Revenue Management Without Commitment: Dynamic Pricing and Periodic Fire Sales*

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Abstract

A profit-maximizing monopolist seller has a fixed number of identical goods to sell before a deadline. Over time, buyers privately enter the market, and they strategically time their purchases. In the unique Markov perfect equilibrium, the seller sporadically holds fire sales to lower the stock of goods. The practice of fire sales increases future buyers' willingness to pay but lowers the willingness to pay of buyers who arrive early in the game. Interestingly, when it is very likely for a buyer to obtain a good in a fire sale, the seller holds a "big" initial fire sale for all but one unit of the good.

Keywords: revenue management, commitment power, dynamic pricing, fire sales, inattention frictions.

JEL Classification Codes: D82, D83

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1 Introduction

Some markets share the following characteristics: (1) goods for sale must be consumed before a certain time, (2) the initial number of goods for sale is fixed in advance, and (3) consumers enter the market over time and choose the timing of their purchases. Examples are the airline, cruise line, hotel, sports ticket, and entertainment industries. In such industries, computer systems based on revenue management algorithms are often employed, which is typically viewed in the revenue management literature as a commitment device used by sellers.

In reality, however, revenue managers frequently adjust prices based on their information and personal experience, instead of strictly following the pricing strategy suggested by the algorithm.[1] Hence, it may not be without loss of generality to assume the seller has perfect commitment power, opening up the question of how the seller's lack of commitment affects the price dynamics. This paper studies revenue management under the assumption that sellers are endowed with no commitment power. We highlight an important effect of the lack of commitment power in such an environment: if the arrival of buyers is slower than expected, the seller finds it optimal to hold a fire sale to lower the stock of goods to sell them at a high price in the future, lowering the willingness to pay of early buyers and resulting in a highly fluctuating price path.

We consider the profit-maximization problem faced by a monopolist (male) seller who has a finite number of identical goods to sell before a deadline. Over time, (female) buyers privately arrive. Each buyer has a single-unit demand and can time her purchasing decision. At any time, the seller posts a price and a quantity (or capacity control) but cannot commit to future offers. Additionally, the seller can hold a *fire sale*, which takes the form of a low price offer that a third party advertises to a broader market as well as to arrived buyers.[2] As a result, the probability that an arrived buyer will obtain the good in a fire sale is less than one.

The model has a unique Markov perfect equilibrium, in which the price exhibits rich dynamics. For most of the time before the deadline, the seller posts the highest price that makes buyers willing to purchase upon arrival. In line with most revenue management models (see, for example, Gallego and Van Ryzin, 1994), the price decreases over time until a transaction happens, and

[1] In the article *Confessions of an Airline Revenue Manager* on FoxNews.com, Goerge Hobica said,"The computer adjusts fares all the way up until the departure time, but as a revenue manager, I can go in and adjust things based on information that the computer may not know. For example, are there specific events taking place at a destination? Are there certain conditions at the departure airport that will allow more than the desired amount of seats to go empty such as weather?"

[2] In practice, the advertisement is a deal-alert email (or text message) from a third party (such as Kayak, Orbitz, etc. in the airline industry) or online real-time bidding (RTB) advertisement that facilitates the seller's ability to target and track "window shoppers." The time windows for these fire sale offers are typically very short compared to the length of the transaction season.

after that, the inventory is reduced and the price jumps up.[3]

In addition to positing a "regular price," the seller occasionally holds a fire sale to sell some goods. In particular, at the deadline, the seller holds a fire sale to obtain some revenue from selling the units not sold before.[4] Similarly, before the deadline, the seller may have an incentive to hold fire sales. The reason is that the seller's decision to make a fire sale involves trading off future transaction prices and future transaction quantities: by reducing the current inventory size, the seller can enhance the willingness to pay of future buyers, but he may end up with insufficient supply in the future (in the event where more buyers arrive than he expects). In equilibrium, when the remaining time is short, it is unlikely that a large number of buyers will arrive before the deadline, so the seller has the incentive to keep the stock of goods small. The equilibrium specifies a decreasing sequence of "cutoff times" $\{t_k : k = 1, ..., K\}$, where K is the initial stock of goods, such that after time t_k, the seller finds it optimal to maintain the inventory size lower than k. If the inventory size is not lower than k at t_k, he holds a fire sale to get rid of the excess stock of goods. As a result, except maybe for an initial fire sale, a single unit is offered in each fire sale that happens on the path of play.

Our results relay on the following key building blocks of our model: the existence of a deadline, the stochasticity of the arrival of buyers, the buyers' ability to time their purchases, the finiteness of the inventory, and the lack of commitment by the seller. First, owing to the stochasticity of arrivals, there are histories where the stock is high and the expected future demand from arriving buyers is low due to the proximity of the deadline, so the seller has an incentive to hold a fire sale. Second, since buyers are forward-looking, their willingness to pay is endogenous. Therefore, to increase the future buyers' willingness to pay, the seller holds fire sales to credibly reduce the future supply, which requires the seller to be unable to produce additional goods over time. Finally, early fire sales are costly for the seller: since buyers have the option of waiting until then to obtain a good at a low price with a positive probability, their reservation price prior to fire sale times is low. As a result, if the seller could commit, he would prefer committing not to sell

[3]The resulting price fluctuation is consistent with extensive empirical evidence. For example, in the airline industry, McAfee and Te Velde (2006) find that the fluctuation of airfares is too high to be explained by the standard monopoly pricing models, Escobari (2012) finds that the airline price declines conditional on the inventory size and jumps up after transactions, and Williams (2015) shows that it leads to substantial revenue loss if the pricing policies are made without taking the remaining inventory into account. Aguirregabiria (1999) shows the important role of inventories by using retail market data. In secondary markets for Major League Baseball tickets, Sweeting (2012) finds that the seller cuts prices dramatically as the deadline approaches.

[4]Last-minute deals or clearance sales are optimal in many dynamic pricing settings. See Nocke and Peitz (2007) as an example. In practice, the last-minute deal strategy is commonly used in many industries. See *The Wall Street Journal*, March 15, 2002, "Airlines now offer 'last-minute' fare bargains weeks before flights," by Kortney Stringer.

some units after some histories to keep the prices high, but he fails to do so in equilibrium due to his lack of commitment.

To illustrate the role of fire sales and the broader market, we consider two limits of our model. If buyers become increasingly unlikely to obtain a good in a fire sale, the seller's commitment problem disappears: he charges a high price, and at the deadline, he holds a fire sale for the remaining units. If, instead, buyers are very likely to obtain a good in a fire sale, the commitment problem of the seller becomes very severe. In this case, even though the expected number of arrived buyers may be high, he holds a fire sale at the beginning of the game for all but one unit and tries to sell the remaining unit to the first buyer to arrive (if one does). These results are reminiscent of the Coase conjecture, albeit with the opposite effect: the more severe the commitment problem is, the less efficient the trade outcome is. In our model, such an inefficiency arises from the misallocation of goods instead of the delay in the trade.

Related Literature. There is a large body of *revenue management* literature that has examined markets with sellers who need to sell a finite number of goods before a deadline to buyers who arrive over time.[5] One of the common assumptions in this literature is that buyers are myopic. However, as Besanko and Winston (1990) argue, mistakenly treating forward-looking customers as myopic may have an important impact on the sellers' revenue, highlighting the need to incorporate strategic buyers in revenue management models.

Recent development of this literature has incorporated forward-looking buyers, assuming that either the seller has perfect commitment power or ignoring the arrival of buyers. Similar to us, Board and Skrzypacz (2016) analyze a dynamic market where a seller sells a finite number of goods before a deadline to agents who arrive in the market over time. They characterize revenue-maximizing mechanism under the assumption that the seller can commit. The optimal allocation mechanism trades off time discounting and incoming demand with higher willingness to pay. In the continuous time limit, the optimal mechanism is implemented via a price-posting mechanism with an auction for the last unit at the deadline.[6] In contrast, we analyze the non-commitment case: the seller cannot commit to not lower the price if the realized demand is low. We show that, even in the absence of discounting, the price exhibits rich dynamics and periodic fire sales. As a result, we not only examine the role of the commitment power of the seller but also provide a theory of fire sales: selling units at very low prices increases the price of the remaining units. In Hörner and Samuelson (2011) and Chen (2012), the seller has no commitment power and sells

[5]See Talluri and Van Ryzin (2006) for a textbook treatment. Gershkov and Moldovanu (2009) extend the benchmark model to address the case of heterogeneous objects.

[6]Also see Mierendorff (2016) and Pai and Vohra (2013). Gershkov, Moldovanu, and Strack (2014) consider a model in which the seller learns the arrival process of the buyer over time.

goods to a fixed number of forward-looking buyers.[7] They show that the seller either replicates a Dutch auction or posts an unacceptable price until the deadline and then charges a static monopoly price. So, in their models, the seller never uses low prices to inefficiently sell to buyers with low valuations to increase the competition between buyers with high valuations. We show that, in contrast, maintaining the assumption that buyers arrive stochastically in line with most revenue management models generates both fluctuating price dynamics and the possibility of fire sales.

Our paper is also related to the *durable goods* literature with arrival of consumers. Conlisk, Gerstner, and Sobel (1984) and Sobel (1991) study pricing dynamics with the arrival of a homogeneous set of buyers per period. In their model, the price fluctuates due to the seller's trade-off between rent extraction from the buyers with a high valuation and the revenue from selling the goods to accumulated buyers with a low valuation.[8] Garrett (2016) shows that similar price dynamics take place in a model where buyers arrive privately and their valuations change stochastically over time. Öry (2017) assumes that it is costly for the seller to use low price offers to lure back arrived buyers and that buyers do not observe the calendar time but adopt a stationary belief over the next low price offer. Similar to ours, her model exhibits discrete price drops at the times where the seller sells to the accumulated low-value buyers.[9] Although aforementioned papers also generates price fluctuations as our model does, the key mechanism is different. In the durable goods literature, the low-type buyers are accumulated until a critical time at which the seller finds it optimal to reap the rent from them. Such an incentive does not exist in our setting due to the finiteness of the horizon and the stationarity of the pool of shoppers. Instead, in our model, the seller holds fire sales to increase the scarcity of the goods, and the willingness to pay of future buyers as a result, after low realizations of the demand.

The insight that sellers may benefit from reducing inventories (or endowments) has been discussed in static competitive equilibrium frameworks such as those of Aumann and Peleg (1974) and Gale (1974). However, our goal in the paper is to investigate how such an insight shapes the equilibrium outcome in a revenue management environment, where dynamic considerations and random demand play an important role in pricing. In particular, the focus of the paper is on the *timing* of the fire sales.

Recently, Deb and Said (2015) show that the seller can also manage *demand*, as opposed

[7] Anton, Biglaiser, and Vettas (2014) illustrate dynamic price competition in a similar setting.

[8] Board (2008) discussed the case where incoming demand varies over time and examined how the seller uses time to discriminate between different generations. Liu, Mierendorff, Shi, and Zhong (2017) consider a durable good setting in which the seller posts the reserved price of an auction in each period.

[9] Heidhues and Kőszegi (2014) illustrate that the stochastic price fluctuation can also be driven by the buyers' "loss aversion."

to supply as in our model, to affect the competition in the future. They consider a two-period dynamic-screening problem where a seller without capacity constraints sells goods to buyers who (non-stochastically) arrive in each period. They show that the seller may deliberately postpone the transaction with some buyers to alter the demand in the second period, and therefore make charging a high price credible.

The rest of this paper is organized as follows. In Section 2, we present the model setting and define the solution concept we are going to use. In Section 3, we analyze equilibria. Section 4 discusses some modeling choices and possible extensions of the baseline model. Section 5 concludes. Appendix A provides the formal model, while Appendix B includes all the proofs.

2 Model

In this section, we describe our model. To prevent technicalities, this section avoids some of the notation-intensive definitions and presents the economically relevant components of our model. Appendix A provides the formal version of the model.

Environment. We consider a dynamic pricing game between a single (male) seller who has $K \in \mathbb{N}$ identical and indivisible goods for sale and many (female) buyers. Time is continuous: $t \in [0, 1]$. Goods are valuable up to time 1 and deliver no value afterward.

Seller. At every time t, the seller makes a *regular offer* specifying a price $P_t \in \mathbb{R}$ and a supply (or capacity control) $Q_t \in \{0, 1, 2, ..., K_t\}$, where K_t is the number of unsold goods until t, with $K_0 = K$. After offering the regular price, he can hold a *fire sale*, discussed in more detail below. The seller values goods at zero. He is a risk-neutral expected-utility maximizer who does not discount the future, so his payoff equals the expected summation of all transaction prices.

Buyers. At time 0, there is no buyer. Over time, buyers arrive privately at a rate $\lambda > 0$. Each buyer has a single-unit demand and values the good at $v_H > 0$. A buyer who buys a good at price p gets a payoff $v_H - p$, and her payoff is 0 if she does not obtain any good. If a buyer purchases a good at time t, she leaves the game. Otherwise, she enters in a waiting state, and she is referred to as an *accumulated buyer*.

When a buyer arrives, she observes the current regular offer and inventory size. She chooses to accept the regular offer or passes it up and becomes an accumulated buyer. An accumulated buyer faces *inattention frictions*: at each moment t, she does not observe the seller's regular offer and inventory unless she incurs a cost $c > 0$ to pay attention. In sum, a buyer is *attentive* at her arrival time and the times when she pays attention.

If there are $N_t \geq 1$ attentive buyers at time t, the game proceeds as follows. They all observe K_t, the price P_t, and the quantity offered Q_t and simultaneously and independently decide whether to purchase. Nature uniformly randomly orders the buyers who decide to purchase a good and allocates the good to the first Q_t of them (to all of them if fewer than Q_t buyers are willing to buy a good).

Fire Sales. Within each instant, after the regular transactions (if any) take place, the seller can hold a fire sale by offering Q_t^D of the remaining units of the good at an exogenously given price $v_L \in (0, v_H)$.[10] A fire sale offer attracts the attention of accumulated buyers as well as additional demand from a broader market. Such an additional demand can be interpreted as a pool of shoppers (or uninterested buyers) who value the good at v_L.

If, in a given period t, the seller offers $Q_t^D \geq 1$ units in a fire sale, the game proceeds similarly as for the regular price offer, although now with the possibility that some goods are purchased by shoppers. First, nature uniformly randomly orders the accumulated buyers (if any). If there is no buyer, all goods are assigned to shoppers. If there is at least one buyer, with some probability $\beta \in (0, 1)$, the first good is assigned to the first buyer, and with probability $1 - \beta$ to a shopper. After the first unit is assigned, if $Q_t^D > 1$, the second unit is assigned analogously: if no buyer remains, it is assigned to a shopper, while if there is at least one buyer, the unit is assigned to the first of the remaining buyers with probability β, and with probability $1 - \beta$ to a shopper. This process is iterated until no unit is left. Note that β is a measure of how "attentive" buyers are with respect to shoppers.

Information. At time t, the seller observes the previous and current stock, $(K_s)_{s=0}^{t}$, as well as the previous prices and quantities offered, $(Q_s, P_s, Q_s^D)_{s=0}^{t-}$, but he does not observe the arrivals of buyers, nor does he see their actions. A buyer arriving at time t only observes (t, K_t, P_t, Q_t). Finally, an accumulated buyer at time t observes K_s, P_s, and Q_s in her previous attention times s (including her arrival time) and whether she tried to purchase the good, as described above.

A heuristic description of the timing within an instant t is depicted as follows. First, nature decides whether a buyer arrives at time t. Then, the seller decides what quantity and price to offer. Next, accumulated buyers decide whether to pay the inspection cost at time t. After this, goods are offered to attentive buyers who accept the offer according to the procedure described above. Finally, the seller decides how many units to offer through a fire sale, and nature assigns them according to the procedure described above.

[10] For simplicity, we do not model the fire sale price as an endogenous choice of the seller. Alternatively, one can consider a model where the willingness to pay of shoppers in the broad market is v_L. By the standard argument in Fudenberg, Levine, and Tirole (1985), the price targeting shoppers must be v_L.

Solution Concept. We say that a strategy profile is *Markovian* if, for any time t and remaining stock value K_t, (i) the price and quantity offered by the seller depend on (t, K_t), and the number of fire sales depend, additionally, on the number of sales at time t;[11] and (ii) a buyer's expected continuation value from rejecting an offer, acceptance decision (that is, the set of price and quantity offers she accepts), and next attention time depend only on (t, K_t). We focus on pure-strategy Markov perfect equilibria (MPE), which we refer to as just "equilibria".[12]

2.1 Discussion of the Assumptions

With the goal of studying the price dynamics in a revenue-management environment without commitment, we make some rather unconventional modeling choices. These modeling choices aim at giving tractability to a problem that otherwise has been proven to be intractable, while keeping the main trade-offs of economic interest. In particular, they allow us to prove that there exists a unique Markov perfect equilibrium, to characterize it using dynamic programing techniques, and to show that it exhibits price dynamics consistent with observed regularities. Before proceeding further, we want to discuss the role of these assumptions.

