 yields equality $\omega''_j = \omega''_j - 1$ at $j = \mu''_1$, and so $\omega''_j = \omega''_j - 1 - i = i - 1$. But then since $\omega''_1 = i - 1$ and each woman has a unique partner, $\omega''_j = i - 1$ implies $\mu'_i = \mu''_1 - 1$. As $\mu''_1 = \mu''_1 - 1 - 1$ by Step 5 and $i \leq \omega''_1 - 1 = \omega''_1$ (by our premise and Step 3), we have $\mu''_1 = \mu''_1 - 1 - 1$. 

An $n$-type pure matching $\mu$ is NAM$^*$ if $\mu_n = n$ and $\mu_i = n - i$ for $i = 1, \ldots, n - 1$, i.e. NAM among types $1, \ldots, n - 1$, so that NAM$^* = $ NAM3 when $n = 3$.

**Step 7.** The matching $\mu'$ is NAM$^*$ among men and women at most $\omega'_1 = \mu''_1 \geq 2$.

**Proof:** Steps 3, 5 and 6 imply $\mu'_i = \mu''_i - 1 = \mu''_i - i$ for $i = 1, \ldots, \omega'_1 = \omega''_1 - 1$. Couple $(\omega''_1, \mu''_1)$ matches under $\mu'$, by Step 4 So $\mu'$ is NAM$^*$ for types $1, \ldots, \mu''_1 = \omega''_1$. 

By Steps 6, 7 $\mu''$ is NAM and $\mu'$ is NAM$^*$ on types $1, \ldots, \omega''_1 = \mu''_1 = k \geq 2$. Since NAM$^* \succ PQD$ NAM, if $k < n + 1$ then Premise $\mathcal{P}_k$ fails. Step 8 finishes the proof by showing that NAM at $\theta''$ and NAM$^*$ at $\theta'$ is also impossible for $k = n + 1$ types.

NAM for men $\{i_1, \ldots, i_\ell\}$ and women $\{j_1, \ldots, i_\ell\}$ is $\{(i_1, j_1), (i_2, j_{\ell-1}), \ldots, (i_\ell, j_1)\}$. Rematching to NAM$^*$, $\{(i_1, j_{\ell-1}), (i_2, j_{\ell-2}), \ldots, (i_\ell, j_1)\}$ changes payoffs by

$$\sum_{u=1}^{\ell-1} (f_{u,j_{\ell-1}} - f_{u,j_{\ell+1-u}}) + f_{i_\ell,j_{\ell}} - f_{i_1,1} = \sum_{u=1}^{\ell-1} [(f_{i_\ell,j_{\ell+1-u}} - f_{i_\ell,j_{\ell-1}}) - (f_{i_\ell,j_{\ell+1-u}} - f_{i_\ell,j_{\ell-1}})]$$
Figure 12: **Step 8 of Induction Proof.** We rule out NAM for $\theta''$ (dots) and NAM* for $\theta'$ (stars) with $n + 1$ types (left). Middle: These matches with $n + k > n + 1$ types, after adding couples weakly higher at $\theta'$ than $\theta''$. Let $K^G, K^O, K^P, K^Y$ be the grey, orange, pink, and yellow regions. By (25), the NAM* minus NAM difference is $S^{n+k}(K^G \cup K^O|\theta') > 0$, as NAM* is optimal for $\theta'$. But $S^{n+k}(K^O|\theta') < 0$, as $K^O$ is the union of rectangles, each below a NAM pair for $\theta''$. So $S^{n+k}(K^G|\theta') > 0$.

By (25), the NAM* minus NAM difference is $S^{n+k}(K^G \cup K^P \cup K^Y|\theta'') < 0$, negative by NAM optimal for $\theta''$. Finally, $S^{n+k}(K^Y|\theta'), S^{n+k}(K^P|\theta') > 0$, as the yellow and pink rectangles are each above a PAM pair for $\theta'$. So $S^{n+k}(K^G|\theta'') < 0$. But since $S^{n+k}(K^G|\theta') > 0$, this contradicts upcrossing summed rectangular synergy in $\theta$. The right panel illustrates Step 8(c).

