Reputation for Quality*

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Abstract

We propose a new model of firm reputation where product quality is persistent and depends stochastically on the firm's past investments. Reputation is then modeled directly as the market belief about quality. We analyze how investment incentives depend on the firm's reputation and derive implications for reputational dynamics.

Reputational incentives depend on the specification of market learning. When consumers learn about quality through perfect good news signals, incentives decrease in reputation and there is a unique work-shirk equilibrium with ergodic dynamics. When learning is through perfect bad news signals, incentives increase in reputation and there is a continuum of shirk-work equilibria with divergent dynamics. For a large class of imperfect Poisson learning processes and low investment costs, we show there exists a work-shirk equilibrium with path-dependent dynamics. We also derive conditions under which this equilibrium is essentially unique.

1 Introduction

In most industries firms can invest into the quality of their products through human capital investment, research and development, or organizational change. While imperfect monitoring by consumers gives rise to a moral hazard problem, the firm can share in the created value by building a reputation for quality, justifying premium prices. This paper analyzes the investment incentives in such a market, characterizing how they depend on the current reputation of the firm and the information structure.

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Our key innovation over classical models of reputation and repeated games is to model product quality as a function of past investments rather than current effort. From a modeling perspective, the introduction of persistence turns quality into a state variable and allows us to model reputation directly as the market's belief about quality. Therefore, our firm works to actually build the quality that underlies reputation, rather than to signal an exogenous type or to avoid punishments by a counterparty. From an economic perspective, persistence alleviates the firm's moral hazard problem. Investment affects quality in a lasting way, and yields rewards even when the firm is believed to be shirking in the future.

The model gives rise to simple Markovian equilibria that explain when a firm builds a reputation, when it invests to maintain its reputation, and when it chooses to run its reputation down. We investigate these incentives for a broad class of Poisson learning processes. Our results explain why the incentives for academics, where the market learns through good news events like publications, should be different from the incentives for clinical doctors, where the market learns through bad news events like malpractice suits.

In the model, illustrated in Figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a stochastic function of the firm's past investments. In particular, the quality at time t is determined by the quality at $t - dt$ and the investment at time t. Consumers' expected utility is determined by the firm's quality, so their willingness to pay is given by the market belief that quality is high; we call this belief the *reputation* of the firm and denote it by x_t.

Consumers observe neither quality nor investment directly, but learn about the firm's quality through Poisson signals. A signal is *good news* if it indicates high quality, and *bad news* if it indicates low quality. Market learning is *imperfect* if no Poisson signal perfectly reveals the firm's quality.[1] The firm's reputation changes as a function of the signals or their absence, and as a function of market beliefs about its investment.

Investment is incentivized by the difference in value between a high and low quality firm, which we call the *value of quality.* Quality derives its value by increasing expected utility to consumers and thereby the firm's reputation. Crucially, as quality is persistent, this reputational payoff does not take the form of an immediate one-off reputational boost, but accrues to the firm as a stream of future *reputational dividends.* Theorem 1 formalizes this idea by writing the value of quality as the present asset value of its future reputational dividends.

In Section 4 we characterize equilibria under perfect Poisson learning. For perfect good news, where high quality gives rise to product *breakthroughs* that boost reputation to one, reputational

[1] MacLeod (2007) coins the terms 'normal goods' for experience goods that are subject to bad news learning and 'innovative goods' for experience goods that are subject to good news learning. Examples abound. Good news signals occur in academia when a paper becomes famous, in the bio-tech industry when a trial succeeds, and for actors when they win an Oscar. Bad news signals occur in the computer industry when batteries explode, in the financial sector when a borrower defaults, and for doctors when they are sued for medical malpractice.