Inattention Frictions. We assume that being attentive is costly for accumulated buyers. This is a simple modeling device to lower the buyers' willingness to monitor the price, and it captures the observation that, in practice, it is costly to closely monitor prices, so buyers check the price occasionally.[13] As we will show, buyers find it suboptimal to check the price in equilibrium, so the seller has a limited ability to influence the beliefs of the buyers by changing the price offered, keeping the analysis of the off-the-equilibrium-path incentives tractable. The fact that all our results are independent of the value of c (as long as it is positive) gives robustness to our model. Alternatively, if accumulated buyers could freely track the price since their arrival, the seller could have the incentive to charge unacceptable prices and manipulate their beliefs about the history of the game, and thus their willingness to pay. As a result, accumulated buyers' beliefs about the previous accumulation would be heterogenous after some histories, preventing the model from being tractable.

Fire Sales. We assume there are two channels through which the seller can sell goods: "regular" sales are accessible to arriving buyers only, and "fire" sales are accessible by both arrived buyers

[11] Formally, in Appendix A, a Markov fire-sales policy defines, for each Markov variable (t, K_t), a function determining the number of units offered in a fire sale depending on the sales at period t. This allows the seller to react to regular sales.

[12] Notice that we allow both the seller and buyers to deviate to non-Markov strategies.

[13] For example, when a consumer is waiting for a discount flight, she may only check the price at agencies' websites once or twice per day.

and shoppers in a broader market. Fire sales play the following two roles in our model. First, they give the seller the incentive to lower the price when the realized demand is low, making the optimal price decision dependent on the previous history. As a result, fire sales allow the seller to increase the competition between future buyers (and the future prices) by lowering the stock. Second, the seller's inability to commit to not holding fire sales lowers the buyers' willingness to pay, as they know that they may get a good at a low price in the future with a positive probability. This both makes them behave non-myopically and makes the commitment problem severe: the seller is forced to lower current prices to give the buyers the incentive to purchase immediately upon arrival.

The assumption that the broader market is stationary is made for tractability. Incorporating a history-dependent population of low-value shoppers would not only complicate the analysis but also prevent the existence of Markov equilibria: the previous history would determine the current distribution of accumulated shoppers, so different buyers with different private histories (i.e., having arrived at different times) would have a different continuation strategy.

Finally, we assume that a fire sale is associated with a *deal alert* that ensures that the fire sale goods are sold immediately. This assumption aims at capturing a trade-off present in dynamic settings between the speed and the price of transactions: in our case, the fire sale offer is able to actively attract more attention and effective demand so the goods are sold more quickly than through the regular transaction. Assuming that fire sale offers lower the stock instantaneously allows us to characterize equilibrium fire sales as a Markov stopping time problem. In Appendix C, we consider an alternative model where it takes time to attract attention to fire sale offers. As the expected sale time it takes to attract attention in a fire sale decreases, we recover our results.

Minor Assumptions. We also make two other minor simplifying assumptions, which can be relaxed without undermining our key results. First, we assume that consumers are "forced" to accept a fire-sale offer (at price v_L). This assumption is made for simplicity, and since it is suboptimal for the seller to charge a price lower than v_L, it would naturally be part of equilibrium behavior to accept such a low price even if buyers were allowed to reject the offer. Second, we assume that holding a fire sale is free. Assuming that offering such a deal would involve a small cost $c' \in (0, v_L)$ (coming, for example, from the cost of advertising) would not qualitatively change our results. Therefore, we include advertising as a way for the seller to lower his stock and increase competition among future buyers. Also see Öry (2017) for a discussion of the role of such advertisements in a stationary model.

3 Analysis

We begin with a lemma that will simplify our analysis. This lemma establishes that if a buyer passes up the offer that the seller makes at her arrival time, she does not check the price again.

Lemma 1. *In any Markov perfect equilibrium, if a buyer does not purchase a good at her arrival time, she never rechecks the price.*

The reasoning behind Lemma 1 is similar to the one in the standard Diamond paradox. Since in a Markov perfect equilibrium the continuation value from rejecting an offer is the same for all buyers, it is optimal for the seller to either charge their reservation price, that is, the price that makes buyers indifferent to accepting it or not, or make a losing offer that is not accepted in equilibrium. To see this, suppose that a buyer rejects the price offer at her arrival time. She has the incentive to recheck the price at some future time only if the payoff she expects from doing so compensates for the rechecking cost c. However, once she rechecks the price, the rechecking cost is sunk. As a result, conditional on rechecking the price at a given time, the buyer is either made indifferent between accepting or rejecting, or she is induced to reject the offer. Hence, there is no gain from rechecking the price: it is costly and it provides the buyer with an information that she cannot use to increase her rents.

Lemma 1 implies that, in any equilibrium, a buyer either purchases a good upon her arrival or at a fire sale. As a result, the equilibrium probability that two or more buyers observe a given price offer at a given time $t \in [0, 1]$ is 0. Thus, in equilibrium, the seller is indifferent on the choice of the quantity offered $Q_t \geq 1$ at any time t. Since the seller can also ensure that no good is bought by offering $P_t > v_H$, and buyers purchase the good only if the price is below their reservation price (independently of the value of Q_t), it can be assumed, without loss of generality, that $Q_t = K_t$ for all $t \in [0, 1]$. Still, even though each buyer enjoys some monopsony power, she competes with other buyers inter-temporally owing to the fact that the inventory is finite: she knows that if she passes up an offer, other buyers may obtain the goods, and she may end up with no good at all. The following lemma establishes that, in any equilibrium, buyers purchase upon arrival.

Lemma 2. *In any Markov perfect equilibrium, in any history where $K_t > 0$, the equilibrium price equals the buyers' reservation price, and buyers accept it at their arrival time.*

Lemma 2 establishes that, in equilibrium, the seller sets a price such that buyers purchase at their arrival time. The reason for this is that, by Lemma 1, if a buyer does not buy a good when she arrives, she never rechecks. Therefore, from the seller's point of view, the presence of an accumulated buyer is irrelevant, as goods sold in fire sales are sold at price v_L independently of the presence of accumulated buyers. Still, for a given number of units left, buyers who arrive late have a lower willingness to pay than buyers who arrive earlier: they expect less future competition. So, even though the seller could be willing to commit to accumulating buyers to increase the reservation price of future buyers, the non-observability of the previous history makes accumulating buyers not incentive compatible, as he has the incentive to deviate and sell to early buyers at a higher price.

Lemmas 1 and 2 imply that the buyers' equilibrium reservation price is determined by the timing of the future fire sales. Indeed, given that rechecking the price is suboptimal, their continuation value from rejecting an offer comes from the opportunity to obtain the good in a future fire sale. As a result, the focus of our analysis is on the timing of fire sales, the incentives of the seller in setting them, and their effect on the resulting equilibrium price dynamics.

3.1 Single-Unit Case

For illustrative purposes, we first analyze the case where the seller has only one good.

Proposition 1. *Suppose $K = 1$. There is a unique Markov perfect equilibrium. In it,*

1. *a fire sale is held at the deadline, and only at the deadline,*
2. *at each $t \in [0, 1]$, an attentive buyer accepts the price offered if and only if it is less than or equal to her reservation $p_1(t)$, which solves*

$$v_H - p_1(t) = e^{-\lambda(1-t)}\beta(v_H - v_L) \ , \tag{1}$$

3. *independently of the previous history, the seller posts a price equal to the buyers' reservation price, $p_1(t)$.*

Proposition 1 establishes that the seller holds a fire sale at time t if and only if $t = 1$. The "at" in the first part of the statement is a consequence of the seller's inability to commit: if the unit has not yet been sold at the deadline, the seller prefers to obtain some revenue $v_L > 0$ by offering a last-minute deal. This lowers the ability of the seller to extract rents from the buyers, as they know that they can wait until the deadline and obtain the good at a low price with a positive probability. Still, owing to the intertemporal competition between buyers and, at fire sale times, between buyers and shoppers, the seller can keep the price above v_L before the deadline. As a result, we obtain the "only at" in the first part of the statement: the lowest acceptable price before the deadline is strictly higher than v_L, so it is optimal not to hold a fire sale before the deadline.

Another implication of Proposition 1 is that the equilibrium price smoothly declines over time. The intuition is simple. The left-hand side of equation (1) represents a buyer's payoff if she arrives at time t and purchases the good at the equilibrium price $p_1(t)$, while the right-hand-side represents her continuation value from rejecting the offer. The (reservation) price $p_1(t)$ makes her indifferent between making the purchase upon arrival and rejecting the offer and (by Lemma 1) waiting for the final fire sale. Consequently, by waiting, she will get the good at a fire sale price v_L only if (1) no other buyer arrives in $(t, 1]$ and (2) the good is assigned to her (instead of to a shopper) at the final fire sale. The former event takes place with probability $e^{-\lambda(1-t)}$, and the probability of the latter event is β, conditional on the former event. Intuitively, as the deadline approaches, the probability that new buyers will arrive shrinks, so a buyer faces less competition; thus her reservation price decreases. In addition, at the deadline, $p_1(1) = (1-\beta)v_H + \beta v_L > v_L$. This gap between $p_1(1)$ and v_L reflects the buyer's willingness to avoid competition with shoppers.

Notice that, because the fire sale only occurs at the deadline, the good is only allocated to a shopper if no buyer arrives. As a result, the allocation rule is efficient. This will not be true in the multi-unit case.

3.2 The Two-Unit Case

Before considering the general multi-unit case, we study the case where the seller has two units to sell. This case illustrates in a simpler way the main trade-offs present in the general multi-unit case, studied afterward. We first prove the necessity of a fire sale before the deadline. We then use this result to characterize the equilibrium time of such a fire sale, which determines the unique equilibrium.

Pre-Deadline Fire Sale. Lemmas 1 and 2 prescribe that the seller does not accumulate buyers: that is, buyers purchase immediately the good on the path of play. Even if, off the path of play, he deviates and accumulates some buyers, such an accumulation is unobservable to future buyers, and accumulated buyers only obtain the good at fire-sale times, so his continuation payoff is independent of the number of buyers accumulated in the past. As a result, independently of the previous history, once one of the units is sold, the equilibrium is characterized by Proposition 1. Our goal is then to characterize the optimal strategy of the seller when $K_t = 2$.

First, we rule out that the seller holds a two-unit fire sale before the deadline. To do this, fix some $t < 1$ and assume that $K_t = 2$. The payoff for the seller of holding a two-unit fire sale at time t is $2v_L$. Instead, the seller has the option of holding a one-unit fire sale and obtaining a payoff equal to $v_L + \Pi_1(t)$ where

$$\begin{aligned}\Pi_1(t) &\equiv \int_t^1 \lambda e^{-\lambda(s-t)} p_1(s)\mathrm{d}s + e^{-(1-t)\lambda} v_L \\ &= v_H - e^{-\lambda(1-t)}(v_H - v_L)(1 + \beta(1-t)\lambda)\end{aligned} \tag{2}$$

is, by Proposition 1, the expected profit for the seller at time t if $K_t = 1$.[14] Since $\Pi_1(t) > v_L$ for $t < 1$ (notice that $p_1(t') > v_L$ for all $t' \in [0, 1]$), holding a two-unit fire sale before the deadline is always dominated by holding a one-unit fire sale. Hence, in equilibrium, if there is a fire sale before the deadline, only one unit is offered in it.

Second, when the inventory is two, the seller can offer the goods at the buyers' reservation price and, at the deadline, to hold a fire sale for the remaining units. This strategy maximizes the probability of selling both units at a price above v_L, and guarantees an efficient allocation. Nevertheless, if the seller holds two units at time $t' > t$, the buyers' reservation price is lower than $p_1(t')$, as they expect a higher probability of obtaining a good in a fire sale when two units instead of one unit are left. Alternatively, the seller may hold a fire sale for one of the units at time t and then offer the one-unit price during the remaining time. In this case, the seller's revenue from the sale of the first unit is low, but the revenue from the sale of the second unit is potentially large.

The next lemma establishes that, when the time is close to the deadline, since the arrival of two buyers is very unlikely, the second strategy dominates, so the seller never holds two units until the deadline.

Lemma 3. *There exists a unique $t_2^* \in [0, 1)$ such that, in any equilibrium, the seller holds a single-unit fire sale ($Q_t^D = 1$) at time $t < 1$ if and only if $K_t = 2$ and $t \geq t_2^*$.*

To gather some intuition for the previous result, assume, for the sake of contradiction, that there is an equilibrium where the seller only holds a fire sale (offering the remaining units) at the deadline. Consider a buyer who arrives at time t and observes $K_t = 2$, for some $t \in [0, 1]$. In this case, she is willing to accept some price p only if it is lower than $\bar{p}_2(t)$, satisfying

$$v_H - \bar{p}_2(t) = \underbrace{e^{-\lambda(1-t)}\beta_2(v_H - v_L)}_{\text{no buyer arrives}} + \underbrace{e^{-\lambda(1-t)}\lambda(1-t)\beta(v_H - v_L)}_{\text{one buyer arrives}},$$

where $\beta_2 \equiv \beta + (1-\beta)\beta$ is the probability that an accumulated buyer of obtains a good at a two-unit fire sale if there is no other accumulated buyer. A simple comparison of the previous

[14] Notice that Π_1 is continuous so the payoff from the seller of holding a fire sale for one unit at time $t < 1$, which is $\lim_{t' \searrow t}(v_L + \Pi_1(t'))$, is equal to $v_L + \Pi_1(t)$.

equation and equation (1) shows that $\bar{p}_2(t) < p_1(t)$ for all $t \in [0,1]$. Indeed, if a buyer arrives and observes that the seller still holds two units instead of one, she obtains a good with a higher probability if no buyer arrives (first term, since $\beta_2 > \beta$) and with a positive probability if one buyer arrives (second term).

Now imagine that the remaining time before the deadline, denoted $\varepsilon > 0$, is small, and the seller still holds two units. It is most likely (with probability $e^{-\lambda\varepsilon}$) that no buyer will arrive at a time in $[1-\varepsilon, 1]$, in which case the seller holds a two-unit fire sale at the deadline and obtains $2v_L$. With probability $e^{-\lambda\varepsilon}\lambda\varepsilon$, exactly one buyer arrives in $[1-\varepsilon, 1]$, in which case the seller sells the second unit in the fire sale, so his payoff is, as $\varepsilon \to 0$,

$$\mathbb{E}\big[\bar{p}_2(t)\big|t \in [1-\varepsilon, 1]\big] + v_L = \bar{p}_2(1) + v_L + O(\varepsilon) \ .$$

The probability that multiple buyers arrive is $O(\varepsilon^2)$ as $\varepsilon \to 0$. Hence, the seller's expected payoff from not holding a fire sale before the deadline is given by

$$e^{-\lambda\varepsilon}2v_L + e^{-\lambda\varepsilon}\lambda\varepsilon\big(\bar{p}_2(1) + v_L\big) + O(\varepsilon^2) \ . \tag{3}$$

Another continuation strategy that the seller may follow in $[1-\varepsilon, 1]$ is given by holding a fire sale at any time $1-\varepsilon$ and then offering $p_1(\cdot)$ afterwards. In this case, he obtains $2v_L$ if no buyer arrives in $(1-\varepsilon, 1]$, while if a buyer arrives, his payoff is $v_L + p_1(1) + O(\varepsilon)$. As before, the probability of the arrival of two or more buyers is $O(\varepsilon^2)$, and therefore becomes negligible as $\varepsilon \to 0$. Hence, her expected payoff from following this strategy is given by

$$e^{-\lambda\varepsilon}2v_L + e^{-\lambda\varepsilon}\lambda\varepsilon\big(v_L + p_1(1)\big) + O(\varepsilon^2) \ . \tag{4}$$

For small ε, the difference between the payoffs of the first and the second strategies is

$$(\bar{p}_2(1) - p_1(1))\varepsilon + O(\varepsilon^2) = -(1-\beta)\beta(v_H - v_L)\varepsilon + O(\varepsilon^2) < 0 \ .$$

As we see, the difference between the payoff mainly comes from the different prices at which the good is sold if only one buyer arrives in each of the strategies, as the payoff if no buyer arrives is the same across both strategies, while the probability that two buyers or more will arrive is very small. This difference is negative: the buyers' reservation price is smaller when there are two units left instead of only one. So, it is never part of equilibrium behavior to not hold fire sales until the deadline: the seller benefits from generating some scarcity to keep the prices high.

The above reasoning highlights an important consequence of the lack of commitment power of the seller. He uses fire sales as a device to "advance" (and therefore increase the probability of a sale at) high prices. Indeed, when the time is close to the deadline, the seller can credibly charge a (relatively) high price only if the stock of goods is low. As the deadline becomes close, the probability of being able to sell all remaining goods at the regular price decreases. Hence, the fire sale ensures not only higher prices for the remaining units, but also a higher probability that they are sold at the regular price.

Optimal Timing of Fire Sales. Now we characterize the equilibrium time at which the first fire sale happens.