So the payoff of NAM* less that of NAM on any subset of $\ell$ types equals (suppressing the superscript on $S$)

$$\sum_{u=1}^{\ell-1} S(i_u, j_{\ell-u}, i_\ell, j_{\ell+1-u})$$

(25)

**Step 8.** NAM at $\theta''$ and NAM* at $\theta'$ is impossible for summed rectangular synergy dyads in $D^*_{n+1}$.

**Part (a): Contradiction Assumption.** For $n + 1$ types, posit NAM* and NAM uniquely optimal at $\theta'$ and $\theta''$ (Figure 12, panel 1). Summed rectangular synergy dyads in $D^*_{n+1}$ are induced by removing $k - 1 \geq 0$ optimally matched couples with higher partners at $\theta'$ than $\theta''$ (building block (f)) from a summed rectangular synergy dyad $S^{n+k}(K|\theta'), S^{n+k}(K|\theta'') \in \hat{D}_{n+k}$. The $\theta'$ matching here is NAM* for men $i' = (i'_1, \ldots, i'_{n+1})$ and women $j' = (j'_1, \ldots, j'_{n+1})$, while the $\theta''$ matching with these $n + k$ types is NAM for men $i'' = (i''_1, \ldots, i''_{n+1})$ and women $j'' = (j''_1, \ldots, j''_{n+1})$, with $(i', j') \leq (i'', j'')$ (Figure 12, panel 2).

**Part (b): couple sets $U', U''$ with $S^{n+k}(U''|\theta'') < 0 < S^{n+k}(U'|\theta')$.** For rectangles $r'_u \equiv (i'_u, j'_{n+1+u-1}, i'_{n+1}, j'_{n+2+u})$ and $r''_u \equiv (i''_u, j''_{n+1+u-1}, i''_{n+1}, j''_{n+2+u})$ define “upper sets”:

- $U' \equiv \bigcup_{u=1}^n r'_u$, the union of the grey and orange rectangles in panel 2 of Figure 12
- $U'' \equiv \bigcup_{u=1}^n r''_u$, the union of the grey, yellow, and pink regions

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As NAM* is uniquely optimal for the subsets of men $i'$ and women $j'$ at $\theta'$, it payoff-dominates NAM. Given linearity of summed rectangular synergy at $\ell = n + 1$ in [25],
\[
S^{n+k}(U'|\theta') = \sum_{u=1}^{n+1} S^{n+k}(r'_u|\theta') = \sum_{u=1}^{n+1} S^{n+k}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u}|\theta') > 0
\]
Likewise, NAM uniquely optimal for subsets $i''$ and $j''$ at $\theta''$ implies $S^{n+k}(U''|\theta'') < 0$.

**Part (c):** $S^{n+k}(K^G|\theta'') > 0$ for $K^G \equiv U'' \cap U''$. First, $U' = \cup_{u=1}^{n}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+1})$, i.e., a union of rectangles with fixed northeast (Figure 12 panel 3). Likewise, we have $U'' \equiv \cup_{u=1}^{n} i''_u$. Since $(i', j') \leq (i'', j'')$ (part (a)), if $(i, j) \in U' \setminus U'' = U' \setminus K^G$ (orange shade, Figure 12 panel 2), then $(i'_u, j'_{n+1-u}) \leq (i, j)$, and $i \leq i''_u$ or $j \leq j''_{n+1-u}$, with at least one strict, at some $u$. So couple $(i, j)$ is below the meet of the $\theta''$ matched NAM pair $(i''_u, j''_{n+2-u})$ and $(i''_{n+1}, j''_{n+1})$. As rectangular synergy is upcrossing in types, $s_{ij}(\theta'') < 0$. Then $s_{ij}(\theta'') < 0$, as synergy is upcrossing in $\theta$. Then $S^{n+k}(U' \setminus K^G|\theta'') < 0$, as this holds for all $(i, j) \in U' \setminus K^G$. As summed rectangular synergy is additive and $S^{n+k}(U'|\theta') > 0$ (part (b)), $S^{n+k}(K^G|\theta') = S^{n+k}(U'|\theta') - S^{n+k}(U' \setminus K^G|\theta') > 0$.