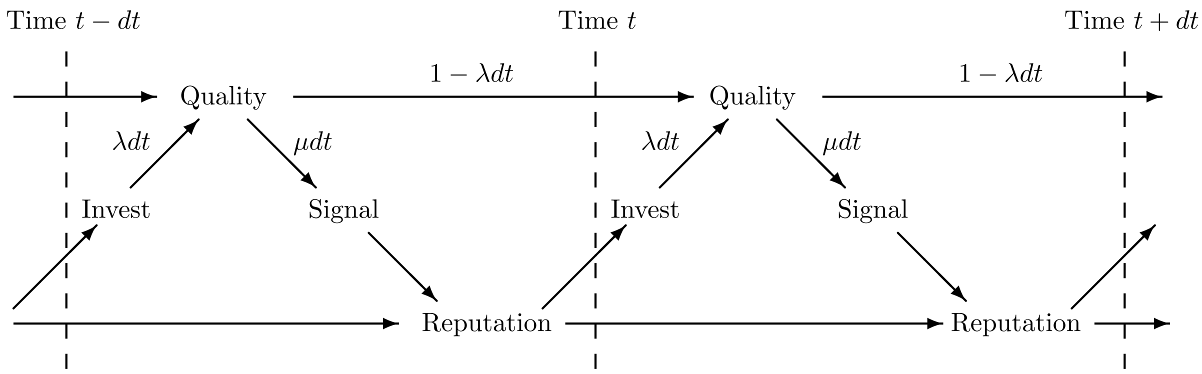

Figure 1: **Timeline.** Quality is persistent and depends stochastically on past investments. The market learns about the firm's quality through Poisson signals. Reputation then evolves as a function of market learning and equilibrium beliefs about the firm's investments. As quality is persistent, current investment affects all future signals rather than just the current signal.

dividends and investment incentives decrease in the firm's reputation. Equilibrium must be *work-shirk* in that the firm works when its reputation lies below some cutoff x^*, and shirks above. Intuitively, a breakthrough that takes the firm's reputation to one is more valuable to a firm with low reputation, so this firm has the highest incentive to invest in quality. Reputational dynamics are ergodic in a work-shirk equilibrium because equilibrium beliefs induce an upward trend for low reputations and a downward trend for high reputations. Under a parametric restriction the work-shirk equilibrium is unique.

For perfect bad news signals, where high quality insures the firm against product *breakdowns* that destroy its reputation, reputational dividends and investment incentives increase in the firm's reputation. Equilibrium must be *shirk-work* in that the firm works when its reputation lies above some cutoff x^*, and shirks below. Intuitively, a breakdown that takes the firm's reputation to zero is more costly to a firm with high reputation, so this firm has the highest incentive to invest in quality. Reputational dynamics are path-dependent in a shirk-work equilibrium because equilibrium beliefs induce a downward trend for low reputations and an upward trend for high reputations. There may be a continuum of shirk-work equilibria: the multiplicity is caused by the divergent reputational drift at the cutoff which creates a discontinuity in the value functions and investment incentives. Intuitively if a firm is believed to be working at the cutoff, there are higher incentives to actually work in order to capitalize on the market's favorable beliefs, creating a self-fulfilling prophecy.

In Section 5 we analyze imperfect Poisson learning processes. When the signal is imperfect, Bayesian learning ceases for extreme reputations and reputational dividends tend to be hump-shaped. While this suggests a shirk-work-shirk equilibrium, we surprisingly show existence of a work-shirk equilibrium for a large range of parameter values. We also derive conditions under which the equilibrium is essentially unique.

The work-shirk result relies on a fundamental asymmetry. For $x \approx 1$, work is not sustainable: If the firm is believed to work, its reputation stays high and reputational dividends stay small, undermining incentives to actually invest. For $x \approx 0$, work is sustainable: If the firm is believed to work, its reputation drifts up and reputational dividends increase, generating incentives to invest. Crucially, a firm with a low reputation works not because of the small, immediate reputational dividends but because of the larger future dividends it expects after its reputation has drifted up. Thus, persistent quality and the endogenous reputational drift jointly introduce an asymmetry that gives rise to the work-shirk equilibrium.