Proposition 2. *Suppose $K_0 = 2$. There exists a unique Markov perfect equilibrium. In this equilibrium, there exists some $t_2^* \in [0, 1)$ such that if $K_t = 2$ for some time $t \in [t_2^*, 1)$ the seller makes a fire sale offer so that $K_{t^+} = 1$.*[15]

We leave the formal proof of Proposition 2 to Appendix B and, instead, we now give some intuitions on how t_2^* is determined. To do this, we first characterize the price offered by the seller at a time t if $K_t = 2$, which we denote using $p_2(t)$. When $t < t_2^*$ this is given by

$$\begin{aligned} v_H - p_2(t) \;\; &= \;\; \overbrace{e^{-\lambda(t_2^*-t)}\big(\beta + (1-\beta)e^{-\lambda(1-t_2^*)}\beta\big)(v_H - v_L)}^{\text{no arrival in } [t, t_2^*]} \\ &\quad + \underbrace{e^{-\lambda(t_2^*-t)}\lambda(t_2^* - t)e^{-\lambda(1-t_2^*)}\beta(v_H - v_L)}_{\text{one arrival in } [t, t_2^*]} \; . \end{aligned} \tag{5}$$

The first term on the right-hand side of the previous equation corresponds to the event that no other buyer arrives in $[t, t_2^*]$. In this case, the buyer may obtain a good at time t_2^* (with probability β) or, if no other buyer arrives in $[t_2^*, 1]$, she may obtain it at the final fire sale (again with probability β). The second term on the right-hand side corresponds to the event that exactly one other buyer arrives in $[t, t_2^*]$, and no buyer arrives in $(t_2^*, 1]$. When, instead, $t \geq t_2^*$, buyers expect an immediate fire-sale, so their continuation payoff is given by

$$v_H - p_2(t) = \beta(v_H - v_L) + (1-\beta)e^{-\lambda(1-t)}\beta(v_H - v_L) \; . \tag{6}$$

Notice that $p_2(\cdot)$ is continuous at t_2^*: as t approximates t_2^* from below, the probability of the arrival of a buyer before the fire sale decreases, so the reservation price of a buyer converges to $p_2(t_2^*)$.

[15] Recall that the seller's fire sale choice is contingent on the units sold at the regular price. So, if no buyer arrives at some time $t \in [t_2^*, 1)$ with $K_t = 2$, then the seller holds a fire sale for one unit, while if a buyer arrives and purchases a good, then the seller makes no fire sale or, equivalently, a fire sale for 0 units. In both cases, $K_{t^+} = 1$.

Given the reservation price of the buyers $p_2(\cdot)$ and a Markov state $(t, 2)$, the seller's problem of finding the optimal time for the next fire sale can be formulated as an optimal stopping time problem as follows:

$$\max_{\tau_2 \geq t} \left\{ \int_t^{\tau_2} e^{-\lambda(s-t)} \lambda[p_2(s) + \Pi_1(s)] \mathrm{d}s + e^{-\lambda(\tau_2 - t)}[v_L + \Pi_1(\tau_2)] \right\} \tag{7}$$

In the unique equilibrium, the seller's optimal choice for the time of the first fire sale is consistent with the buyers' belief, so the previous problem has $\tau_2 = t_2^*$ if $t < t_2^*$ and $\tau_2 = t$ if $t \geq t_2^*$ as a solution. The value of t_2^* balances the willingness to delay the fire sale to increase the probability of selling the first unit at a high price and the incentive to anticipate it to increase the price of the remaining unit.

Equation (7) implies that, when $t < t_2^*$, the expected profits of the seller at time t if $K_t = 2$ are given by

$$\Pi_2(t) \equiv \int_t^{t_2^*} e^{-\lambda(s-t)} \lambda[p_2(s) + \Pi_1(s)] \mathrm{d}s + e^{-\lambda(t_2^* - t)}[v_L + \Pi_1(t_2^*)] \ .$$

Notice that $\lim_{t \nearrow t_2^*} \Pi_2(t) = \Pi_1(t_2^*) + v_L$, which is sometimes called the *value-matching* condition. We can differentiate previous equation to obtain the so-called *Hamilton-Jacob-Bellman (HJB)* equation for the seller's profit function:

$$\Pi_2'(t) = -\lambda[p_2(t) + \Pi_1(t) - \Pi_2(t)], \quad \forall t \in [0, t_2^*) \ . \tag{8}$$

The intuition behind equation (8) is as follows: as $p_2(t) + \Pi_1(t) > \Pi_2(t)$, the arrival of a buyer is (strictly) preferable than no buyer arriving, so the continuation value decreases over time if there are no arrivals.

Because p_2 is continuous and Π_1 is differentiable, the expression on the right-hand side of equation (7) is differentiable for all $\tau_2 > t$. So, for any $t < t_2^*$, its derivative with respect to τ_2 at $\tau_2 = t_2^*$ is

$$e^{-\lambda(t_2^* - t)} \big(\lambda(p_2(t_2^*) - v_L) + \Pi_1'(t_2^*) \big) \ .$$

This first order condition for t_2^* requires the previous expression be equal to 0. So, using the HJB equation for $t < t_2^*$ and the value-matching condition, $\lim_{t \nearrow t_2^*} \Pi_2'(t) = \Pi_1'(t_2^*)$, that is, the *smooth pasting* condition holds. Figure 1 depicts Π_1 and Π_2, and shows the suboptimality of not holding a fire sale before the deadline.

3.3 The Multi-Unit Case

We now consider the general case, where K is an arbitrary natural number. The following proposition characterizes the unique equilibrium:

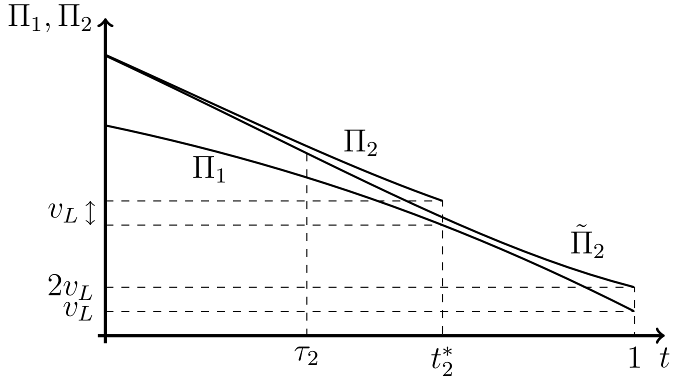

Figure 1: Illustration of the continuation expected profits of the seller. If there is no fire sale until the deadline, the profit from holding two units at a time $t \in (\tau_2, 1)$ (given by $\tilde{\Pi}_2(t)$) is lower than the profit from immediately holding a fire sale (given by $\Pi_1(t) + v_L$), so the seller benefits from deviating in this range (see Lemma 3). In equilibrium, the seller holds a fire sale at t_2^* (see Proposition 2), which generates a profit from holding to units equal to Π_2.

Proposition 3. *There exists a unique Markov perfect equilibrium for any $K \geq 1$. In this equilibrium, there exists a decreasing sequence of threshold times $(t_k^*)_{k=1}^K$ such that*

1. *$t_1^* = 1$ and, for all $k \in \{2, ..., K\}$, $t_k^* < t_{k-1}^*$ whenever $t_{k-1}^* > 0$, and*
2. *if (t, K_t) is such that $t_k^* \leq t < t_{k-1}^*$ and $K_t \geq k$, the seller holds a fire sale so that $K_{t^+} = k-1$.*

Proposition 3 establishes that, in a state (t, k) with $t \leq t_k^*$ and $k \geq 2$, the seller offers a price $p_k(t)$. Such a price makes buyers indifferent between accepting it and rejecting it. He does so until either there is a regular sale or t_k^* is reached and the seller holds a fire sale for one unit. As a result, the continuation value of a buyer arriving at t is:

$$v_H - p_k(t) = \int_t^{t_k^*} \lambda e^{-\lambda(s-t)}(v_H - p_{k-1}(s))\,\mathrm{d}s \\ + e^{-\lambda(t_k^*-t)}\big(\beta(v_H - v_L) + (1-\beta)(v_H - p_{k-1}(t_k^*))\big) \ ,$$

The previous equation uses the fact that if a buyer arrives at some time $s \in (t, t_k^*]$ then k decreases in one unit to $k-1$. So, the continuation value of an accumulated buyer decreases from $v_H - p_k(t)$ to $v_H - p_{k-1}(t)$. This allows us to obtain the following recursive equation for the continuation value of the buyers (and, therefore, the price) for each $k \in \{2, ..., K\}$:

$$0 = (0 - p_k'(t)) - \lambda\big(v_H - p_{k-1}(t) - (v_H - p_k(t))\big) \\ \Rightarrow \qquad p_k'(t) = \lambda(p_k(t) - p_{k-1}(t)) \ . \tag{9}$$

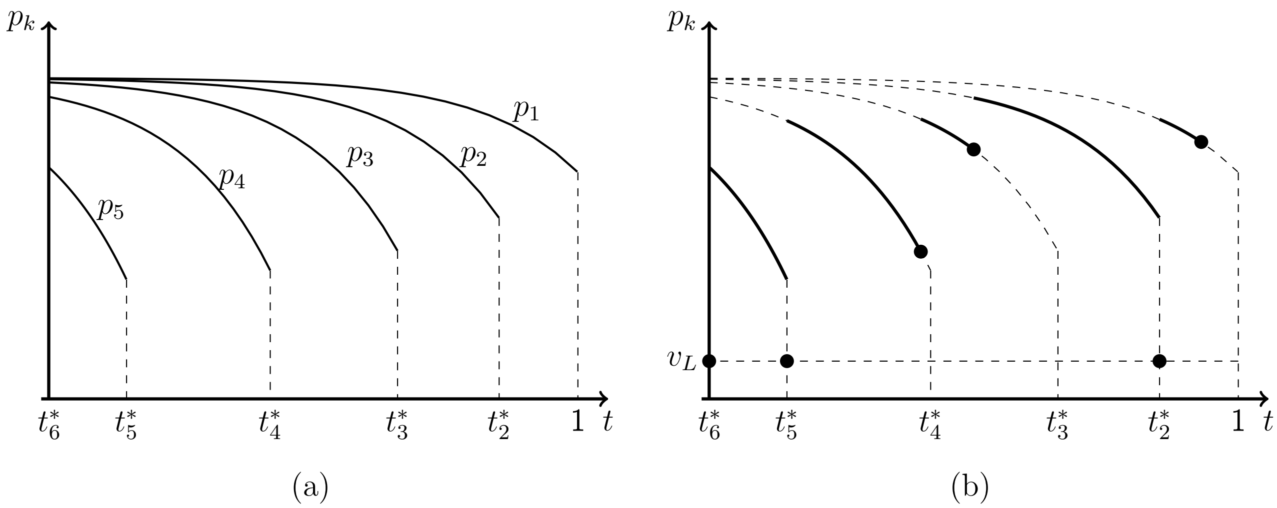

Figure 2: (a) depicts p_K in $[0, t_K^*]$ for different values of K. (b) depicts a possible realization of the price path, where the dots denote transaction prices.

If, instead, $t_{k'+1}^* \leq t < t_{k'}^*$ for some $k' < k$, then the seller immediately holds a fire sale for $k - k'$ units. As a result, in this case, the following equation holds:

$$v_H - p_k(t) = \beta_{k-k'}(v_H - v_L) + (1 - \beta_{k-k'})(v_H - p_{k'}(t)) \ , \tag{10}$$

where $\beta_{k-k'}$ denotes the probability that a given buyer in the waiting state obtains the good at time t if $k - k'$ goods are offered and she is the only buyer in the waiting state at time t. Notice that $\beta_{k-k'}$ is increasing in both β and $k - k'$, for all $k - k' \geq 1$.

Figure 2 provides some simulated price paths. Conditional on the inventory size, the price declines as the deadline approaches. However, transactions occur over time, and each transaction triggers an uptick in price. As a result, the price goes up or down depending on the arrival of buyers.

The payoff of the seller is computed in an analogous way. So, in a state (t, k) with $t \leq t_k^*$ and $k \geq 2$, a buyer arrives at a Poisson rate and purchases the good at price $p_k(t)$. As a result, the corresponding HJB equation for the seller becomes

$$0 = \Pi_k'(t) + \lambda\big(p_k(t) + \Pi_{k-1}(t) - \Pi_k(t)\big) \ . \tag{11}$$

If, instead, $t_{k'+1}^* \leq t < t_{k'}^*$ for some $k' < k$, the seller holds a fire sale for $k - k'$ units, so she obtains

$$\Pi_k(t) = (k - k')v_L + \Pi_{k-k'}(t) \ . \tag{12}$$

Finally, as in the two-unit case, t_k^* is found by ensuring that the seller does not have the incentive to change the timing of a fire sale when the remaining stock of units is k. As in Section 3.2, if $t_k^* > 0$, this implies the smooth-pasting condition – that is, the equilibrium profit function for k units is smooth at t_k^*:

$$\lim_{t \searrow t_k^*} \Pi_k'(t) = \lim_{t \nearrow t_k^*} \Pi_k'(t) = \Pi_{k-1}'(t_k^*) \ . \tag{13}$$

If no such $t_k^* > 0$ exists, then $t_k^* = 0$.

The proof of Proposition 3 shows that the equilibrium can be solved recursively. For each $k \geq 2$, having found the unique set of price and profits continuous functions and fire sale times $(p_{k'}, \Pi_{k'}, t_{k'}^*)_{k'=1}^{k-1}$ satisfying equations (9)-(13), we can find a unique (p_k, Π_k, t_k^*) satisfying the same equations.

Remark 1. *In our model, only the rate of arrival of buyers sets the pace. The reason is that, at a given instant t, the relevant state variables for both a the seller and the buyers are the stock K_t and aggregated arrival of buyers from t to 1, that is, $(1-t)\lambda$. So, if the arrival rate of buyers increases from λ to $\lambda' > \lambda$ and if k is such that $t_k^* > 0$ for λ, then its new value under λ', $t_k'^*$, satisfies $\lambda(1-t_k^*) = \lambda'(1-t_k'^*)$.*[16] *As a result, fire sales get close to the deadline when more buyers arrive over time. The rationale for this result is that when the expected demand increases, the incentive to hold a fire sale at a given time is lowered so, in equilibrium, the seller postpones it.*

3.4 The Role of Competition of Consumers in Fire Sales

In our model, the access to a broader market in a fire sale has a dual role. First, it serves as an "outside option" to the seller, as he can immediately sell some units (at a low price). Second, it increases the buyers' willingness to pay, as they know that obtaining the good at a fire sale is not guaranteed. To further understand the importance of the broader market, this section studies some comparative statics results for β.

When β is large. First, we consider the case where β is large. This is the case where, conditional on a fire sale, a buyer (if any) obtains the good with a very high probability. The following result claims that the seller sells most of the goods through an initial fire sale due to his inability to commit to future high prices:

[16] More generally, if the arrival rate changes to a potentially time-dependent function $\lambda(\cdot)$, the time variable can be "stretched" so that, under the normalized time, the arrival is constant. So, the strategies of the seller and the buyer in a model with arrival rate $\lambda(\cdot)$ at state (K, t) are the same as in a model with fixed rate $\bar{\lambda}$ at state (K, s) with s satisfying $\int_t^1 \lambda(t)dt = \bar{\lambda}(1-s)$ and $\bar{\lambda}$ defined by $\bar{\lambda} = \int_0^1 \lambda(t)dt$.

Proposition 4. *There exists some $\bar{\beta} < 1$ such that, if $\beta > \bar{\beta}$, then $t_k^* = 0$ for all $k > 1$.*

Proposition 4 establishes that when buyers are very likely to obtain a good in a fire sale, the seller can only generate intertemporal competition for one unit. In equilibrium, the seller holds a fire sale at time 0 for $K - 1$ units and keeps only one unit after then. This is indeed incentive-compatible: if, for example, a buyer arrives at some time $t \in (0, 1)$ and $K_t = 2$, she believes that the seller is going to hold a fire sale immediately and therefore she is going to obtain the good with a high probability. Thus she has a very low willingness to pay. Even if λ is large (that is, if there is a high chance that more buyers are going to arrive in the future, and therefore there is a high intertemporal competition), this belief depresses the expected profit of the seller, so he has the incentive to hold fire sales at the beginning.