**Part (d):** $S^{n+k}(K^G|\theta'') < 0$. Since $(i', j') \leq (i'', j'')$ (part (a)), define rectangles $K^Y \equiv (i'_1, j'_{n+1}, i'_{n+1}, j'_{n+1})$ and $K^P \equiv (i'_1, j'_{1}, i'_{n+1}, j'_{n+1})$ (resp., yellow and pink regions, Figure 12 panel 2). Then $U'' \setminus K^G = K^Y \cup K^P$. As summed rectangular synergy is linear:
\[
S^{n+k}(K^G|\theta') = S^{n+k}(U''|\theta'') - S^{n+k}(K^Y|\theta') - S^{n+k}(K^P|\theta')
\]
Rectangle $K^Y$ is above the rectangle defined by the $\theta'$ PAM pair $(i'_1, j'_n)$ and $(i'_{n+1}, j'_{n+1})$. So $S^{n+k}(K^Y|\theta'') > 0$, as summed rectangular synergy is upcrossing on rectangles and $\theta$. Likewise, $K^P$ is above the rectangle defined by the $\theta'$ PAM pair $(i'_1, j'_1)$ and $(i'_{n+1}, j'_{n+1})$. So $S^{n+k}(K^P|\theta'') > 0$. Then $S^{n+k}(K^G|\theta'') < 0$, as $S^{n+k}(U''|\theta'') < 0$ (part (b)) and (26).

Since $S^{n+k}(K^G|\theta') > 0$ (part (c)), we cannot have $(S^{n+k}(K|\theta'), S^{n+k}(K|\theta'')) \in \mathcal{D}_{n+k}$; and thus, by part (a) we have contradicted dyads $(S^{n+1}(K|\theta'), S^{n+1}(K|\theta'')) \in \mathcal{D}_{n+1}$, and thus conclude that NAM at $\theta''$ and NAM* at $\theta'$ is impossible.

**E.3 Proof of Proposition 2** for a Continuum of Types

**Step 1.** Uniquely optimal finite type matchings exist for a payoff perturbation with summed rectangular synergy upcrossing in $\theta$.

**Proof:** Let $X^n = \{x^n_1, \ldots, x^n_n\}$ and $Y^n = \{y^n_1, \ldots, y^n_n\}$ be equal quantile increments, with $G(x^n_i) = H(y^n_i) = 1/n$ and $G(x^n_{i-1}) = G(x^n_i)+1/n$ and $H(y^n_{j-1}) = H(y^n_j)+1/n$. Let $G^n$ and $H^n$

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29This last step assumes upcrossing synergy sums on connected join semi-lattices (sets that contain the join of any pair of elements). All of our results only require this weaker time series assumption.
$H^n$ be cdfs on $[0, 1]$, stepping by $1/n$ at $X^n$ and $Y^n$ (resp.). Put $f^n_{ij}(\theta) = \phi(x^n_i, y^n_j | \theta)$. The set $M^n(\theta)$ of pure optimal matchings is non-empty, by Lemma \[2\]

Since unique optimal matchings are pure, we restrict to pure matchings. These are uniquely defined by the male partner vector $\mu = (\mu_1, \ldots, \mu_n)$. Call the pure matching $\bar{M}$ lexicographically higher than $M$ iff its male partner vector $\bar{\mu}$ lexicographically dominates $\mu$. Let $\bar{M}^n(\theta)$ (resp. $\bar{\mu}^n(\theta)$) be the optimal pure matching highest in the lexicographic order, and $\bar{M}^n(\theta)$ (resp. $\bar{\mu}^n(\theta)$) the lowest. Easily, each is well-defined.

Fix $\theta'' > \theta'$. Let $i(j) = \bar{\mu}^n(\theta'') - 1$ and pick $\varepsilon > 0$. Perturb synergy down at $\theta'$:

$$s_{ij}^{e\varepsilon}(\theta') \equiv s_{ij}(\theta') - \varepsilon^j 1_{(i,j)=i(j),j}$$

(27)

We prove that $\bar{M}^n(\theta')$ is uniquely optimal at $\theta'$ for any production function with $\varepsilon$-perturbed synergy [27], for all small $\varepsilon > 0$. Similar logic will prove that $\bar{M}^n(\theta'')$ is uniquely optimal at $\theta''$ with $s_{ij}^{e\varepsilon}(\theta'') \equiv s_{ij}(\theta'') + \varepsilon^j 1_{(i,j)=i(\theta''),j}$ for all small $\varepsilon > 0$.