The work-shirk equilibrium is essentially unique if the firm's reputation may rise, even when it is believed to be shirking. This condition is satisfied if the signal is good news where reputation jumps up at a signal arrival; it is also satisfied if the signal is bad news and positive reputational drift from the absence of a bad signal overcomes the negative drift from adverse equilibrium beliefs. Under this condition, putative shirk-work-shirk equilibria unravel, as the favorable beliefs in the work-region guarantee high investment incentives for a firm around the shirk-work cutoff. To the contrary, if the condition is not satisfied, adverse beliefs below a shirk-work cutoff are self-fulfilling and support a continuum of shirk-work-shirk equilibria.

1.1 Literature

The key feature distinguishing our paper from classical models of reputation and repeated games is that product quality is a function of past investments rather than current effort. This difference is important. In classical models, the firm exerts effort to convince the market that it will also exert effort in the future. In our model, a firm's investment increases its quality and future revenue independent of market beliefs about future investment since quality is persistent.

The two reputation models closest to ours are Mailath and Samuelson (2001) and Holmström (1999), which both model reputation as the market's belief about some exogenous state variable. The mechanisms linking effort, type and utility are depicted in Figure 2. In Mailath and Samuelson (2001) a competent firm, that can choose to work or shirk, tries to distinguish itself from an incompetent type, that always shirks. A reputation for competence benefits the firm to the degree that the market expects a competent firm to work. With imperfect monitoring, a firm with a high reputation shirks because updating is slow. This causes effort to unravel from the top as a firm just below a putative work-shirk cutoff finds it unprofitable to further invest into its reputation. In our model, persistent quality prevents this unraveling because current investment affects the firm's future reputation and revenue irrespective of beliefs about its future investments.

Holmström's (1999) signal-jamming model is similar to ours in that the firm's type directly affects consumers' utility. In this model, the firm works to induce erroneous market beliefs that its exogenous ability type is higher than in reality. This is in stark contrast to our model, where a firm invests to actually improve its endogenous quality type.

In both of these papers, learning about a fixed type eventually vanishes and so do reputational incentives. These are instances of a more general theme: Cripps, Mailath, and Samuelson (2004)

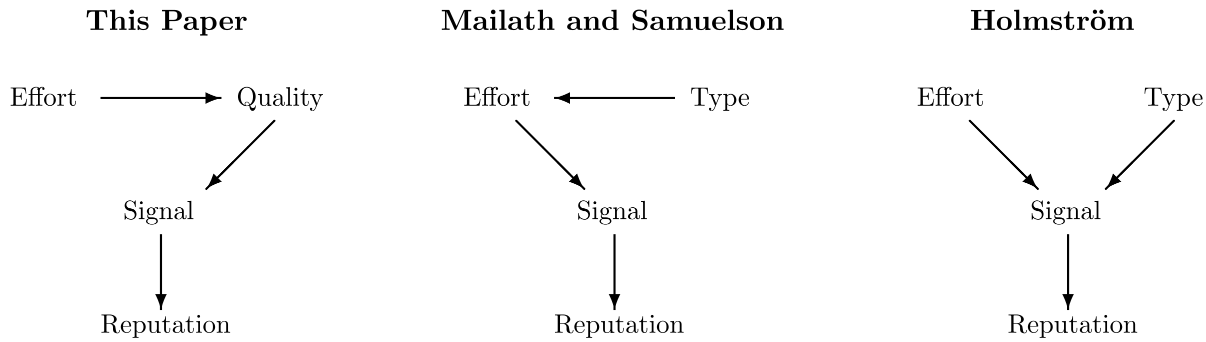

Figure 2: **Comparison of reputation models.** The literature usually models reputation as belief over some exogenous type. This type affects consumer utility either directly, as in Holmström (1999) or indirectly through the cost of effort, Mailath and Samuelson (2001). In contrast, our firm controls its type endogenously through its investment.