Proposition 4 is another consequence of the lack of commitment of the seller and the stochastic arrival of buyers. Assume, for the sake of contradiction, that $t_2^* > 0$ when $\beta = 1$. Consider a history such that $K_{t_2^* - \varepsilon} = 2$ for some $\varepsilon > 0$ small. In this case, a large value of β depresses the low willingness to pay of a buyer before a fire sale: she knows that if she rejects the offer she is very likely to obtain the good at a very low price. More formally, $p_2(t_2^* - \varepsilon) = v_L + O(\varepsilon)$ because, if a buyer arrives at $t_2^* - \varepsilon$ and rejects the offer, she almost surely obtains the good at t_2^* (as the probability that another buyer arrives in $(t_2^* - \varepsilon, t_2^*]$ is $O(\varepsilon)$). This implies that, if a buyer arrives in $(t_2^* - \varepsilon, t_2^*]$, the payoff of the seller is given by

$$v_L + e^{-\lambda(1-t_2^*)}\mathbb{E}\left[\, p_1(t) \,\middle|\, t > t_2^* \,\right] + \left(1 - e^{-\lambda(1-t_2^*)}\right) v_L + O(\varepsilon) \,, \tag{14}$$

where the expectation is over the time t when the next buyer arrives. The seller can instead hold a fire sale at $t_2^* - \varepsilon$. In this case, if a buyer arrives in $(t_2^* - \varepsilon, t_2^*]$, the payoff of the seller is $v_L + p_1(t_2^*) + O(\varepsilon)$, which is higher than the payoff in (14) for ε small enough, given that $p_1(\cdot)$ is decreasing. So, conditional on one buyer arriving in $(t_2^* - \varepsilon, t_2^*]$, the seller prefers holding a fire sale at $t_2^* - \varepsilon$ rather than at t_2^*. If, instead, no buyer arrives in $(t_2^* - \varepsilon, t_2^*]$, the deviation does not change the seller's payoff. Finally, it is very unlikely that two or more buyers will arrive in $(t_2^* - \varepsilon, t_2^*]$ (its probability is $O(\varepsilon^2)$). Then, the expected gain from deviating is positive, which shows that the initial assumption $t_2^* > 0$ is not valid.

In our model, $1 - \beta$ can be interpreted as "friction" from the buyer's perspective: it is the probability of not obtaining a good in a fire sale. In this sense, Proposition 4 has some resemblance with the outcome in the Coase conjecture literature (Gul, Sonnenschein, and Wilson 1986 and Ausubel and Deneckere 1989): when the frictions are low, most of the trade happens at time 0 at a low price. However, the implication of Proposition 4 differs from that of the Coase conjecture literature in the following aspects.

First, in our model, the seller still enjoys positive rent as $\beta \to 1$, as

$$\lim_{\beta \to 1} \Pi_K(0) = (K-1)v_L + \Pi_1^{\beta=1}(0) \ ,$$

where $\Pi_1^{\beta=1}(0)$ is the payoff of the firm when the seller initially holds one unit and $\beta = 1$ in equation (2). Although the seller does not obtain any rent from the first $K-1$ units, $\Pi_1(0) > v_L$ since $\lambda > 0$: buyers still face inter-temporal competition from future arriving buyers, so the price is higher than v_L before the deadline. This implies that the transaction of the last unit takes place when the first buyer arrives and, if no buyer arrives, at the deadline. In contrast, the Coase conjecture implies that, in the limit where the seller makes offers very frequently, there is no trade delay, and the seller's profit converges to the lower bound of the buyer's value.

Second, the welfare implications derived from Proposition 4 are significantly different from those obtained from the Coase conjecture. To get some intuition, suppose the demand from the boarder market come from shoppers whose valuation on the good is v_L. If β is large, the first $K-1$ goods are purchased by (low-value) shoppers, rather than arriving (high-value) buyers. Hence, in our model, there exists significant inefficiency owing to the misallocation of most of the goods. Remarkably, endowing the buyers with a bigger advantage in the fire sale (larger β) hurts, in equilibrium, both the seller and buyers.

When β is small. The next proposition considers the limit where β becomes small – that is, when the probability that a buyer obtains the good in a fire sale shrinks. It claims that, in this limit, the commitment problem of the seller disappears: he extracts all the rent from the buyers and holds fire sales only at the deadline.

Proposition 5. *As $\beta \to 0$, $t_K^* \to 1$ and $p_K(t) \to v_H$ for all $K \in \mathbb{N}$ and $t \in [0,1]$.*

The result is intuitive. In the limit where $\beta \to 0$, a buyer expects a zero continuation payoff by waiting, so her willingness to pay increases to v_H. As a result buyers behave as myopic players as in Gallego and Van Ryzin (1994), so the seller's profit achieves its first-best value.

The parameter β captures the likelihood that accumulated buyers notice a deal. Therefore, in practice, it is determined by multiple factors. For example, it depends on the effective demand from the boarder market, that is, the number of people who only consider purchasing the good if it is cheap enough. In the airline example, the demand from the broader market may be contributed by the leisure passengers who search for low price tickets for vacations, so one can expect β to be lower in the vacation seasons. From Proposition 4, in this case, the regular price tends to be high and fire sale offers are made only when the deadline is close. Also, the value of β is likely to depend on the third party's ability (technology) to attract accumulated buyers' attention. Nowadays, it has become easier for price aggregators to track accumulated buyers (window shoppers) by using "cookies", so they can send deal-alert emails or text messages, or bid on personalized online advertisements in real time. Therefore, one should expect that β has become higher with the development of information technology.[17] The fact that, by Proposition 5, the seller benefits from low values of β rationalizes the observation in the airline industry that deal alert services are only provided by some third parties instead of airlines, even though they could do it at a lower cost.

4 Further Discussion

This section discusses some possible extensions of our model.

4.1 Multiple Types

We assume that arriving buyers have the same valuation for the good, v_H, so they only differ because of their private histories. This assumption ensures that buyers purchase immediately upon arrival, which is necessary to keep (t, K_t) as the only payoff-relevant variable because, in equilibrium, the seller does not have any payoff-relevant private information. Our no-accumulation result is robust to perturbing the heterogeneity in buyers' value: as long as the valuations of the buyers are not too spread, there is a Markov equilibrium where the regular price is immediately accepted by all types of buyers. The following result states that, as long as the buyers' valuation is bounded away from v_L and is not too sparse, there are equilibria as described in section 3.

Proposition 6. *Fix $\underline{v_H} > v_L$. Then, there exists some $\overline{v_H}^* > \underline{v_H}$ such that if the types of buyers are uniformly distributed in $[\underline{v_H}, \overline{v_H}]$ for $\overline{v_H} \in (\underline{v_H}, \overline{v_H}^*)$ then there is an equilibrium without accumulation as described in Proposition 3.*

The logic behind Proposition 6 is the following. In a static model where a monopolist makes a take-it-or-leave-it offer to a buyer, if the buyer's valuation is uniformly distributed in $[\underline{v_H}, \overline{v_H}]$ and $\overline{v_H} \leq 2\underline{v_H}$, it is optimal for the seller to charge a price equal to $\underline{v_H}$. Indeed, the decrease in the probability of trade that increasing the price above $\underline{v_H}$ generates is high enough that the monopolist prefers to ensure trade by setting the price equal to $\underline{v_H}$. The intuition is similar

[17] Öry (2017) assumes that it is costly to attract accumulated buyers' attention and argues that such a cost has been lowered with the development of the Internet. In this sense, her cost of attracting accumulated buyers has the similar flavor as the $1 - \beta$ in our model: they capture inefficiencies that help the seller to increase his commitment power and, as a result, the price.

in our dynamic model: increasing the price above the reservation price of the buyers with the lowest valuation effectively implies losing them as potential buyers, since they do not recheck the price in the future. Even though increasing the price increases the revenue if a buyer with a high valuation arrives, as in the static model, the negative effect dominates the positive one. This result can be generalized to distributions of types with a support bounded away from v_L and with a probability density function bounded away from 0.

A higher buyer heterogeneity would imply that, at any given instant, there would be a history-dependent stochastic stock of "accumulated buyers," which would affect the continuation strategy of the seller and the (private-history dependent) continuation strategy of the buyers, making the model intractable. However, the key incentive highlighted by our paper is likely to remain in a more general type space for the buyers. Intuitively, when the stock is high, a low price increases the incentive to further lower the price (or to hold a fire sale) to lower the stock soon and increase the price. So, the lack of commitment of the seller is likely to accelerate the price decay at some isolated times, and generate price dynamics similar to the ones in our model.[18]

4.2 More on Inattention Frictions

We assume that buyers "freely" check the regular offer at their arrival time to ensure that, in equilibrium, some goods are sold through regular offers. There are other assumptions that give similar results. For example, one can study a model model where buyers get the opportunity to check for free the price at an arbitrarily large number of times. From the proofs of Lemmas 1 and 2, it is easy to see that the unique equilibrium in our model remains an equilibrium. In the equilibrium, buyers decide to check the price at their arrival time.[19]

4.3 Observability of Inventory

We assume that the remaining inventory size K_t is observable. This is necessary to avoid the usual signaling problem: if the seller has payoff-relevant private information, he can use prices to signal this information. This results in the usual multiplicity of equilibria and off-the-equilibrium-path beliefs. It is worthwhile to point out, however, that a simple unraveling argument shows that if it is costless and verifiable, the seller always reveals the remaining stock.

In practice, buyers may observe some imperfect but informative signals of the inventory in

[18] Notice that if $\underline{v}_H = v_L$ the discontinuous price drop implied in our model may not be present. See Biyalogorsky, Gerstner, and Libai (2001) for a discussion of a similar result in a different setting. We thank a referee for asking us to clarify this point.

[19] We thank the referee who suggests us to consider this alternative model.

some markets, providing the seller some incentive to hold a fire sale. For example, in the airline industry, airlines sometimes report online the number of remaining available seats. Even though the advertised number of available seats may not coincide with the actual inventory (airlines sometimes block some seats for elite passengers), it is an informative proxy. Escobari (2012) uses the number of available seats as a proxy for the real inventory and empirically studies the price patterns in the airline industry. He finds that the price significantly increases as the number of available seats decreases. This suggests that the number of available seats is a signal of inventory which is interpreted as credible. Recently, Williams (2015) analyzes a new airline dataset that allows one to distinguish between blocked and occupied seats. He finds that seat maps observed by buyers are a useful proxy for bookings even though they overstate the latter by 10%.

4.4 Creating Scarcity by Discarding Inventory

In our model, the seller sometimes finds it optimal to create scarcity by holding fire sales. Holding a fire sale is good for the seller because it reduces the available stock and allows him to increase the price in the future. Nevertheless, since the seller cannot commit to the timing of the fire sales and buyers may obtain goods through them, the prospect of future fire sales lowers the reservation value of the buyers and also their willingness to pay. Thus, after some histories, the seller would like to commit not to sell some of the units (or, equivalently, discarding them) instead of holding fire sales, which would allow him to effectively lower the stock without lowering the willingness to pay of the current buyers.

Our results do not change if the seller has access to other mechanisms (in addition to fire sales) that allow him to lower the stock of goods as long as they generate a lower revenue. Consider, for example, adding to our base model the possibility that the seller reduces the stock obtaining a revenue $r < v_L$ from each unit coming, for example, from giving them away in special promotions. Given that our seller lacks commitment power and has access to the market of shoppers, if there were an equilibrium where, after some histories, the seller would give away some goods, he would rather hold fire sales as, because $v_L > r$, he obtains a higher revenue.

4.5 Returned Goods and Overbooking

One of the reasons that fire sales can serve as an endogenous commitment device is that transactions are irreversible in our baseline model: the inventory K_t weakly declines over time. In reality, however, there exist mechanisms that allow K_t to occasionally increase.

One such possibility is that buyers may experience regret after having bought the good and request to return the good. Imagine that returns may occur with a positive probability. In such an event, K_t increases. Our model and equilibrium can easily accommodate the return goods as long as the probability of a good being returned is independent of the value that its buyer attached to it (at the purchase time). The reason is that, in this case, a Markov equilibrium with (t, K_t) as the state variable remains: if a unit is returned when the stock is large, the seller immediately holds a fire sale to get rid of the extra unit. Note that the possibility of these stock increases (and corresponding random fire sales) lowers the buyers' willingness to pay; as a result, the fire sale as an endogenous commitment device is less effective.

When the seller is allowed to overbook and buy back sold goods, K_t may also go up. In this case, the seller may endogenously buy back sold goods from low-type buyers and sell it to high-value buyers. See Ely, Garrett, and Hinnosaar (2017) for an interesting discussion of this topic. Obviously, in this case, the fire sale is less effective in increasing the buyers' willingness to pay, but the mechanism remains as long as such reallocation is sufficiently costly.[20]

5 Conclusions

This paper studies the role of commitment in revenue management with strategic arriving buyers. When sales are low, the seller finds it optimal to hold fire sales to lower the inventory and to increase future prices. Still, as buyers anticipate the possibility of obtaining a good at a bargain price, the price decreases before fire sales, and jumps up immediately afterwards. The lack of commitment by the seller, then, provides a new channel to generate price fluctuations and fire sales. This insight can contribute to our understanding of the price fluctuations in industries such as airlines, cruise lines and hotel services.

[20]In fact, the overbooking exercise is very costly in the airline industry. When a passenger is involuntarily bumped, the airline is required to pay four times the cost of the one-way fare.

A Formal Model

We present here the formal version of our model.

Players: The players in the model are a seller and a unit mass of buyers, indexed by $b \in [0,1]$, which denotes their arrival time.

Strategy of the buyers: For each t, a buyer b's history h_t^b, is a finite (potentially empty) sequence of attention times $b = \tau_1^b < ... < \tau_N^b < t$, for some $N \in \{0\} \cup \mathbb{N}$, and the corresponding prices, capacity controls, stocks and decision to accept the offer or not, so $h_t^b = (\tau_n^b, P_{\tau_n^b}, Q_{\tau_n^b}, K_{\tau_n^b}, \tilde{A}^b_{\tau_n^b})_{n=1}^N$, where $\tilde{A}^b_{\tau_n^b} = 1$ if the buyer accepted the offer at time τ_n^b (but no good was assigned to him), while $\tilde{A}^b_{\tau_n^b} = 0$ is interpreted as the buyer did not accept it. Notice that, if $b \geq t$ then the sequence is necessarily empty, $N = 0$ (so $h_t^b = \emptyset$), meaning that the buyer has still yet to arrive. Also, $N = 0$ (or $h_t^b = \emptyset$) for a buyer $b < t$ is interpreted as "buyer b did not arrive" or, equivalently, "no buyer arrived at time b". A buyer's strategy is a map from each time t and non-empty buyer's history $h_t^b \equiv (\tau_n^b, P_{\tau_n^b}, Q_{\tau_n^b}, K_{\tau_n^b}, \tilde{A}^b_{\tau_n^b})_{n=1}^N$ to

1. a re-checking decision $RC_t^b(h_t^b) \in \{0,1\}$, where $RC_t^b(h_t^b) = 0$ denotes that the buyer does not recheck and $RC_t^b(h_t^b) = 1$ denotes that she does, with the restriction that $RC_t^b(\emptyset) = 0$ if $b \neq t$ (no re-check if no arrived) and $RC_t^b(\emptyset) = 1$ when $t = b$ (check the price upon arrival), and,

2. if $RC_t^b(h_t^b) = 1$, a decision to accept the price of the good or not for each possible price and remaining stock (P_t, Q_t, K_t), $A_t^b(P_t, Q_t, K_t; h_t^b) \in \{0,1\}$, where $A_t^b(P_t, Q_t, K_t; h_t^b) = 0$ indicates that the price is rejected, while $A_t^b(P_t, Q_t, K_t; h_t^b) = 1$ indicates that the price is accepted.

In order to ensure that the payoff for each buyer is finite for all strategies of the seller and the other buyers, we add the condition that a strategy of a buyer generates a finite number of re-checking times independently of the realized path $(P_t, Q_t, K_t)_{t=0}^1$.[21]

A strategy of a buyer is Markov if the re-checking decision at time t and history $h_t^b \equiv (\tau_n^b, P_{\tau_n^b}, Q_{\tau_n^b}, K_{\tau_n^b}, \tilde{A}^b_{\tau_n^b})_{n=1}^N$, $rc_t^b(h_t^b)$, only depends on t and $(\tau_N^b, K_{\tau_N^b})$, and also $A_t^b(P_t, Q_t, K_t; h_t^b)$ only depends on (P_t, t, K_t), so there exist two functions $rc : [0,1] \times [0,1] \times \{0, ..., K\}$ and $a : \mathbb{R}_+ \times [0,1] \times \{0, ..., K\}$ so $RC_t^b(h_t^b) = rc(t, \tau_N^b, K_{\tau_N^b})$ and $A_t^b(P_t, Q_t, K_t; h_t^b) = a(P_t; t, K_t)$.

Strategy of the seller: For a fixed strategy of the buyers, the seller solves a dynamic optimization (or optimal control) problem. Given that the seller has full control on the prices and

[21]For example, requiring two checking times to be separated by at least $\varepsilon > 0$, for any $\varepsilon > 0$ small, is enough to guarantee that this condition is satisfied.

quantities offered, a (pure) strategy of the seller is formed by a price, a quantity control and a fire-sale decision as a function of the current time and the previous times of sale and units sold.