Pick a matching $M$ that is not optimal at $\varepsilon = 0$. Since $\bar{M}^n(\theta')$ is optimal at $\varepsilon = 0$, $\bar{M}^n(\theta')$ yields a higher payoff than $M$ for all small $\varepsilon > 0$.

As $\bar{\mu}^n(\theta')$ is the lexicographically highest optimal matching at $\theta'$, another optimal $\mu$ obeys $(\bar{\mu}_1^n(\theta'), \ldots, \bar{\mu}_{n-1}^n(\theta')) = (\mu_1, \ldots, \mu_{n-1})$, and first diverges at $\bar{\mu}_n^n(\theta') > \mu_n$, for some woman $\ell < n$. Using $M_{ij} = \sum_{k=1}^n 1_{\mu_k \leq j}$, equation [4], and [27], the payoff $\bar{M}^n(\theta')$ exceeds that of $M \in M^n(\theta')$ by $\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}^{e\varepsilon}(\theta') \left[ \bar{M}^n_{ij}(\theta') - M_{ij} \right]$. This expands to:

$$\sum_{j=1}^{n-1} \varepsilon^j \left[ M_{i(j),j} - M^n_{i(j),j}(\theta') \right] = \varepsilon^\ell + \sum_{j=\ell+1}^{n-1} \varepsilon^j \sum_{k=\ell+1}^{n-1} 1_{\mu_k \leq i(j)} - 1_{\bar{\mu}_k^n \leq i(j)}$$

Altogether, $\lim_{\varepsilon \to 0} \varepsilon^{-\ell} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}^{e\varepsilon}(\theta') \left[ \bar{M}^n_{ij}(\theta') - M_{ij} \right] = 1 > 0$.

**Step 2.** If $\theta'' > \theta'$, then $\bar{M}^n(\theta'') \geq_{PQD} \bar{M}^n(\theta')$ for all $n$.

**Proof:** Since $S^{e\varepsilon}(r|\theta)$ is continuous in $\varepsilon$, there exists $\bar{\varepsilon}_n > 0$ such that, for all $r = (i_1, j_1, i_2, j_2)$ and $0 \leq \varepsilon < \bar{\varepsilon}_n$, if $S^{0\varepsilon}(r|\theta) \leq 0$ then $S^{\varepsilon\varepsilon}(r|\theta) \leq 0$. By the contrapositives:

$$S^{\varepsilon\varepsilon}(r|\theta) \geq 0 \Rightarrow S^{0\varepsilon}(r|\theta) \geq 0 \quad \text{and} \quad S^{\varepsilon\varepsilon}(r|\theta) \leq 0 \Rightarrow S^{0\varepsilon}(r|\theta) \leq 0. \quad (28)$$

We claim that $S^{\varepsilon\varepsilon}(r|\theta)$ is strictly upcrossing in $r$ for all $0 < \varepsilon < \bar{\varepsilon}_n$. For if not, then $S^{\varepsilon\varepsilon}(r''|\theta) \leq 0 \leq S^{\varepsilon\varepsilon}(r'|\theta)$ for some $r'' >_{NE} r'$. But then $S^{0\varepsilon}(r''|\theta) \leq 0 \leq S^{0\varepsilon}(r'|\theta)$ by (28), contradicting $S^{0\varepsilon}(r'|\theta)$ strictly upcrossing in $r$, as follows from Step 1.