show that with imperfect monitoring and fixed types, reputation is always a short-run phenomenon. A long-run analysis of reputation requires '... some mechanism by which the uncertainty about types is continually replenished'. Our stochastic investment into quality is a natural candidate for this mechanism. Unlike models of exogenous shocks, such as Mailath and Samuelson (2001) and Holmström (1999), in which reputation simply trails the shocks, the reputational dynamics of our model are endogenously determined by the forward-looking reputational incentives.[2]

There is a wider literature on lifecycle effects in reputation models, as surveyed in Bar-Isaac and Tadelis (2008). Some of these results can be understood through our analysis of different learning processes: With perfect good news learning, firms with low reputation try to build, or buy a reputation (Tadelis (1999)). With perfect bad news learning, firms with high reputation have high incentives to maintain them (Diamond (1989)). With imperfect learning, reputational incentives are hump-shaped (Benabou and Laroque (1992), Mailath and Samuelson (2001)).[3]

In contrast to the repeated games literature (e.g. Fudenberg, Kreps, and Maskin (1990)), our model is distinguished by an evolving state variable. Investment directly feeds through to future reputation and revenue in our model, rather than preventing deliberate punishment by a counterparty.[4]

Our model has clear empirical predictions concerning the dynamics of reputations. While

[2] Liu (2009) gives an alternative explanation of long-run reputational dynamics that is driven by imperfect, costly recall and lack of a public posterior.

[3] From a technical perspective our paper differs from other recent reputation models in continuous-time, Faingold and Sannikov (2010) and Atkeson, Hellwig, and Ordonez (2010), in that we directly analyze firm value and investment incentives as integrals over profits and reputational dividends, rather than deriving ODEs for firm value as a function of reputation.

[4] Our model is related to other literatures. Fishman and Rob (2005) use a repeated game with imperfect monitoring to explain the dynamics of firm size. In the contract design literature, models with persistent effort have been studied by Fernandes and Phelan (2000) and Jarque (2010). Finally, the contrast between classical reputation models and our model is analogous to the difference between models of industry dynamics with exogenous types (Jovanovic (1982), Hopenhayn (1992)) and those with endogenous capital accumulation (Ericson and Pakes (1995)).

there is a growing empirical literature concerning reputation (Bar-Isaac and Tadelis (2008)), most of these papers are static, focusing on quantifying the value of reputation. One notable exception is Cabral and Hortaçsu (2010) which shows that an eBay seller who receives negative feedback becomes more likely to receive additional negative feedback, and is more likely to exit. This is consistent with our bad news case where a seller who receives negative feedback stops investing.

2 Model

Overview: There is one firm and a continuum of identical consumers. Time $t \in [0, \infty)$ is continuous[5] and the common interest rate is $r \in (0, \infty)$. At time t the firm produces one unit of a product that can have high or low *quality* $\theta_t \in \{L, H\}$, where $L = 0$ and $H = 1$. The firm also chooses to invest into future product quality with intensity $\eta_t \in [0, 1]$ at a flow cost of $c\eta_t$.

The firm and consumers are risk-neutral. The expected flow value of the product to a consumer equals θ_t. Consumers' common belief at time t about product quality at time t is called the firm's *reputation* $x_t = \mathbb{E}_t[\theta_t]$. At time t the firm sets price equal to the expected value x_t, so consumers' expected utility is 0 and the firm's flow profit is $x_t - c\eta_t$.