For each t, a seller's history h_t^s is a *finite* sequence $((\tau_1^s, R_1, F_1), ..., (\tau_N^s, R_N, F_N))$, for some $N \in \{0\} \cup \mathbb{N}$, with $(\tau_n^s)_n$ being a strictly increasing sequence of sales times, with $\tau_N^s < t$, which we interpret as the times at which regular sales happened before time t, and where, for all n, $R_n, F_n \in \{0\} \cup \mathbb{N}$ are interpreted, respectively, as the number of sales through the regular price and a fire sale at time τ_n, and satisfy $R_n + F_n \geq 1$ and $\sum_{n=1}^{N}(R_n + F_n) \leq K$.[22] Then, a strategy of the seller is a map which associates each time t and seller's history h_t^s to

1. a price $P_t(h_t^s) \in \mathbb{R}$,
2. a capacity control $Q_t(h_t^s) \in \{0, ..., K\} \to \{0, ..., K\}$ and
3. a contingent number of fire sales, $F_t(\cdot; h_t^s) : \{0, ..., K\} \to \{0, ..., K\}$, where $F_t(R; h_t^s)$ denotes the number of units offered in a fire sale if R units are sold at the regular price at time t, and which satisfies that $\lim_{t' \searrow t} F_{t'}(0; \hat{h}_{t'}^s(h_t^s, t, R)) = 0$ for any R, where

$$\hat{h}_{t'}^s(h_t^s, t, R) \equiv \big(\underbrace{(\tau_1^s, R_1, F_1), ..., (\tau_N^s, R_N, F_N)}_{=h_t^s}, (t, R, F_t(R, h_t^s))\big)$$

is the history where R and $F_t(R; h_t^s)$ goods are sold at t through, respectively, a regular and a fire sale.[23]

For a time t and a seller's history $h_t^s \equiv (\tau_n^s, R_n, F_n)_{n=1}^N$, we let $K_t(h_t^s)$ denote the number of units not sold until t, that is, $K - \sum_{n=1}^{N}(R_n + F_n)$. We say that a strategy is Markov if for any t and seller histories $h_t^s, h_t'^s$ such that $K_t(h_t^s) = K_t(h_t'^s)$, we have $p_t(h_t^s) = p_t(h_t'^s)$, $Q_t(h_t^s) = Q_t(h_t'^s)$ and $F_t(h_t^s) = F_t(h_t'^s)$. This implies that there exist three functions $p : [0, 1] \times \{0, ..., K\} \to \mathbb{R}$, $q : [0, 1] \times \{0, ..., K\} \to \mathbb{R}$ and $f : \{0, ..., K\} \times [0, 1] \times \{0, ..., K\}$ we can write $P_t(h_t^s) \equiv p(t, K_t(h_t^s))$, $Q_t(h_t^s) \equiv q(t, K(h_t^s))$ and $F(\cdot; h_t^s) = f(\cdot; t, K(h_t^s))$.

Payoffs: For a given strategy of the seller and the buyers, there is a distribution over outcomes, that is, over total (and terminated) histories of the game. Fix a strategy for the buyers and the seller. A total history of the game consistent with strategy profile is:

[22] In our specification, for a fixed seller's strategy and realization of the seller's history $h_t^s \equiv (\tau_n^s, R_n, F_n)_{n=1}^N$, the offer process of the seller is uniquely defined, and therefore there is no need to condition his current decision on it. So, the fact that the previous history of sales is finite-dimensional avoids the well-known continuous-time problem on the uniqueness the outcome of the game for a given strategy profile (see Bergin and MacLeod (1993)).

[23] The condition ensures that, after any given history, the time where the next fire sale happens (if not buyer arrives) is well defined, that is, the remaining stock *after regular and fire sales* is almost surely right continuous.

- **Times of sales:** two finite sets of times $\tau^s \equiv (\tau_i^s)_{i=1}^I$ and $\hat{\tau}^s \equiv (\hat{\tau}_j^s)_{j=1}^J$ when regular and fire sales happened, respectively. Let R_t and F_t denote the number of regular and fire sales at time t, respectively, and $h_t^s \equiv (\tau_t^s | \tau_t^s < t)$.
- **Offers:** a triple of functions (p_t, K_t, Q_t) such that $p_t = p_t(h_t^s)$, $K_t = K_t(\tau_t^s)$ and $Q_t = Q_t(h_t^s)$, for all t.
- **Buyers:** a finite set of time of birth of buyers, $(\theta_1, ..., \theta_M) \in [0,1]^M$ for some $M \in \{0\} \cup \mathbb{N}$, trade times $(t_1, ..., t_M) \in ([0,1] \cup \{\infty\})^M$, with $t_m \geq \theta_m$ for all m and where $t_m = \infty$ indicates that buyer m did not purchase a good, and the type of trade, $(d_1, ..., d_M) \in \{0,1\}^M$, where $d_m = 1$ means that the buyer obtained the good at a regular price, while $d_m = 0$ means that she got it at a regular price.
- **Checking times:** a set of checking times for each buyer θ_m, $(\tau_n^m)_{n=1}^{N_m}$, for some $N_m \in \mathbb{N}$, where $\tau_1^m = \theta_m$ (checking upon arrival), $\tau_n^m = \min\{t | RC_t((\tau_{n'}^m, P_{\tau_{n'}^m}, Q_{\tau_{n'}^m}, K_{\tau_{n'}^m}, \tilde{A}_{\tau_{n'}^m}^m)_{n'=1}^{n-1}) = 1\}$ (consistency with the strategy) and where $\tau_n^{N_m} \leq t_m$ (no re-checks after trade). We use h_t^m to denote the checking times before t and the corresponding values of p, Q and K.
- **Regular sales:** at each moment in time, $R_t = \min\{K_t, Q_t, |\{m | A^m(P_t, Q_t, K_t; h_t^m) = 1\}|\}$.
- **Fire sales:** at each moment in time, $F_t = \min\{K_t - R_t, F_t(h_t^s, R_t)\}$,

The payoff that the seller seller obtains from a total history of the game is

$$V_s = \sum_{i=1}^{I} p_{\tau_i^s} + J v_L \ .$$

Also, for a buyer with $m = 1, ..., M$,

$$V_b^m = \begin{cases} v_H - (d_m p_{t_m} + (1-d_m) v_L) - (N_m - 1)c & \text{if } t_m < \infty, \\ -(N_m - 1)c & \text{otherwise.} \end{cases}$$

B Proofs

B.1 Proof of Results prior to Section 3.2

The Proof of Lemma 1. Fix an equilibrium and a Markov state (t, K_t). Let $V(t, K_t)$ be the buyers' expected continuation value from rejecting an offer. Then, it is optimal for a buyer to accept a price strictly below $V(t, K_t)$ and reject a price strictly above $V(t, K_t)$. As a result, since the price offer does not affect the buyers' continuation strategy (conditional on rejecting the offer), the standard pricing argument implies that if the equilibrium offer at state (t, K_t) is accepted by attentive buyers, such an offer is necessarily equal to $\bar{p}_{K_t}(t) \equiv v_H - V(t, K_t)$. Therefore, never purchasing the good at attention times (and waiting to obtain it in a fire sale) is an optimal best response independently of the re-checking strategy.[24] This further implies that re-checking is suboptimal, so if a buyer does not purchase the good at her arrival time, she only obtains it in a fire sale. $\square$

The Proof of Lemma 2. Fix an equilibrium and, for any two values t^1, t^2 such that $0 \leq t^1 \leq t^2 \leq 1$, let $B_{[t^1,t^2]} \in [0, t^2 - t^1]$ be the measure of times between t^1 and t^2 where, in the equilibrium, the regular price offer is accepted by attentive buyers when there is one unit left. Formally, $B_{[t^1,t^2]}$ is defined as

$$B_{[t^1,t^2]} \equiv \mu\big(\{\, t \in [t^1, t^2] \,\big|\, a(p(t,1); t, 1) = 1 \,\}\big) \ ,$$

where μ is the Lebesgue measure in $\mathbb{R}$, and a and p are the equilibrium Markov acceptance and pricing strategies established in Appendix A. It will also be useful to use the following result:

Lemma B.1. *In any equilibrium, if $K_t = 1$, the seller never holds a fire sale at time t if $t < 1$, and holds a fire sale at time $t = 1$ if the good is not sold through at the regular price.*

Proof. We begin by noting that, in any equilibrium, if at the deadline the seller holds a fire sale for the goods that have not been sold. Formally, in terms of the formal strategies defined in Appendix A, we have that $f(R_1; 1, K_1) = K_1 - R$ for all stock levels $K_1 \in \{1, ..., K\}$ and regular sales $R_1 \in \{0, ..., K_1\}$. This is because, a fire sale offers the seller a payoff of $v_L > 0$ per unit, while the payoff per unsold unit is 0.

Notice now that, at any time $t < 1$ with $K_t = 1$, a buyer accepts for sure an offer equal to $\underline{p} \equiv \min\{v_L + c, v_H - \beta(v_H - v_L)\} > v_L$. The reason is that rejecting such an offer implies either waiting until a fire sale takes place (and obtaining an expected payoff of at most $\beta(v_H - v_L)$) or

[24] Note that it does not immediately imply that the value of re-checking is 0: a buyer who rechecks obtains information about the Markov state and, as a result, information about when it is best to re-check again.

paying c at some time in the future and receiving a price offer no lower than v_L.[25] In consequence, the seller never holds a fire sale before the deadline when $K_t = 1$. To see why assume, for the sake of contradiction, that there is an equilibrium where the seller holds a fire sale at some time $t < 1$, so $f(0; t, 1) = 1$. In this case, the seller can deviate to not holding a fire sale at time t and posting a price equal to $\underline{p}$ until the deadline (and then holding a fire sale). This provides him an expected payoff higher than v_L, so we have a contradiction. $\square$

(Proof of Lemma 2 continues.)

Monotone Reservation Price When $K_t = 1$. Fix a pair t^1, t^2 such that $0 \leq t^1 \leq t^2 \leq 1$, Notice that the probability that no buyer arrives in a time when an acceptable price is offered between t^1 and 1 is $e^{-\lambda(B_{[t^1,t^2]}+B_{[t^2,1]})}$. So, the reservation value of a buyer arriving at t^1 is

$$e^{-\lambda(B_{[t^1,t^2]}+B_{[t^2,1]})}\beta_{t^1}(v_H - v_L) \ , \tag{15}$$

where β_{t^1} is the buyer's believed probability to obtain the good at the final fire sale (which, by Lemma B.1, takes place if the unit has not been sold before) conditional on reaching the deadline and conditional on $K_{t^1} = 1$. This depends, in general, on the belief about the previous history and, in particular, about the number of accumulated in the past. In the formula, $e^{-\lambda(B_{[t^1,t^2]}+B_{[t^2,1]})}$ is the probability that no buyer arrives in $[t^1, 1]$ at a time where the price is acceptable.

Consider a buyer who, having arrived at time t^1, having observed $K_{t^1} = 1$, and having rejected the price, rechecks (by deviating) at time t^2 and observes that $K_{t^2} = 1$. Her continuation value at time t^2 is

$$e^{-\lambda B_{[t^2,1]}}\beta_{t^1,t^2}(v_H - v_L) \ , \tag{16}$$

where β_{t^1,t^2} is her new believed probability to obtain the good at the final fire sale conditional on the good not having been sold through the regular price. Notice that $\beta_{t^1,t^2} = \beta_{t^1}$. This is the case because, from time t^1 perspective, we have that the probability of obtaining a good

$$\underbrace{e^{-\lambda(B_{[t^1,t^2]}+B_{[t^2,1]})}}_{\Pr(\text{no sale in } [t^1,1])}\beta_{t^1} = \underbrace{e^{-\lambda B_{[t^1,t^2]}}}_{\Pr(\text{no sale in } [t^1,t^2])}\underbrace{e^{-\lambda B_{[t^2,1]}}}_{\Pr(\text{no sale in } [t^2,1])}\beta_{t^1} \ .$$

Given that arrivals are independent, the probability of obtaining a good after t^2 conditional on no sale in $[t^1, t^2]$ is just $e^{-\lambda B_{[t^2,1]}}\beta_{t^1}$, and therefore $\beta_{t^1,t^2} = \beta_{t^1}$. As a result, since $B_{[t^2,1]} \leq B_{[t^1,1]}$, the reservation value at t^2 is weakly lower than the reservation value at time t^1. Using the notation in the proof of Lemma 1, $\bar{p}_1(t^2) \leq \bar{p}_1(t^1)$.

[25]Notice that since the seller always can sell the good at a price v_L through a fire sale, posting a regular price strictly below v_L is strictly suboptimal.

No Accumulation When $K_t = 1$. Suppose that after a history at time $t^1 < 1$ where there is only one unit left, $K_{t^1} = 1$, and there is accumulation in $[t^1, 1]$, that is, $B_{[t^1,1]} < 1 - t^1$. As we showed before, the reservation price of a buyer arriving in $t \in [t^1, 1]$ is necessarily (weakly) decreasing in t. So, it is a profitable deviation for the seller to offer $\bar{p}_1(t)$ for all $t \in [t^1, 1]$. Indeed, the deviation strictly increases the probability of selling the good at a regular price and, conditional on at least one buyer arriving, it ensures that the transaction price is, at least, the price at which the good is sold if there is no deviation. So, when there is only one unit left, the seller never accumulates buyers, that is, $B_{[t^1,1]} = 1 - t^1$.

Notice that, for any $t \in (0, 1]$, if a buyer observes $K_t = 1$, then she believes that, in equilibrium, there is not going to be accumulation in $[t, 1]$. Given that there is no accumulated buyer at time 0, the belief about the number of accumulated buyers has to converge to 0 as $t \searrow 0$. Since the continuation payoff for a buyer at any $t \in [0, 1]$ has to be independent of her private history, then necessarily any buyer arriving at any $t \in [0, 1]$ and observing $K_t = 1$ believes, in equilibrium, that there is no accumulated buyer.

No Accumulation When $K_t > 1$. Finally, to close the argument, assume that there is accumulation for some history leading to Markov state (t, K_t). If a buyer arrives at time t and observes K_t, she has some non-degenerated belief about the accumulated number of buyers. Still, if she re-checks the offer at time $t' > t$ and observes $K_{t'} = 1$, her reservation price at t' is higher than the reservation price of an arriving buyer (who believes that there is no accumulated buyer), which contradicts the Markov assumption. Therefore, in equilibrium, there is no accumulation at any history.

As argued in the proof of Lemma 1, the seller charges the reservation price of the arriving buyers. □

The Proof of Proposition 1. The proof is immediate given Lemma B.1 (in the proof of Lemma 2) and the arguments in the main text. □

B.2 Proof of Results in Section 3.2

The Proof of Lemma 3 and Proposition 2. In the proof, we characterize the necessary conditions that an equilibrium satisfies if it exists. Then we verify these conditions are, indeed, satisfied by a unique equilibrium. Notice that, in any equilibrium, if $K_t = 1$ at some time $t \in [0, 1]$, then the continuation play is given by Proposition 1. Indeed, by Lemma 2 there is no accumulation in equilibrium, so the reservation value of a buyer arriving at t and observing $K_t = 1$ satisfies equation (1), and therefore the seller charges $p_1(t)$.

For each time t we use $\tau_2(t) \geq t$ to denote the next time where a fire sale occurs if $K_t = 2$ and there is no regular sale, that is, $\tau_2(t) \equiv \inf\{t' \geq t | f(0; t', 2) > 0\} \in [t, 1] \cup \{\infty\}$, where we use the convention $\inf \emptyset = \infty$.[26] Notice finally that if $\tau_2(t) > t$, then $\tau_2(t') = \tau_2(t)$ for all $t' \in [t, \tau_2(t)]$.

We divide the proof in 3 steps. The first sets some properties that τ_2 has in an(y) equilibrium. The second step establishes the uniqueness of an equilibrium τ_2 and, as a result, a unique equilibrium. The third step proves that the profit function satisfies the smooth pasting condition.

Step 1. *In any equilibrium, for any $t < 1$, if $\tau_2(t) < 1$ then, $f(0; \tau_2(t), 2) = 1$, that is, there is never a fire sale for two units. Also, $f(0, 1, 2) = 2$.*

Proof. First, notice that using an argument similar to the proof in Lemma B.1, at the deadline, if $K_1 = 2$ and there is no transaction at the regular sale stage, then the seller holds a fire sale for two units, so $f(0, 1, 2) = 2$.