Continuum summed rectangular synergy is upcrossing in $\theta$ by assumption; and thus, finite summed rectangular synergy $\sum_{k=1}^n S^{0\varepsilon}(r_k|\theta)$ for all finite approximations. Then, perturbed summed rectangular synergy $\sum_{k=1}^n S^{\varepsilon\varepsilon}(r_k|\theta)$ is upcrossing in $\theta$, since synergy $s_{ij}^{\varepsilon\varepsilon}(\theta')$ is non-increasing in $\varepsilon$ and $s_{ij}^{e\varepsilon}(\theta'')$ is non-decreasing in $\varepsilon$ by construction (27).
So for $\varepsilon \in (0, \varepsilon_n)$, rectangular synergy $S^{n\varepsilon}(r|\theta)$ is strictly upcrossovering in $r$ and summed rectangular synergy $\sum_{k=1}^{\infty} S^{n\varepsilon}(r_k|\theta)$ upcrossovering in $\theta$, for couple sets $K \subseteq \mathbb{Z}_n^2$. Given $\bar{M}^n(\theta'), M^n(\theta'')$ uniquely optimal, $M^n(\theta'') \geq_{P,Q,D} \bar{M}^n(\theta') \forall n$, by Proposition 2. \qed

**Step 3.** There exists a subsequence of groupings $\{M^n_k(\theta)\}$ that converges to an optimal matching in the continuum model.

**Proof:** Define a step function matching in the continuum model. There exists a subsequence of matchings $\{M^n_k(\theta)\}$ where $\epsilon_n = \hat{\epsilon}_n/n$. By construction, $\{G^n\}$ and $\{H^n\}$ weakly converge to $G$ and $H$ as $n \to \infty$, while $\phi^n$ uniformly converges to $\phi$. By Theorem 5.20 in Villani (2008), the associated optimal matching cdfs have a convergent subsequence $\{M^n_k(\theta)\}$ with limit point $M^\infty(\theta)$ optimal in the continuum model. \qed

**Step 4.** $M^\infty(\theta'') \geq_{P,Q,D} M^\infty(\theta')$ for all $\theta'' \geq \theta'$

**Proof:** Fix $\theta'' \geq \theta'$, and let $\{n_k\}$ be a subsequence along which the sequence of finite type matchings $\{M^{n_k}(\theta')\}$ converges to $M^\infty(\theta')$, as defined in Step 3. Now, since cdfs $\{G^n\}$ and $\{H^n\}$ weakly converge to $G$ and $H$, and $\phi^{n_k}(x,y|\theta'')$ converges uniformly to $\phi(x,y|\theta'')$, there exists a subsequence $\{n_{k_l}\}$ of $\{n_k\}$, along which the sequence of finite type matchings $\{M^{n_{k_l}}(\theta'')\}$ converges to $M^\infty(\theta'')$ by Theorem 5.20 in Villani (2008). Further, by Step 2, $M^{n_{k_l}}(\theta'') \geq_{P,Q,D} M^{n_{k_l}}(\theta')$. But then, the limits must be ordered $M^\infty(\theta'') \geq_{P,Q,D} M^\infty(\theta')$ by Theorem 9.A.2.a in Shaked and Shanthikumar (2007). \qed

### E.4 Marginal Rectangular Synergy: Proof of Proposition 3

We assume marginal rectangular synergy is upcrossovering in types. The steps for downcrossing marginal rectangular synergy are symmetric. We use the relationship:

$$S(x_1, x_2, y_1, y_2|\theta) = \int_{x_1}^{x_2} \Delta_x(x|y_1, y_2, \theta) dx = \int_0^1 \Delta_x(x|y_1, y_2, \theta) 1_{x \in [x_1, x_2]} dx \quad (29)$$

**Step 1.** *(Strictly) upcrossovering marginal rectangular synergy $\Rightarrow$ (strictly) upcrossovering rectangular synergy.*

**Proof:** We prove the continuum types case, which implies the finite type result. First, we show the non-strict claim. Indeed, any indicator function $1_{x \in [x_1, x_2]}$ is a log-supermodular function of $\Delta_x(x|y_1, y_2, \theta)$ and $1_{x \in [x_1, x_2]}$. By Karlin and Rubin’s classic 1956 result: 

\[ \phi(x, y) \geq 0 \text{ is log-supermodular (LSPM) if } \phi(x', y') \phi(x'', y'') \geq \phi(x', y'') \phi(x'', y') \text{ for all } x' \leq x'' \text{ and } y' \leq y''. \]

Easily, we can check that the indicator is LSPM: If $x \in [x_1, x_2]$ and $x' \in [x_1', x_2']$ then $\max(x, x') \in [\max(x_1, x_1'), \max(x_2, x_2')]$ and $\min(x, x') \in [\min(x_1, x'), \min(x_2, x_2')]$.
result, if $\Delta_x(x|y_1, y_2, \theta)$ is upcrossing in $x$, then the last integral in (29) is upcrossing in $x_1$ and $x_2$, and so in $(x_1, x_2)$. Symmetrically, rectangular synergy is upcrossing in $(y_1, y_2)$ when the $y$-marginal rectangular synergy is upcrossing in $y$. Altogether, rectangular synergy is upcrossing in types if both MPIs are upcrossing.