Technology: Initial quality is $\theta_0 \in \{L, H\}$. Investment controls future product quality via a Poisson process with arrival rate λ that models quality obsolescence through unpredictable technology shocks. Quality at time t is determined by the firm's investment at the most recent technology shock $s \leq t$, i.e. $\Pr(\theta_t = H) = \eta_s$; between shocks quality is constant, so if there has been no shock in $[0, t]$ then $\theta_t = \theta_0$.[6] At time t, the time of the last shock $s \leq t$ is distributed with density $\lambda e^{-\lambda(t-s)}$ so that expected quality at time t is a geometric sum of past investments $\eta^t = (\eta_s)_{s \in [0,t]}$:

$$\mathbb{E}_{\eta^t}[\theta_t] = \int_0^t \lambda e^{\lambda(s-t)} \eta_s ds + e^{-\lambda t} \mathbb{E}[\theta_0]. \tag{2.1}$$

A Markovian formulation of the same stochastic process is described by an infinitesimal quality transition matrix with diagonal entries $1 - \lambda\eta_t dt$ for low quality and $1 - \lambda(1 - \eta_t) dt$ for high quality. This formulation shows that we can interpret the firm's investment as a low quality firm buying the arrival rate of a quality improvement, and a high quality firm abating the arrival rate of a quality deterioration.

[5] There is an obvious analogue of our model in discrete time. While this discrete-time model has many expositional advantages and many of our results remain true in discrete time, finite intervals between actions mean than simple properties like monotonicity (Lemma 3) may fail since a firm with a low reputation may leap-frog a firm with a high reputation.

[6] This formulation provides a tractable way to model product quality as a function of past investments. One can interpret investment as the choice of absorptive capacity, determining the ability of a firm to recognise new external information and apply it to commercial ends (Cohen and Levinthal (1990)). Equivalently, one could assume the firm observes arrivals of technology shocks, and then chooses whether to adopt the new technology at cost $k = c/\lambda$ to become high quality, or to forgo the opportunity and become low quality.

Information: Investment η_t and actual product quality θ_t are observed only by the firm. Initially the firm's reputation is given by $x_0 = \mathbb{E}_0[\theta_0]$. Consumers learn about quality through a second Poisson process. The arrival rate of these Poisson signals at time t depends on current quality and equals μ_L if $\theta_t = L$, and μ_H if $\theta_t = H$. Conversely, the arrival rate of technology shocks is independent of past signals. A public history h_t at time t thus consists of a sequence of past signals $0 \leq t_1 \leq \cdots \leq t_n \leq t$.[7] We say that the learning process is *good news* if the net arrival rate $\mu := \mu_H - \mu_L$ is positive, *perfect good news* if $\mu_L = 0$, *bad news* if $\mu < 0$, and *perfect bad news* if $\mu_H = 0$. Market learning is *imperfect* if $\mu_L, \mu_H > 0$ and $\mu \neq 0$.

Reputation updating: The firm's reputation x_t evolves as a function of believed investments $\widetilde{\eta}^t$ and the public history of signals h_{t^-} up to time t. Formally $x_t = \mathbb{E}_t[\theta_t] = \mathbb{E}_{\widetilde{\eta}^t, h_{t^-}}[\theta_t]$, where the '$-$' indicates that reputation at time t does not take account of signals at time t. Suppose the firm is believed to invest at intensity $\widetilde{\eta}_t$ over the time interval $[t, t+dt)$.

First, if no signal arrives then reputation at the end of the interval is:

$$x_{t+dt} = \lambda dt \widetilde{\eta}_t + (1 - \lambda dt) \frac{x_t (1 - \mu_H dt)}{x_t (1 - \mu_H dt) + (1 - x_t)(1 - \mu_L dt)}.$$

The term $\lambda dt \widetilde{\eta}_t$ reflects the possibility that quality became obsolete in $[t, t+dt)$ and is newly determined based on $\widetilde{\eta}_t$. The second term reflects learning about quality at time t based on the absence of a signal in $[t, t+dt)$ and Bayes' rule. In the limit as $dt \to 0$, absent a signal, reputation evolves smoothly with *reputational drift:*

$$d(x_t) = \lambda (\widetilde{\eta}_t - x_t) - \mu x_t (1 - x_t). \tag{2.2}$$

In Section 2.1 we impose restrictions on $\widetilde{\eta}_t$ to ensure that the ODE $\dot{x}_t = d(x_t)$ has a unique solution $x_t^{\varnothing}$ that describes the reputational trajectory in the absence of signals.