Suppose that there is an equilibrium in which $\tau_2(t) = 1$ for some $t < 1$. By Lemma 1, if $K_t = 2$ at time t, a buyer who decides to wait expects a continuation value:

$$\underbrace{e^{-\lambda(1-t)}\beta_2(v_H - v_L)}_{\text{no buyer arrives before deadline}} + \underbrace{e^{-\lambda(1-t)}\lambda(1-t)\beta(v_H - v_L)}_{\text{exactly one buyer arrives before deadline}}$$

where β_2 is, as in the main text, the probability that an accumulated buyer of obtains a good if there is a two-unit fire sale if there is no other accumulated buyer. In equilibrium, the buyer is indifferent between waiting and accepting the price posted by the seller, which is given by

$$p_2(t; 1) = v_H - e^{-\lambda(1-t)}\beta_2(v_H - v_L) - e^{-\lambda(1-t)}\lambda(1-t)\beta(v_H - v_L) \ .$$

Hence, the payoff of the seller at time t is then given by

$$\Pi_2(t; 1) \equiv \int_t^1 (p_2(t'; 1) + \Pi_1(t'))e^{-\lambda(t'-t)}\lambda \mathrm{d}t' + e^{-\lambda(1-t)}2v_L \ .$$

where Π_1 is defined in equation (2). Differentiating the previous expression minus $\Pi_1(t)$ with respect to t and evaluating it at $t = 1$ yields

$$\begin{aligned}\left.\frac{\mathrm{d}(\Pi_2(t;1) - \Pi_1(t))}{\mathrm{d}t}\right|_{t=1} &= \lambda\big(-p_2(1;1) - \Pi_1(1) + e^{-\lambda(1-t)}2v_L\big) - \lambda(v_L - p_1(1))\big) \\ &= \lambda\big(p_1(1) - p_2(1;1)\big) > 0 \ .\end{aligned}$$

[26]Recall that $f(Q; t, K)$ (defined in Appendix A) indicates the amount of units in a fire sale at time t if Q units are sold through a regular price at this time and $K_t = K$. Notice that, given the condition on the right-limit of F_t in its definition, we have that, in fact, the infimum is a minimum.

The second equality holds because $\Pi_1(1) = v_L$ and $\Pi_1'(1) = \lambda(v_L - p_1(1))$, and the formula is strictly positive because

$$p_2(1;1) = \beta_2(v_H - v_L) = (\beta + \beta(1-\beta))(v_H - v_L) < \beta(v_H - v_L) = p_1(1) \ .$$

This generates a contradiction, since $\Pi_2(t-\varepsilon;1) - \Pi_1(t-\varepsilon) < v_L$ for $\varepsilon > 0$ small enough, but then the seller prefers to hold a fire sale at $1-\varepsilon$ than waiting (see Figure 1, where $\Pi_2(t-\varepsilon;1)$ is referred to as $\tilde{\Pi}_2$).

Finally, to prove that for any $t < 1$, if $K_{\tau_2(t)} = 2$ then $f(0;\tau_2(t),2) = 1$, assume for the sake of contradiction that $f(0;\tau_2(t),2) = 2$ for some $t < 1$. Notice that, if $K_{\tau_2(t)} = 2$, the payoff of the seller at $\tau_2(t)$ is $2v_L$. Still, if he deviates and holds only one fire sale, he obtains a payoff of $v_L + \Pi_1(\tau_2(t)) > 2v_L$, so we reached a contradiction. □

Step 2. *There is a unique equilibrium. In this equilibrium,* $\tau_2(t) = \max\{t, t_2^*\}$ *for some* $t_2^* \in [0,1)$.

Proof. Suppose that $K_t = 2$ for some $t < 1$. Recall that the seller offers one unit at the next fire sale time $\tau_2(t)$, so a buyer's continuation value is given by

$$\overbrace{e^{-\lambda(1-t)}(\beta + (1-\beta)\beta)(v_H - v_L)}^{\text{no buyer arrives before deadline}} + \overbrace{e^{-\lambda(\tau_2(t)-t)}\lambda(\tau_2(t)-t)e^{-(1-\tau_2(t))\lambda}\beta(v_H - v_L)}^{\text{exactly one buyer arrives before } \tau_2(t) \text{ and no buyer after}}$$
$$+ \underbrace{e^{-\lambda(\tau_2(t)-t)}\beta(v_H - v_L)(1 - e^{-(1-\tau_2(t))\lambda})}_{\text{no buyer arrives in } [t,\tau_2(t)] \text{ and one or more in } [\tau_2(t),1]} \ .$$

As before, we define $p_2(t;t_2)$ as the highest price that a buyer is willing to pay if the next fire sale is at $t_2 \geq t$ by setting the previous expression equal to $v_H - p_2(t;t_2)$. So

$$p_2(t;t_2) \equiv v_H - \beta(v_H - v_L)\left(e^{-\lambda(1-t)}(1 - \beta + \lambda(t_2 - t)) + e^{-\lambda(t_2 - t)}\right) \tag{17}$$

for all $t_2 \in [0,1]$ and $t \leq t_2$. Notice that $p_2(\cdot;t_2)$ has a bounded slope and it is continuous in $t \in [0, t_2]$.

Given t_2, the seller's payoff is

$$\Pi_2(t;t_2) \equiv \int_t^{t_2} (p_2(t',t_2) + \Pi_1(t'))e^{-\lambda(t'-t)}\lambda \mathrm{d}t' + e^{-\lambda(t_2-t)}(v_L + \Pi_1(t_2)) \tag{18}$$

for all $t_2 \in [0,1]$ and $t \leq t_2$. Notice that $\Pi_2(t_2;t_2) = \Pi_1(t_2) + v_L$. The left-derivative of $\Pi_2(t;t_2)$ at t_2 is as follows:

$$\lim_{t' \nearrow t_2} \frac{\mathrm{d}(\Pi_2(t';t_2) - \Pi_1(t'))}{\mathrm{d}t'} = \underbrace{(v_H - v_L)\lambda\big(\beta - 1 + e^{-(1-t_2)\lambda}(1 - \beta^2 + \beta(1-t_2)\lambda\big)}_{\equiv \phi_2(t_2)} \ . \tag{19}$$

Simple algebra shows that the derivative of the right-hand side of the previous expression with respect to t_2, $\phi_2'(t_2)$, equals 0 at $\hat{t}_2 \equiv 1 - \frac{\beta^2+\beta-1}{\beta\lambda}$, and the second derivative at this point is negative, so it is a maximum. By Step 1, $\phi_2(1) > 0$. So, there exists at most one $\hat{t}_2^* \in [0,1)$ such that $\phi_2(\hat{t}_2^*) = 0$. If $\hat{t}_2^* \in [0,1)$ exists, then $\phi_2(t_2) > 0$ for all $t_2 > \hat{t}_2^*$ and $\phi_2(t_2) < 0$ if $t_2 < \hat{t}_2^*$, while if $\hat{t}_2^*$ does not exist, then $\phi_2(t_2) > 0$ for all $t_2 \in [0,1]$. We study the two cases separately:

- **Case 1, $\phi_2(t_2) > 0$ for all $t_2 \in [0,1]$:** If $\phi_2(t_2) > 0$ for all $t_2 \in [0,1]$, we claim that, in the unique equilibrium, $\tau_2(t) = t$ for all $t \in [0,1]$, that is, the seller holds a fire sale immediately at any time t if $K_t = 2$ (by Step 1, for one unit when $t < 1$ and for two units when $t = 1$). To see this assume, for the sake of contradiction, that some time $t \in [0,1)$ exists such that $\tau_2(t) > t$. Nevertheless, in this case equation (19) indicates that $\Pi_2(t';\tau_2(t)) - \Pi_1(t')) < v_L$ for $t' < \tau_2(t)$ close enough to $\tau_2(t)$, so the seller would have a strict incentive to hold a fire sale before $\tau_2(t)$. Therefore, if an equilibrium exists, it is unique, and satisfies $\tau_2(t) = t$ for all t. To show existence notice that, if $\tau_2(t) = t$ for all t is an equilibrium, the reservation price of a buyer arriving at time t and observing $K_t = 2$ (given in (17)) now is

$$p_2(t;t) = v_H - \beta(v_H - v_L) + (1-\beta)\beta e^{-(1-t)\lambda}(v_H - v_L) \ . \tag{20}$$

So, the payoff of the seller from holding a fire sale at time $\hat{t}_2 > t$ instead of at time $t < 1$ is

$$\hat{\Pi}_2(t;\hat{t}_2) = \int_t^{\hat{t}_2} (p_2(t',t') + \Pi_1(t'))e^{-\lambda(t'-t)}\lambda \mathrm{d}t' + e^{-\lambda(\hat{t}_2 - t)}(v_L + \Pi_1(\hat{t}_2)) \ . \tag{21}$$

Notice that the main difference between the previous equation and (18) is the fact that now, in the integral term, we have $p_2(t',t')$ instead of $p_2(t',\tau_2(t))$. Given that $\hat{\Pi}_2(\cdot;\hat{t}_2)$ is differentiable, we can differentiate it and obtain that

$$\left.\frac{\partial \hat{\Pi}_2(t;\hat{t}_2)}{\partial \hat{t}_2}\right|_{\hat{t}_2 = t} = -\phi_2(t) < 0 \ . \tag{22}$$

Then, we see that the seller has no incentives to delay the fire sale, and therefore $\tau_2(t) = t$ is part if an equilibrium. Also, as we showed before, holding two fire sales is suboptimal, since $\Pi_1(t) + v_L > 2v_L$ for all $t < 1$.

- **Case 2, $\phi_2(t_2) = 0$ for some $t_2 \in [0,1]$:** Now assume $\phi_2(t_2^*) = 0$ for some (unique) $t_2^* \in [0,1]$, so $\phi_2(t_2) < 0$ for all $t < t_2^*$ and $\phi_2(t_2) > 0$ for all $t_2 > t_2^*$. We claim that, in the unique equilibrium, $\tau_2(t) = t_2^*$ for all $t < t_2^*$, and $\tau_2(t) = t$ for all $t \geq t_2^*$, that is, $\tau_2(t) = \max\{t, t_2^*\}$. The same argument as in Case 1 can be used to show that, in all equilibria, necessarily $\tau_2(t) = t$ for all $t \geq t_2^*$. So, it is only left to verify that, in all equilibria, $\tau_2(t) = t_2^*$ for all $t < t_2^*$, and that the seller does not have the incentive to deviate (by holding an early fire-sale).

Notice first that for any $t_2 \in [0, 1]$ and $t < t_2$, using some algebra, we have that

$$\frac{\partial}{\partial t}\left(e^{-\lambda t}(\hat{\Pi}_2(t; t_2) - \Pi_1(t) - v_L)\right) = e^{-\lambda t}\phi_2(t) < 0 \ .$$

As a result, it is clear that $\hat{\Pi}_2(t; t_2) > \Pi_1(t) + v_L$ for all $t < t_2 \leq t_2^*$.

Assume first that there is an equilibrium and $t < t_2^*$ such that $\tau_2(t) < t_2^*$. Notice first that we have $\Pi_2(t; \tau_2(t)) \geq \hat{\Pi}_2(t; t_2^*)$. Indeed, it is easy to verify, using equations (17) and (20), that for all $t' \in [t, t_2^*)$ we have that $p_2(t'; \tau_2(t')) \geq p_2(t'; t')$. This is intuitive: the reservation value of a buyer arriving at time t is maximal when she expects the first fire sale to be held immediately. Nevertheless, if offering the good at time $\tau_2(t)$ was part of an equilibrium, we would have both $\Pi_2(\tau_2(t); \tau_2(t)) = \Pi_1(\tau_2(t)) + v_L$ and $\Pi_2(\tau_2(t); \tau_2(t)) \geq \hat{\Pi}_2(\tau_2(t); t_2^*) > \Pi_1(\tau_2(t)) + v_L$, which is a contradiction. So, in any equilibrium, $\tau_2(t) = t_2^*$ for all $t < t_2^*$.

Now, fix $\tau_2(t) \equiv \max\{t, t_2^*\}$, and we want to show that the seller has no incentive to deviate. Take some time $t < t_2^*$. Notice first that, by the previous argument, we have $\Pi_2(t; t_2^*) \geq \hat{\Pi}_2(t; t_2^*)$. Then, $\Pi_2(t; t_2^*) \geq \hat{\Pi}_2(t; t_2^*) > \Pi_1(t) + v_L > 2v_L$, and therefore there is no incentive to deviate (to hold a fire sale for either one or two units at time t). As a result, in the unique equilibrium, $\tau_2(t) = t_2^*$ for all $t < t_2^*$.

□

Given that an equilibrium exists and it is unique, we use $\Pi_2(t)$ to denote the seller's expected payoff at time t if $K_t = 2$, which is $\Pi_2(t; t_2^*)$ if $t < t_2^*$, and $\Pi_1(t) + v_L$ if $t \geq t_2^*$. Notice that $\Pi_2(t)$ is continuous.

Step 3. *The smooth-pasting condition holds at t_2^*, that is, $\lim_{t \nearrow t_2^*} \Pi_2'(t) = \lim_{t \searrow t_2^*} \Pi_2'(t) = \Pi_1'(t_2^*)$.*

Proof. To conclude the proof, notice that when $t_2^* \in (0, 1)$, we have $\phi_2(t_2^*) = 0$. So, when approaching t_2^* from the left, we obtain

$$\lim_{t \nearrow t_2^*} \frac{\mathrm{d}(\Pi_2(t; t_2^*) - \Pi_1(t))}{\mathrm{d}t} = \phi_2(t_2^*) = 0 \ .$$

Furthermore, it is easy to obtain that, when t approaches t_2^* from the right (so the profits at t when $K_t = 2$ are $\Pi_2(t; t)$), we have that

$$\lim_{t \searrow t_2^*} \frac{\mathrm{d}(\Pi_2(t; t) - \Pi_1(t))}{\mathrm{d}t} = \lim_{t \searrow t_2^*} \frac{\mathrm{d}(v_L)}{\mathrm{d}t} = 0 \ .$$

Therefore, when $t_2^* \in (0, 1)$, the *smooth pasting condition* holds: the marginal change of Π_2 and Π_1 around t_2^* is the same. □

$\square$

B.3 Proof of Results in Section 3.3

The Proof of Proposition 3. We conduct the rest of our proof by induction, fixing $K \geq 3$. We assume that for all $k \leq K-1$ there exists a unique equilibrium as described in the statement of the proposition. We want then to prove that the result also applies to $k = K$. In particular, we will assume that for all $2 \leq k \leq K-1$ (and we will prove for K) there is some $t_k^* \in [0,1]$ satisfying $t_k^* < t_{k-1}^*$ if $t_{k-1}^* > 0$ and $t_k^* = 0$ if $t_{k-1}^* = 0$ such that:

- **Property 1:** If $K_t = k$ and $t \in [t_{k'}^*, t_{k'-1}^*)$ for some $1 \leq k' \leq k$, then the seller holds a fire sale for $k - k'$ units.
- **Property 2:** If $t \in [t_k^*, t_{k-1}^*)$ it is the case that $\Pi_{k'}(t) > \Pi_{k'-1}(t) + v_L$, for all $1 \leq k' < k$, and $\Pi_{k'}(t) = \Pi_{k'-1}(t) + v_L$ for all $k \leq k' \leq K-1$.
- **Property 3:** If $t \in [0,1]$, Π_k is differentiable at t and, if $t_k^* > 0$, $\Pi_k'(t_k^*) = \Pi_{k-1}'(t_k^*)$.

In the proof, we use $\tau_K(t) \geq t$ to denote the first time where there is a fire sale if $K_t = K$ in a hypothetical equilibrium (assuming it exists) and we give properties of it. Then, constructively, we prove that an equilibrium exists, that it is unique, that is such that there is some t_K^* such that $\tau_K(t) = t_K^*$ for $t < t_K^*$ and $\tau_K(t) = t$ for $t \geq t_K^*$. Finally, we prove that Properties 1, 2 and 3 hold for K.

Step 1. $\tau_K(t) < 1$ *for all* $t < 1$.