Now assume $\Delta_x(x|y_1, y_2)$ is strictly upcrossing; and so, if $S(x', y_1, x'_2, y_2) = 0$ then $\Delta_x(x'_1|y_1, y_2) < 0 < \Delta_x(x'_2|y_1, y_2)$. So $S_{x_1}(x'_1, y_1, x'_2, y_2) = -\Delta_x(x'_1|y_1, y_2) > 0$ and $S_{x_2}(x'_1, y_1, x'_2, y_2) = \Delta_x(x'_2|y_1, y_2) > 0$. Then $S(x''_1, y_1, x''_2, y_2) > 0$ for all $(x''_1, x''_2) > (x'_1, x'_2)$. By symmetric reasoning, $S$ strictly upcrosses in $(y_1, y_2)$.

**Step 2.** The optimal matching is unique in the continuum types model.

*Proof:* By Theorem 5.1 in Ahmad, Kim, and McCann (2011), there is a unique optimal matching when: (i) $G$ is absolutely continuous, (ii) $\phi$ is $C^2$, and (iii) the critical points of (their “twist difference”) $\phi(x, y_2) - \phi(x, y_1)$ include at most one local max and one local min, for all $y_1, y_2$. Our continuum types model imposes (i) and (ii). We claim that (iii) follows from marginal rectangular synergy $\Delta_x(x|y_1, y_2) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in $x$, for $y_2 > y_1$. In particular, if $y_2 > y_1$, then $\Delta_x(x|y_1, y_2)$ is upcrossing in $x$, and any critical point of the twist difference is a global minimum. Similarly, then any critical point is a global maximum if $y_2 < y_1$.

**Step 3.** Sorting increases in $\theta$.

*Proof:* Proposition 2 and Proposition 3 share the same time series assumption. Step 1 establishes that the cross-sectional premise of Proposition 3 implies the cross-sectional premise of Proposition 2. Finally, the optimal matching is generically unique for finite type models and unique for continuum type models by Step 2. Altogether, all assumptions in Proposition 2 are met; and thus, sorting increasing in $\theta$.

**E.5 Increasing Sorting: Proof of Proposition 4**

**Finite Types Proof.** We verify the premise of Proposition 2. First, by Theorem 3, total synergy $\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)1_{(i,j)\in Z}$ on any set of couples $Z \subseteq \mathbb{Z}_n^2$ is upcrossing in the parameter $t = \theta$. Thus, summed rectangular synergy $\sum_k S(r_k|\theta)$ is upcrossing in $\theta$ for any non-overlapping set of rectangles $\{r_k\}$. Next, to see that rectangular synergy $S(r|\theta) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)1_{(i,j)\in r}$ is upcrossing in $r$, we apply Theorem 3 to the parameter $t = r \in \mathbb{R}^4$. By a similar proof to footnote 31, the indicator function $1_{(i,j)\in r}$ is a non-negative LSPM function of $(i, j, r)$, since a rectangle $r$ is a sublattice.

---

32 Theorem 3 assumes $t \in \mathcal{T}$, a poset. Here we exploit the fact that the space of rectangular sets of couples is a sublattice of $\mathbb{Z}^2$, even though the PQD order on distributions over couples is not a lattice.
Then \( s_{ij}(\theta) \mathbb{1}_{(i,j) \in r} \) obeys inequality \( \ref{eq:original} \) in \( z = (i, j) \) and \( r \), since \( s_{ij}(\theta) \) obeys \( \ref{eq:original} \) for fixed \( \theta \). Rectangular synergy is then upcrossing in \( r \), by Theorem \ref{thm:rectangular}