Second, if there is a signal at time t then reputation jumps from x_{t^-} (the limit of the reputation before the jump) to:

$$x_t = j(x_{t^-}) := \frac{\mu_H x_{t^-}}{\mu_H x_{t^-} + \mu_L (1 - x_{t^-})} = x_{t^-} + \frac{\mu x_{t^-} (1 - x_{t^-})}{\mu_H x_{t^-} + \mu_L (1 - x_{t^-})} \tag{2.3}$$

With good news the signal indicates high quality and $j(x) > x$; with bad news we have $j(x) < x$.

Believed investment $\widetilde{\eta}$ controls the reputational drift, and actual investment η controls the distribution of signal arrivals. We call the resulting stochastic process that governs reputation x_t

[7] The public history h_t can be interpreted as consumers' realized utilities that consumers share perfectly. Alternatively, it may represent information distinct from utility realizations; under this interpretation we need to assume that consumers are short-lived and do not share their experiences.

reputational dynamics.

Markov perfect equilibrium: We assume that market beliefs about investment $\widetilde{\eta}_t = \widetilde{\eta}(x_t)$ depend on calendar time t and the public history h_{t^-} only via the firm's reputation x_t. As a result, optimal investment $\eta = \eta(\theta, x)$ also depends on history only via the firm's reputation.[8] The firm's value is a function of the initial quality and reputation, the firm's investment and the market's beliefs,

$$V_{\theta_0}(x_0; \eta, \widetilde{\eta}) := \mathbb{E}_{\theta_0, x_0, \eta, \tilde{\eta}}\left[\int_{t=0}^{\infty} e^{-rt}(x_t - c\eta_t)dt\right]. \tag{2.4}$$

When the context is clear, we often drop η and $\widetilde{\eta}$ from the notation and write firm value as a function of its quality and its reputation, $V_\theta(x)$. A Markov perfect equilibrium $\langle \eta, \widetilde{\eta} \rangle$, or simply *equilibrium* consists of a Markovian investment function $\eta : \{L, H\} \times [0,1] \to [0,1]$ for the firm and Markovian market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that:

(a) Investment maximizes firm value, $\eta \in \arg\max_\eta \{V_{\theta_0}(x_0; \eta, \widetilde{\eta})\}$,

(b) Market beliefs are correct, $\widetilde{\eta}(x) = x\eta(H, x) + (1 - x)\eta(L, x)$,

and additionally the following technical regularity conditions are satisfied:

(c) Market beliefs $\widetilde{\eta}(x)$ give rise to a well-defined law of motion (2.2)

(d) Investment as a function of time is *forward-continuous*: $\eta(x_0) = \lim_{\delta \to 0} \eta\left(x_\delta^\varnothing\right)$ for all x_0.

Condition (d) ensures that reputational dynamics and firm value are responsive to instantaneous investment by ruling out strategies where the firm 'pulses' its investment, such as $\eta_0 = 0$ and $\eta_t = 1$ for $t > 0$. Below we discuss conditions on $\widetilde{\eta}$ that ensure that (c) and (d) are satisfied.

We solve for equilibrium as follows. Consider a Markovian *candidate equilibrium*, i.e. Markovian investment $\eta : \{L, H\} \times [0,1] \to [0,1]$ and beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that beliefs are correct (b), and conditions (c) and (d) are satisfied. We then verify whether $\langle \eta, \widetilde{\eta} \rangle$ is a Markov perfect equilibrium by calculating investment incentives according to Theorem 1 and checking whether the firm's investment choice is optimal as characterised by Lemma 4, below. That lemma implies that we can restrict attention to candidate equilibria where investment does not depend on quality, i.e. $\eta(\theta, x) = \eta(x) = \widetilde{\eta}(x)$, and we simply denote candidate equilibria by η thereafter.