Proof. The first time where the seller holds a fire sale if $K_t = K$ is strictly below one. The argument is similar to the proof of Lemma 3. Assume now, for the sake of contradiction, that $\tau_K(t) = 1$ for some $t < 1$, so $\tau_K(t') = 1$ for all $t' \in [t, 1]$. Therefore, assume without loss of generality that $t \in (t_2^*, 1)$. If a buyer arrives and purchases one good, then the seller immediately holds a fire sale for $K-2$ units (as specified by the continuation play). So, the reservation value of a buyer at time $t \in (t_2^*, 1)$ is $p_K(t;1)$ satisfying

$$\begin{aligned} v_H - p_K(t;1) = {} & e^{-(1-t)\lambda}\beta_K(v_H - v_L) + e^{-(1-t)\lambda}(1-t)\lambda\big(\beta_{K-2} + (1-\beta_{K-2})\beta\big)(v_H - v_L) \\ & + \big(1 - e^{-(1-t)\lambda} - e^{-(1-t)\lambda}(1-t)\lambda\big)\beta_{K-2}(v_H - v_L) \end{aligned}$$

where, as in the main text, for each $k \leq K$ we use β_k to denotes the probability that a given buyer in the waiting state obtains at a fire sale where k goods are offered if she is the only buyer in the waiting state. Notice that if the buyer at time t rejects the offer and no other buyer arrives, she gets the good with probability β_K in the fire sale at the deadline. If, instead, exactly only one buyer arrives in $[t, 1]$, say at time t', the t-buyer gets a good with probability β_{K-2} at the time-t' fire sale and, if she does not, she gets the remaining good with probability β in the deadline fire sale, that is, she gets a good with a total probability $\beta_{K-2} + (1 - \beta_{K-2})\beta = \beta_{K-1}$. Finally, if more than two buyers arrive (and the first of them arrives at time t'), the t-buyer gets a good only with probability β_{K-2} at the time-t' fire sale. Then, the payoff of the seller at time t is then given by

$$\Pi_K(t; 1) \equiv \int_t^1 \big(p_K(t'; 1) + (K-2)v_L + \Pi_1(t')\big)e^{-\lambda(t'-t)}\lambda \mathrm{d}t' + e^{-\lambda(1-t)}Kv_L \ .$$

Then, proceeding analogously to the $K = 2$ case, we can differentiate $\Pi_K(t; 1) - \Pi_1(t) - (K-1)v_L$ with respect to t and obtain that the derivative is negative at $t = 1$, obtaining a contradiction. □

Step 2. $\tau_K(t) = t$ *for all* $t > t^*_{K-1}$ *and, if* $t \in [t^*_{k+1}, t^*_k)$*, then* $f(0; t, K) = K - k$*.*

Proof. For the sake of contradiction, assume $\tau_K(t) > t$ for some $t > t^*_{K-1}$, and assume that $q \geq 1$ units are offered in a fire sale at time $\tau_K(t)$ if $K_{\tau_K(t)} = K$. Let $k < K - 1$ denote the unique natural number lower than K such that $\tau_K(t) \in (t^*_{k+1}, t^*_k]$. There are two cases:

- If $\tau_K(t) \in (t^*_{k+1}, t^*_k)$ then, necessarily, $q = f(0; \tau_K(t), K) = K - k$, since by Property 1 k is the unique maximizer $(K - k')v_L + \Pi_{k'}(\tau_K(t))$ among all $k' \in \{1, ..., K-1\}$. Now, the reservation price of a buyer at any time $t \in [t^*_{k+1}, \tau_K(t)]$ after observing $K_t = K$ is equal to

$$v_H - p_K(t; \tau_K(t)) = \int_t^{\tau_K(t)} \big(\beta_{K-k-1}(v_H - v_L) + (1 - \beta_{K-k-1})(v_H - p_k(t'))\big)e^{-(t'-t)\lambda}\lambda \mathrm{d}t'$$
$$+ e^{-(\tau_K(t)-t)\lambda}\big(\beta_{K-k}(v_H - v_L) + (1 - \beta_{K-k})(v_H - p_k(\tau_K(t)))\big) \ . \quad (23)$$

  Notice that $p_K(\tau_K(t); \tau_K(t)) = \beta_{K-k}(v_H - v_L) + (1 - \beta_{K-k})(v_H - p_k(\tau_K(t)))$. Also, at any time $t \in (t^*_{k+1}, t^*_k)$, we have $p_{k+1}(t) = \beta(v_H - v_L) + (1 - \beta)(v_H - p_k(t)) > p_K(\tau_K(t); \tau_K(t))$. Let $\bar{t} < \tau_K(t)$ be the highest time t' in $[t^*_{k+1}, \tau_K(t)]$ where $p_K(t'; \tau_K(t)) \geq p_{k+1}(t')$. So, for all $t' \in [\bar{t}, \tau_K(t)]$, the revenue from a regular sale is lower when $K_{t'} = K$ than when $K_t = 1$. As a result, since making a fire sale at time $t \in (\bar{t}, \tau_K(t)]$ is strictly optimal when $K_t = k + 1$, it is strictly suboptimal not to hold it whenever $K_t = K > k + 1$.

- If $\tau_K(t) = t^*_k$ then, necessarily, $q = f(0; \tau_K(t), K) = K - k + 1$. Now, given that $\Pi_k(\tau_K(t)) = \Pi_{k-1}(\tau_K(t)) + v_L$, we have that $(K - k')v_L + \Pi_{k'}(\tau_K(t))$ is maximized at both k and $k - 1$. Still, since $\tau_k(t) = t$ for all $t \geq t^*_k$, $q = K - k$ would violate the condition of no successive fire sales in the definition of the seller's strategies, and hence $q = K-k+1$.[27] Then, we have

$$v_H - p_K(t;\tau_K(t)) = \int_t^{\tau_K(t)} \big(\beta_{K-k-1}(v_H - v_L) + (1-\beta_{K-k-1})(v_H - p_k(t'))\big)e^{-(t'-t)\lambda}\lambda \mathrm{d}t' \\ + e^{-(\tau_K(t)-t)\lambda}\big(\beta_{K-k+1}(v_H - v_L) + (1-\beta_{K-k+1})(v_H - p_k(\tau_K(t)))\big) \ .$$

Notice that the only difference between the previous expression and (23) is that, at $\tau_K(t)$, the number of goods offered in the fire sale is $K-(k-1)$ instead of $K-k$. Still, the same argument as before applies.

□

Step 3. *If $t^*_{K-1} > 0$ then $\tau_K(t) < t^*_{K-1}$ for all $t < t^*_{K-1}$*

Proof. Assume, for the sake of contradiction, that $t^*_{K-1} > 0$ and $\tau_K(t) = t^*_{K-1}$ for some $t < t^*_{K-1}$ (notice that, by Step 2 of this proof, $\tau_K(t) \le t^*_{K-1}$ for all $t < t^*_{K-1}$). So, if $K_{t^*_{K-1}} = K$ at time t^*_{K-1}, the seller holds a fire sale for two units. The continuation payoff of a buyer at time t is

$$v_H - p_K(t; t^*_{K-1}) = \int_t^{t^*_{K-1}} (v_H - p_{K-1}(t'))e^{-(t'-t)\lambda}\lambda \mathrm{d}t' \\ + e^{-(t^*_{K-1}-t)\lambda}\big(\beta_2(v_H - v_L) + (1-\beta_2)(v_H - p_{K-2}(t))\big) \ .$$

The corresponding profits of the seller are given by

$$\Pi_K(t; t^*_{K-1}) = \int_t^{t^*_{K-1}} (p_K(t', t^*_{K-1}) + \Pi_{K-1}(t'))e^{-\lambda(t'-t)}\lambda \mathrm{d}t' \\ + e^{-\lambda(t^*_{K-1}-t)}(2v_L + \Pi_{K-2}(t^*_{K-1})) \ .$$

Now, as we did for the case $K=2$, we differentiate the difference between $\Pi_K(t; t^*_{K-1})$ and $\Pi_{K-1}(t)$, for $t < t^*_{K-1}$, and we take the limit of t converging to t^*_{K-1}:

$$\lim_{t \nearrow t^*_{K-1}} \frac{\mathrm{d}(\Pi_K(t; t^*_{K-1}) - \Pi_{K-1}(t))}{\mathrm{d}t} = \lambda(p_{K-1}(t^*_{K-1}) - p_K(t^*_{K-1}; t^*_{K-1})) > 0 \ ,$$

where the inequality holds because

$$p_K(t^*_{K-1}; t^*_{K-1}) = \beta_2(v_H - v_L) + (1-\beta_2)(v_H - p_{K-2}(t^*_{K-1})) \\ < \beta(v_H - v_L) + (1-\beta)(v_H - p_{K-2}(t^*_{K-1})) = p_{K-1}(t^*_{K-1}) \ .$$

Since $\Pi_K(t^*_{K-1}; t^*_{K-1}) - \Pi_{K-1}(t^*_{K-1}) = v_L$ the inequality implies that there exists a $t < t^*_{K-1}$ such that $\Pi_K(t^*_{K-1}; t^*_{K-1}) - \Pi_{K-1}(t^*_{K-1}) < v_L$, which is a contradiction. □

[27] Condition 3 in the definition of the seller's strategies establishes that, if there is a fire sale at some time t, the time of the next fire sale t' if there is no regular sale is well defined, and therefore bounded away from t.

Step 4. *There is a unique equilibrium for $K_0 = K$. In this equilibrium, $\tau_K(t) = \max\{t, t_K^*\}$ for some t_K^* such that $t_K^* \le t_{K-1}^*$ and $t_K^* < t_{K-1}^*$ if $t_{K-1}^* > 0$.*

Proof. We proceed similarly to the proof of Lemma 3. Assume then that there is a time t such that $\tau_K(t) > t$. Since, as we have shown in Steps 2 and 3, it is the case that $\tau_K(t) > t$ only if $t < t_{K-1}^*$, we have that the equilibrium price at time t satisfies

$$v_H - p_K(t;\tau_K(t)) = \int_t^{\tau_K(t)} \big(\beta_{K-k-1}(v_H - v_L) + (1-\beta_{K-k-1})(v_H - p_k(t'))\big)e^{-(t'-t)\lambda}\lambda \mathrm{d}t' \\ + e^{-(\tau_K(t)-t)\lambda}\big(\beta(v_H - v_L) + (1-\beta)(v_H - p_k(\tau_K(t)))\big)\ . \tag{24}$$

The equilibrium payoff of the seller at state (t, K_t), denoted $\Pi_K(t;\tau_K(t))$, can be expressed as

$$\Pi_K(t;\tau_K(t))) = \int_t^{\tau_K(t)} (p_K(t';\tau_K(t)) + \Pi_{K-1}(t'))e^{-\lambda(t'-t)}\lambda \mathrm{d}t' \\ + e^{-\lambda(\tau_K(t)-t)}(v_L + \Pi_{K-1}(\tau_K(t)))\ . \tag{25}$$

We can compute the right-derivative of $\Pi_K(t;\tau_K(t))-\Pi_{K-1}(t)$ at $\tau_K(t)$ as follows:

$$\begin{aligned}\lim_{t'\nearrow\tau_K(t)} \frac{\mathrm{d}(\Pi_K(t';\tau_K(t))-\Pi_{K-1}(t'))}{\mathrm{d}t'} &= -(1-\beta)\lambda(p_{K-1}(\tau_K(t)) - v_L) - \Pi'_{K-1}(\tau_K(t)) \\ &= \underbrace{\lambda((1-\beta)v_L - \Pi_{K-1}(\tau_K(t)) + \Pi_{K-2}(\tau_K(t)) + \beta p_{K-1}(\tau_K(t)))}_{\equiv\phi_K(\tau_K(t))}\ .\end{aligned}$$

In the limit $\tau_K(t) \to t_{K-1}^*$, the previous expression converges to $\lambda\beta(p_{K-1}(t_{K-1}^*) - v_L) > 0$, so $\phi_K(t_{K-1}^*) > 0$. As in the proof of Lemma 3 and Proposition 2 we have two cases:

- **Case 1, $\phi_K(t) > 0$ for all $t \in [0, t_{K-1}^*)$:** To show existence of an equilibrium notice that, if $\tau_K(t) = t$ is part of an equilibrium, the reservation price of a buyer arriving at time $t < t_{K-1}^*$, $p_K(t;t)$, satisfies

  $$v_H - p_K(t;t) = \beta_{K-1}(v_H - v_L) + (1-\beta_{K-1})(v_H - p_{K-1}(t))\ . \tag{26}$$

  So, the payoff of the seller from holding a fire sale at time $\hat{t}_K \in (t, t_{K-1}^*)$ (instead of at time t) is

  $$\hat{\Pi}_K(t;\hat{t}_K) \equiv \int_t^{\hat{t}_K} (p_K(t',t') + \Pi_{K-1}(t'))e^{-\lambda(t'-t)}\lambda \mathrm{d}t' + e^{-\lambda(\hat{t}_K-t)}(v_L + \Pi_{K-1}(\hat{t}_K))\ . \tag{27}$$

  We can differentiate the previous expression with respect to $\hat{t}_K$ and evaluate t at $\hat{t}_K$, and obtain that

  $$\left.\frac{\partial\hat{\Pi}_K(t;\hat{t}_K)}{\partial\hat{t}_K}\right|_{\hat{t}_K=t} = -\phi_K(t)\ . \tag{28}$$

Then, we see that the seller has no incentives to delay the fire sale, and therefore $\tau_K(t) = t$ for all t. As a result, we have a unique equilibrium in this case.

- **Case 2, $\phi_K(t) = 0$ for some $t \in [0, t^*_{K-1})$:** Now assume $\phi_K(t) = 0$ for some $t \in [0, t^*_{K-1})$. Let $t^*_K < t^*_{K-1}$ be the maximum $t < t^*_{K-1}$ such that $\phi_K(t) = 0$ (notice that, by continuity of ϕ_K, it exists). Clearly, by the same argument as in Case 1 (and, in particular, equation (28)), in any equilibrium $\tau_K(t) = t$ for all $t \in [t^*_K, 1]$. The following theorem establishes that the seller does not have an incentive to hold a fire sale before t^*_K:

  Lemma B.2. $\Pi_K(t; t^*_K) - \Pi_{K-1}(t) > v_L$ *for all* $t < t^*_K$.

  Given that the proof of Lemma B.2 is technical and long, we leave it for the end of the proof. Notice that Lemma B.2 leaves $\tau_K(t) = t^*_K$ for all $t < t^*_K$ and $\tau_K(t) = t$ for all $t \geq t^*_K$ as the unique candidate for an equilibrium. Furthermore, it ensures that the seller has the incentive to hold a fire sale at t^*_K and not before. Therefore, there is a unique equilibrium also in this case.

Given that, in the unique equilibrium, $\tau_K(t) = \max\{t, t^*_K\}$, we use $p_K(t)$ and $\Pi_K(t) \equiv \Pi_K(t; t^*_K)$ to denote the corresponding equilibrium values for the price and the profits in state (t, K). □

Step 5. *Properties 1, 2 and 3 hold for K.*

Proof. Property 1 holds for K because, as we showed in Steps 1 and 2, the seller holds $K - k$ fire sales when $K_t = K$ and $t \in [t^*_k, t^*_{k-1})$ for all $k \leq K - 1$. Property 2 holds for K because, by Step 4, $\tau_K(t) = t$ for all $t \geq t^*_K$ (so $\Pi_K(t) = \Pi_{K-1}(t) + v_L$) and $\Pi_K(t) > \Pi_{K-1}(t) + v_L$ for all $t < t^*_K$ by Step 4. Finally, Property 3 holds for K because of the definition of t^*_K (notice that t^*_K solves $\phi_K(t^*_K) = 0$, which implies $\Pi'_K(t^*_K) = \Pi'_{K-1}(t^*_K)$). □

The only result left to prove to conclude the proof is Lemma B.2.

The Proof of Lemma B.2. Define, for each $k \in \{1, ..., K\}$, $\Delta\Pi_k(t) \equiv \Pi_k(t) - \Pi_{k-1}(t)$, with the convention that $\Pi_0(t) = 0$ for all t. We want to prove that $\Delta\Pi_k(t) - v_L > 0$ for all $t < t^*_k$. To do this, notice that using some algebra we can obtain equations (9) and (12) from equations (25) and (24) (using $\tau_K(t) = t^*_K$). Furthermore, using these equations and some algebra, it is easy to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda t}(\Delta\Pi_k(t) - v_L)\right) = -\lambda e^{-\lambda t}\left(\Delta\Pi_{k-1}(t) - \Delta p_k(t) - v_L\right) ,$$

where it is convenient to define $\Delta p_k(t) = -(p_k(t) - p_{k-1}(t))$ to ensure it is positive. The following result ensures that the right hand side of the previous expression is negative for $t < t^*_K$.

Lemma B.3. *For all $k = 2, ..., K$ and $n \in \{0, ..., k-1\}$, there exists some $t_{k,n} \in [0, t_k^*]$ such that $\Delta\Pi_{k-1}(t) - n\Delta p_k(t) > v_L$ for all $t < t_{k,n}$ and $\Delta\Pi_{k-1}(t) - n\Delta p_k(t) < v_L$ for all $t \in (t_{k,n}, t_k^*]$.*

Proof. Consider first the case $k = 2$. In this case, by Proposition 1, we have that, for $n = 0$, $\Delta\Pi_1(t) = \Pi_1(t)$ is decreasing for all t, and equal to v_L only at $t_1^* = 1$. For $n = 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-\lambda t}(\Delta\Pi_1(t) - \Delta p_2(t) - v_L)\big) = \big(2\beta e^{-\lambda} - e^{-\lambda t}\big)\lambda(v_H - v_L) \ .$$

This is increasing in t. Also, the right hand side of the expression is equal to 0 at $\bar{t} \equiv 1 - \frac{\log(2\beta)}{\lambda}$. It is only left to show that $t_2^* \leq \bar{t}$. If $\beta \leq \frac{1}{2}$, then the result is clear, so assume $\beta > \frac{1}{2}$. In this case, notice that t_2^* is defined by $\phi_2(t_2^*) = 0$, where $\phi_2(t)$ is defined in (19), which is negative for all $t < t_2^*$. We then have

$$\phi_2(\bar{t}) = (v_H - v_L)\lambda\frac{1}{2}\left(\beta + \frac{1}{\beta} - 2 + \log(2\beta)\right) \ .$$

Notice that since $\beta > \frac{1}{2}$ we have that both $\beta + \frac{1}{\beta} > 2$ and $\log(2\beta) > 0$, so $\phi_2(\bar{t}) > 0$ and therefore $\bar{t} > t_2^*$, which proves the result.