**Continuum of Types Proof.** We apply Proposition \ref{prop:continuum}. By Theorem \ref{thm:rectangular} total synergy \( \int_Z \phi_{12}(x, y; \theta) dx dy \) is upcrossing in \( t = \theta \) for any measurable set \( Z \subseteq [0, 1]^2 \). Thus, summed rectangular synergy \( \sum_k S(R_k; \theta) \) is upcrossing in \( \theta \) for any non-overlapping set of rectangles \( \{R_k\} \). Next, the \( x \)-marginal rectangular synergy \( \int \phi_{12}(x, y) \mathbb{1}_{y \in [y_1, y_2]} dy \) is strictly upcrossing in \( x \). Let \( x'' > x' \). Posit for a contradiction:

\[
\int \phi_{12}(x'', y) \mathbb{1}_{y \in [y_1, y_2]} dy \leq 0 \leq \int \phi_{12}(x', y) \mathbb{1}_{y \in [y_1, y_2]} dy
\]  

As synergy \( \phi_{12}(x, y) \) is strictly upcrossing in \( x \) and \( y \), by \( \ref{eq:continuum} \), there exist zeros \( y', y'' \in (y_1, y_2) \) such that \( \phi_{12}(x', y) \leq 0 \) for \( y \leq y' \) and \( \phi_{12}(x'', y) \leq 0 \) for \( y \leq y'' \). Easily, these zeros are ordered \( y'' < y' \). But then inequalities in \( \ref{eq:continuum} \) are simultaneously impossible, for:

\[
0 \leq \int \phi_{12}(x', y) \mathbb{1}_{y \in [y_1, y_2]} dy < \int \phi_{12}(x', y) \mathbb{1}_{y \in [y_1, y'']} \mathbb{1}_{y \in [y', y_2]} dy
\]

\[
0 < \int \phi_{12}(x'', y) \mathbb{1}_{y \in [y_1, y']} \mathbb{1}_{y \in [y', y_2]} dy < \int \phi_{12}(x'', y) \mathbb{1}_{y \in [y_1, y_2]} dy
\]

by Theorem \ref{thm:rectangular}, since \( \int \phi_{12}(x, y) \lambda(y) dy \) is upcrossing in \( t = x \) for any non-negative \( \lambda(y) \) — because \( \phi_{12}(x, y) \) is proportionately upcrossing in types and upcrossing in \( y \). ∎

**E.6 Type Distribution Shifts: Proof of Corollary \ref{cor:continuum}**

**Proof:** Throughout we WLOG assume types shift up in the parameter \( \theta \).

**Summed Rectangular Quantile Synergy is Upcrossing in \( \theta \).** We make the stronger claims that total quantile synergy \( \ref{eq:quantile} \) is upcrossing in \( \theta \) on any measurable set of quantile pairs \( Z \subseteq [0, 1]^2 \). In the continuum type model, total synergy is:

\[
\Upsilon(\theta) \equiv \int \int \varphi_{12}(p, q; \theta) \mathbb{1}_{(p, q) \in \mathcal{Z}} dp dq = \int \int \phi_{12}(x, y) \mathbb{1}_{(G(x; \theta), H(y; \theta)) \in \mathcal{Z}} dx dy
\]

by the change of variables \( x = X(p, \theta) \) and \( y = Y(q, \theta) \) (equivalently, \( p = G(x; \theta) \) and \( q = H(y; \theta) \)); and thus, \( dx = X_p dp \) and \( dy = Y_q dq \). Since distributions \( G \) and \( H \) fall in \( \theta \), the cdf associated with pdf \( \lambda(x, y; \theta) \equiv \mathbb{1}_{(G(x; \theta), H(y; \theta)) \in \mathcal{Z}} / [\int \int \mathbb{1}_{(G(x; \theta), H(y; \theta)) \in \mathcal{Z}} dx dy] \) is stochastically increasing in \( \theta \). And thus, since \( \phi_{12}(x, y) \) is strictly increasing:

\[
0 \leq \Upsilon(\theta) \Rightarrow 0 \leq \int \int \phi_{12}(x, y) \lambda(x, y; \theta) dx dy \leq \int \int \phi_{12}(x, y) \lambda(x, y; \theta') dx dy \Rightarrow 0 \leq \Upsilon(\theta')
\]

Identical yields total synergy upcrossing in \( \theta \) on any set of couples with finite types.

**Case (a): Quantile Rectangular Synergy is Upcrossing.** Rectangular synergy is upcrossing in types when synergy is non-decreasing in types. Because types
$X(p, \theta)$ and $Y(q, \theta)$ are non-decreasing in the quantiles $p$ and $q$, quantile rectangular synergy $S(X(p_1, \theta), Y(q_1, \theta), X(p_2, \theta), Y(q_2, \theta))$ upcrosses in $(p_1, q_1, p_2, q_2)$. Hence, the quantile sorting increases in $\theta$ by Proposition 2.

**Case (b): Quantile Marginal Rectangular Synergy Strictly Upcrosses.**

Non-decreasing synergy is proportionately upcrossing; and thus $\Delta_x(x|y_1, y_2)$ strictly upcrosses in $x$ as shown in \[E.5\] Given $G(x|\theta)$ absolutely continuous $X_p > 0$; and so,

$$\Delta_p(p|q_1, q_2, \theta) = \Delta_x(X(p, \theta)|Y(q_1, \theta), Y(q_2, \theta))X_p(p, \theta)$$

is strictly upcrossing in $p$. Similarly, $\Delta_q(q|p_1, p_2, \theta)$ is strictly upcrossing in $q$. All told, we’ve seen that quantile sorting increases in $\theta$, by Step 1 and Proposition 3.

\[\square\]

**F Omitted Analysis for Economic Applications**

**Concave Transformations:** Let $\phi(x, y|\theta) = \psi(xy|\theta)$ with $\psi'' < 0 < \psi'$ with “relative risk aversion” $R(z, \theta) \equiv -z\psi''(z|\theta)/\psi'(z|\theta)$ decreasing in $z$ and $\theta$. Synergy is then:

$$\phi_{12}(x, y|\theta) = \psi'(xy|\theta)((1 - R(xy, \theta))$$

By assumption $\psi' > 0$ and $R(xy, \theta)$ is decreasing in $x, y$, and $\theta$. Thus, synergy is strictly upcrossing in $x, y$, and $\theta$. Further, $R(z, \theta)$ decreasing in $z$ and $\theta$ implies that $\psi'(xy|\theta) > 0$ is smoothly LSPM in $(x, y, \theta)$. Altogether, synergy is the product of a positive smoothly LSPM function and an increasing function. Thus, synergy $\phi_{12}$ is proportionately upcrossing. Altogether, sorting rises in $\theta$ by Proposition 4.

**Imperfect Credit Markets.** Differentiating production function (10) in $x$ yields:

$$\phi_1(x, y) = \frac{(\pi - c - d)(1 - d y^2) + 2 c y}{(1 - d x - d y + d x y)^2} > 0 \quad (31)$$

As $\partial[\phi_1(x, y_2)/\phi_1(x, y_1)]/\partial x < 0$ for all $y_2 > y_1$\[33\] the $x$-marginal rectangular synergy is strictly downcrossing in $x$ — and symmetrically, for the $y$-marginal rectangular synergy.

Next consider synergy as a function of parameters $\theta = (c, d, \pi)$. Differentiating (31) yields:

$$\phi_{12}(x, y|\theta) = ca(x, y) + (\pi - d)b(x, y)$$

for functions $a(x, y) > 0$ and $b(x, y) \in \mathbb{R}$. Synergy is increasing in $c$; and thus, so is summed rectangular synergy. Synergy is linear in $\pi - d$ with a positive intercept; thus,\[33\]

\[
\text{Indeed, } D_x[\log (\phi_1(x, y_2)/\phi_1(x, y_1)) = \frac{2\delta(1-\delta)(y_1-y_2)}{(1-\delta(x+y_1)(1-x))(1-\delta(x+y_2)(1-x))}. \text{ The numerator is negative by } \delta \in (0, 1) \text{ and } y_1 < y_2, \text{ and the denominator is positive since } x, y_1, \text{ and } y_2 \text{ are probabilities.}
\]
summed rectangular synergy is downcrossing in $\pi - d$ by Claim 1 in Appendix E.1. Altogether, sorting is increasing in $c$ and decreasing in $\pi - d$ by Proposition 3.

References