Now we proceed by induction: take $k > 2$ and assume that for all $\tilde{k} < k$ and for all $n \in \{0, ..., \tilde{k} - 1\}$, $\Delta\Pi_{\tilde{k}-1}(t) - n\Delta p_{\tilde{k}}(t)$ satisfies the property of the statement of the proposition for some $t_{\tilde{k},n}$. Simple algebra (again from equations (9) and (12)) shows that the following formula is true for all $k \geq 3$ and $n \in \{0, ..., k-1\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-\lambda t}(\Delta\Pi_{k-1} - n\Delta p_k(t))\big) = -\lambda e^{-\lambda t}\big(\Delta\Pi_{k-2}(t) - (n+1)\Delta p_{k-1}(t)\big) \ . \tag{29}$$

We first want to show that the statement of the lemma holds for k and $n = 0$. In this case, note first that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}(e^{-\lambda t}\Delta\Pi_{k-1}(t))\right|_{t=t_{k-1}^*} = -\lambda e^{-\lambda t_{k-1}^*}\Delta\Pi_{k-1}(t_{k-1}^*) + e^{-\lambda t_{k-1}^*}\Delta\Pi_{k-1}'(t_{k-1}^*) = -e^{-\lambda t_{k-1}^*}v_L$$

where we used that the value-matching and smooth pasting conditions hold for $\Pi_{k-1}(t)$ at $t = t_{k-1}^*$. Also, we can write (29) for $n = 0$ to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-\lambda t}\Delta\Pi_{k-1}(t)) = -\lambda e^{-\lambda t}(\Delta\Pi_{k-2}(t) - \Delta p_{k-1}(t)) \ .$$

By assumption, the right hand side of the previous equation equals to $-\lambda e^{-\lambda t}v_L$ in at most one time in $[0, t_{k-1}^*]$, denoted $t_{k-1,1}$, and due to the minus sign, it is strictly below $-\lambda e^{-\lambda t}v_L$ for all $t < t_{k-1,1}$. Since it equals $-\lambda e^{-\lambda t_{k-1}^*}v_L$ at t_{k-1}^* we have that, in fact, $t_{k-1,1} = t_{k-1}^*$. Then, $\frac{d}{dt}(e^{-\lambda t}\Delta\Pi_{k-1}(t))$ is negative for all $t \leq t_k^* < t_{k-1}^*$, so the result holds for $n = 0$.

Notice now that for all $n \in \{1, ..., k-1\}$ we have (again from equations (9) and (12))

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-\lambda t}(\Delta\Pi_{k-1}(t) - n\Delta p_k(t))\big) = -\lambda e^{-\lambda t}(\Delta\Pi_{k-2}(t) - (n+1)\Delta p_{k-1}(t)) \ .$$

By the induction argument, again, the right hand side of the previous equation equals to $-\lambda e^{-\lambda t} v_L$ in $[0, t^*_{k-1}]$ in at most one time, $t = t_{k-1,n+1}$, and due to the minus sign, it is strictly below $-\lambda e^{-\lambda t} v_L$ for all $t < t_{k-1,n+1}$. Now, given that $\Delta p_k(t) \geq 0$ for all $t < t^*_k$, we have

$$\Delta\Pi_{k-1}(t^*_k) - n\Delta p_k(t^*_k) \leq \Delta\Pi_{k-1}(t^*_k) - \Delta p_k(t^*_k) < 0 \ .$$

So, we have that the derivative of $e^{-\lambda t}(\Delta\Pi_{k-1}(t) - n\Delta p_k(t)))$ is negative for all $t < t^*_k$, and then the result holds. □

□

□

B.4 Proof of Results in Sections 3.4 and 4

The Proof of Proposition 4. We only need to prove that there exists some $\bar{\beta} < 1$ such that if $\beta > \bar{\beta}$ then $t^*_2 = 0$. Recall that t^*_2 is the solves $\phi_2(t^*_2) = 0$ (where ϕ_2 is defined equation (19) in the proof of Lemma 3) when a positive solution exists, and $t^*_2 = 0$ otherwise. Notice that, $e^{-\lambda t}\phi_2(t)$ is a concave function:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(e^{-\lambda t}\phi_2(t)) = -(v_H - v_L)\lambda^3(1-\beta)e^{-\lambda t} < 0 \ .$$

Also, $e^{-\lambda}\phi_2(1) = (v_H - v_L)\lambda(1-\beta)\beta e^{-\lambda} > 0$, and $\lim_{\beta\to 1}\phi_2(0) = (v_H - v_L)\lambda^2 e^{-\lambda} > 0$. So, by continuity of ϕ_2, if β is close to 1, $\phi_2(0), \phi_2(1) > 0$ and, since $e^{-\lambda t}\phi_2(t)$ is concave, $\phi_2(t)$ is also strictly positive for all $t \in [0, 1]$, so $t^*_2 = 0$ in the unique equilibrium. □

The Proof of Proposition 5. Using equation (1) it is clear that the result is true for $K = 1$. Then, one can proceed analogously to show $\lim_{\beta\to 0} t^*_2 = 1$ and $\lim_{\beta\to 0} p_2(t) = v_H$ for all $t \in [0, 1]$. The reason is that when β is low, even when the time is close to the fire sale, the willingness to pay of the buyers is again, close to v_H, since obtaining the good in a fire sale is unlikely. As a result the seller has the incentive to post-pone fire sales toward the deadline. This argument can be applied iteratively for any K. □

Proof of Proposition 6. First, consider the case where $K_0 = 1$, and for notational convenience, let $G(v_H) = \frac{v_H - \underline{v_H}}{\overline{v_H} - \underline{v_H}}$ denote the CDF of a uniform distribution in $[\underline{v_H}, \overline{v_H}]$. Assume that there is an equilibrium as described in Proposition 1, where now $p_1(t)$ is the price corresponding to the case where the valuation of the buyers is $\underline{v_H}$. Assume that $\overline{v_H}$ is sufficiently low that no type of buyer rechecks the price. Therefore, if a buyer with valuation v_H arrives at time t, the price she is willing to pay equals

$$p_1(t, v_H) \equiv v_H - e^{-\lambda(1-t)}(v_H - v_L)\beta > \underline{v_H} - e^{-\lambda(1-t)}(\underline{v_H} - v_L)\beta = p_1(t) \ .$$

Consider the seller's gain from deviating. By increasing the price from $p_1(t)$ to $p_1(t) + \varepsilon$ the seller increases the price conditional on acceptance, but lowers the probability of acceptance conditional on arrival to $1 - G(v_H(t, p_1(t) + \varepsilon))$, where

$$p = v_H(t,p) - e^{-\lambda(1-t)}(v_H(t,p) - v_L)\beta \quad \Rightarrow \quad v_H(t,p) = v_L + \frac{p - v_L}{1 - e^{-(1-t)\lambda}\beta} \ .$$

Consider the benefit from deviating at all $s \in [t, t+\Delta]$ to offering $p_1(s) + \varepsilon$, for $\Delta > 0$ small. This deviation is not profitable only if $p_1(t)$ is higher than $(p_1(t)+\varepsilon)(1 - G(v_H(t, p_1(t)+\varepsilon)))$ for all $\varepsilon > 0$. The derivative of this last expression with respect to ε gives

$$1 - G(v_H(t, p_1(t) + \varepsilon)) - \frac{(p_1(t)+\varepsilon)G'(v_H(t,p_1(t)+\varepsilon))}{1 - e^{(1-t)\lambda}\beta} < 1 - \frac{v_L}{\overline{v_H} - \underline{v_H}} \ .$$

If $\overline{v_H} - \underline{v_H}$ is small enough, the last term to the right of the previous function is negative, so it is optimal to choose $\varepsilon = 0$.

For a general K, a similar argument applies. Indeed, at any given time, the reservation price of a buyer with valuation is $v_H - b_{K_t}(t)(v_H - v_L)$, where $b_k(t)$ is the probability of obtaining a good in the future (in a fire sale) if at time t the stock is k, for all $k = 1, ..., K$. So, in general, the derivative of the payoff from offering a price equal to $p_1(t) + \varepsilon$ with respect to ε is

$$1 - G(v_H(t, p_1(t) + \varepsilon)) - \frac{(p_1(t)+\varepsilon)G'(v_H(t,p_1(t)+\varepsilon))}{1 - b_{K_t}(t)} < 1 - \frac{v_L}{\overline{v_H} - \underline{v_H}} \ .$$

Hence, again, if $\overline{v_H} - \underline{v_H}$ is sufficiently small, the right-hand side of the inequality of the previous function is negative, so it is optimal to choose $\varepsilon = 0$. $\square$

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C Online Appendix: A Random Attention Model

In this section we relax the assumption that fire sales allow the seller to sell the offered units instantaneously. To do this, we discuss an alternative model where, whenever the seller holds a fire sale, buyers and shoppers obtain the offer at a random time. This implies that the seller cannot guarantee the sale of all units, specially when the time is close to the deadline.

Consider a model identical to our base model except for the following feature. We now assume that, when the seller decides to hold a fire sale, she does not sell the goods offered in it immediately. Instead, while a fire sale takes place, accumulated buyers observe it at exponentially distributed times, arriving at rate $\beta\gamma$ for some $\gamma > 0$. Also, a shopper observes (and accepts) the fire sale at a Poisson rate $(1-\beta)\gamma$. Notice that such observation rates ensure that, if there is (exactly) one buyer in the market and there is a fire sale, conditional on the good being sold through the fire sale, the probability that she gets a good is β, as in our base model.

To simplify our analysis, we assume that the fire-sales strategy of the seller is limited to deciding a decreasing sequence of times $(t_k^\gamma)_{k=1}^K$ such that if $K_t = k$ and $t \geq t_k^\gamma$ then, additionally to offering the good at the regular price, the seller holds a fire sale. Notice that if there is a unit being offered in a fire sale at time t and a buyer arrives, she observes the regular offer $(P(t), Q(t))$ and the remaining stock $K(t)$, and only if she rejects such a price she observes the fire sale at a rate $\beta\gamma$. We use $(t_k^*)_{k=1}^K$ to denote the unique values of our base model, established in Proposition 3.

The following proposition establishes that, as the frequency with which buyers and shoppers observe a fire sale increases, the equilibrium outcomes of the model with random attention converge to the equilibrium outcome of our base model.

Proposition 7. *For all $\varepsilon > 0$ there is $\bar{\gamma}$ such that $|t_k^\gamma - t_k^*| < \varepsilon$ for all equilibria, $k \in \mathbb{N}$ and $\gamma > \bar{\gamma}$.*

Proof. It is analogous to prove that Lemmas 1 and 2 still hold in the model with random attention, that is, the seller never accumulates buyers in equilibrium and accumulated buyers never re-check the price.

We begin in the case $K = 1$. Assume that the equilibrium first time where a fire sale is offered is t_1^γ. The price that makes a buyer indifferent between purchasing the good or not at any time $t \in [0, 1]$ is

$$v_H - p_1(t) = \underbrace{e^{-(t_1^\gamma \vee t - t)\lambda}}_{\text{no arrival before } t_1^\gamma} \underbrace{\left(1 - e^{-(1 - t_1^\gamma \vee t)(\gamma+\lambda)}\right)}_{\text{arrival or sampling in } [t_1^\gamma, 1]} \underbrace{\frac{\beta\gamma}{\gamma + \lambda}}_{\text{prob. get good}} (v_H - v_L) \;, \tag{30}$$

since this is the highest price that a buyer is willing to pay, where $t_1^\gamma \vee t \equiv \max\{t_1^\gamma, t\}$. The expression has 4 terms. The first is the probability that no buyer arrives before t_1^γ. The second is that, in $[t_1^\gamma, 1]$, either a buyer arrives (at arrival rate λ), or a shopper observes the fire sale (at rate $(1-\beta)\gamma$) or the accumulated buyer observes the fire sale (at rate $\beta\gamma$). Conditional on one of these events happening, the third term is the probability that the accumulated buyer gets the good is $\frac{\beta\gamma}{\gamma+\lambda}$. Finally, the last term is the payoff conditional on obtaining the good in the fire sale. As in our model, p_1 is decreasing, and it is computed as the one that makes the buyer indifferent between purchasing now or waiting until he has the chance of obtaining the good at price equal to v_L.

Fix a time $\hat{t}_1^\gamma$ when buyers believe that a fire sale takes place first. The profits of the seller from setting a fire sale at some $t_1 \in [0,1]$ are given by

$$\Pi_1(0; t_1, \hat{t}_1^\gamma) = \int_0^{t_1} p_1(t; \hat{t}_1^\gamma) e^{-\lambda t} \lambda dt + e^{-\lambda t_1} \int_{t_1}^1 e^{-(\lambda+(1-\beta)\gamma)(t-\hat{t}_1^\gamma)} \big((1-\beta)\gamma v_L + \lambda p_1(t; \hat{t}_1^\gamma)\big) dt \ .$$

Differentiating the previous with respect to t_1, using (30), and replacing t_1 by $\hat{t}_1^\gamma$ we obtain

$$\frac{(1-\beta)\gamma e^{-\lambda \hat{t}_1^\gamma}}{\gamma+\lambda} \left(\lambda(v_H - v_L)(1 - e^{-(\gamma+\lambda)(1-\hat{t}_1^\gamma)}) - v_L(\gamma+\lambda) e^{-((1-\beta)\gamma+\lambda)(1-\hat{t}_1^\gamma)} \right) \ . \tag{31}$$

For a fixed $\hat{t}_1^\gamma < 1$, the previous expression is positive for γ big enough. Similarly, for $\hat{t}_1^\gamma = 1$, the previous expression is negative. Also, the derivative of the big parenthesis term in the previous expression is

$$(\gamma+\lambda) e^{\hat{t}_1^\gamma(\gamma+\lambda)} \left(\lambda(v_H - v_L) + v_L((1-\beta)\gamma + \lambda) e^{\beta\gamma(1-\hat{t}_1^\gamma)} \right) > 0 \ .$$

Then, a unique solution for $\hat{t}_1^\gamma \in (0,1)$ exists for γ big enough, denoted t_1^γ, and such a solution converges to 1 as $\gamma \to \infty$. Furthermore, necessarily $e^{-((1-\beta)\gamma+\lambda)(1-t_1^\gamma)} \to 0$, in order to keep the big parenthesis term in the previous expression (31) equal to 0 as γ increases, which implies that the probability of transaction converges to 1 as $\gamma \to \infty$.

Now, consider a general $K > 1$. For each fixed γ and set of dates $\vec{t} \equiv (t_k)_{k=1}^K$, we use $p_k^\gamma(t; \vec{t})$ to denote the reservation price of a buyer at time t if the number of units left is k and the buyer expects fire sales to begin happening at times in $\vec{t}$. Similarly, we let $\Pi_k^\gamma(t; \vec{t}, \vec{t}')$ denote the payoff of the seller if, at any time t, he posts the reservation price $p_{K_t}(t, \vec{t})$, and begins holding fire sales at times $\vec{t}'$ (instead of at the times believed by the buyers, $\vec{t}$).

Fix a sequence $(\gamma_n)_n$ such that $\lim_{n\to\infty} \gamma_n = \infty$. For each n, fix a sequence decreasing $\vec{t}^{\gamma_n} \equiv (t_k^{\gamma_n})_{k=1}^K$, interpreted as a sequence of candidate equilibrium values for sales, with $t_1^{\gamma_n} = t_1^{*\gamma_n}$ for all n. So,

$$\lim_{n\to\infty} \Pi_k^{\gamma_n}(t; \vec{t}^{\gamma_n}, \vec{t}^{\gamma_n}) = \Pi_k(t; \vec{t}^{\infty}, \vec{t}^{\infty}) \ ,$$

where $\vec{t}^{\infty} \equiv \limsup_{n\to\infty} \vec{t}^{\gamma_n}$ and Π_k is the corresponding payoff of the seller in our base model. Notice that, since $e^{-(\beta\gamma+\lambda)(1-t_k^{\gamma_n})} \to 0$ for all $k = 1, ..., K$ implies that the probability of selling all units converges to 1.

Now, consider a sequence of corresponding equilibria, characterized by fire sales times $(\vec{t}^{*\gamma_n})_n$. Assume that $(\vec{t}^{*\gamma_n})_n$ does not converge to $(t_k^*)_{k=1}^K$, so there is a convergent subsequence with limit equal to $\vec{t}^{\dagger} \equiv (t_k^{\dagger})_{k=1}^K \neq (t_k^*)_{k=1}^K$. This implies that, for any sequence $(\vec{t}'^{\gamma_n})_n$ satisfying $e^{-(\beta\gamma+\lambda)(1-t_1^{\gamma_n})} \to 0$ (so transaction happens asymptotically for sure) and converging to some $\vec{t}'$, we have that, for all k,

$$\Pi_k(t; \vec{t}^{\dagger}, \vec{t}^{\dagger}) \geq \Pi_k(t; \vec{t}^{\dagger}, \vec{t}') \ .$$

This implies that $\vec{t}^{\dagger}$ is an equilibrium of our model, which contradicts the uniqueness of $(t_k^*)_{k=1}^K$.
$\square$