[8] In principle, investment η and beliefs $\widetilde{\eta}$ could depend on time t and the entire public history h_{t-}, and investment additionally on past realizations of quality. We assume that market beliefs $\widetilde{\eta}$ are Markovian because we think of the continuum of consumers as sharing their experience in a sufficient yet incomplete manner, e.g. through consumer reports.

2.1 Reputational Dynamics

In this section we impose restrictions on the Markovian beliefs $\widetilde{\eta}(x)$ to ensure that conditions (c) and (d) are satisfied, we introduce the 'work-shirk' terminology, and we show that reputational dynamics are ergodic in any work-shirk equilibrium.

First, we assume that $\widetilde{\eta}(x)$ is piecewise constant and equal to 0 or 1 for all x but a finite number of cutoffs $0 \leq x_1^* < \cdots < x_n^* \leq 1$ where $\widetilde{\eta}$ is discontinuous. This restriction on investment as a function of reputation does not yet mean that dynamics are well-defined: if reputational drift is strictly positive below x^* and strictly negative above x^*, then $\widetilde{\eta}(x^*)$ must be such that the drift at x^* is zero. To account for this complication we assume that

$$d(x_i^*) > 0 \;\Rightarrow\; d(x_i^*) = \lim_{\epsilon \to 0} d(x_i^* + \epsilon) \qquad \text{and} \qquad d(x_i^*) < 0 \;\Rightarrow\; d(x_i^*) = \lim_{\epsilon \to 0} d(x_i^* - \epsilon) \tag{2.5}$$

at any cutoff x_i^*. Given any x_i^* we can choose $\widetilde{\eta}(x_i^*)$ such that (2.5) holds. Following Klein and Rady (2010), this implies that conditions (c) and (d) are satisfied.

We call x_i^* a *work-shirk cutoff* if $\widetilde{\eta}$ jumps up at x_i^*; otherwise x_i^* is a *shirk-work cutoff*. We say that reputational drift is *convergent* at work-shirk cutoff x_i^* if it is positive below the cutoff and negative above. Conversely, we say that reputational drift is *divergent* at shirk-work cutoff x_i^* if it is negative below the cutoff and positive above. A cutoff is *permeable* if reputational drift is either strictly positive in a neighborhood of the cutoff or strictly negative. If reputational drift at x_i^* is convergent or x_i^* is permeable, investment $\widetilde{\eta}(x_i^*)$ is uniquely determined by (2.5). If reputational drift at x_i^* is divergent, multiple values of $\widetilde{\eta}(x_i^*)$ are compatible with (2.5). A candidate equilibrium is *work-shirk* if there exists a single work-shirk cutoff $x^* \in (0,1)$, so $\widetilde{\eta} = 1$ below x^* and $\widetilde{\eta} = 0$ above x^*. Conversely, a candidate equilibrium is *shirk-work* if there exists a single shirk-work cutoff $x^* \in [0,1]$, so $\widetilde{\eta} = 0$ below x^* and $\widetilde{\eta} = 1$ above x^*.[9] Finally, a candidate equilibrium is *full work* if $\widetilde{\eta} \equiv 1$ and *full shirk* if $\widetilde{\eta} \equiv 0$.

Finally, we call reputational dynamics in equilibrium $\langle \eta, \widetilde{\eta} \rangle$ *ergodic* if there exists a probability distribution F over $[0,1]$ such that for any starting values x_0 reputation x_t converges to F in distribution as $t \to \infty$.

Lemma 1 *In any candidate equilibrium $\langle \eta, \widetilde{\eta} \rangle$ that is work-shirk, full work or full shirk, reputational dynamics are ergodic.*

Proof. In Appendix A.1. □

[9] We do not allow for work-shirk cutoffs $x^* = 0$ or 1, because (2.5) implies $\widetilde{\eta}(x^*) = 0$ for $x^* = 0$, so a work-shirk candidate equilibrium with cutoff $x^* = 0$ is identical to the full shirk candidate equilibrium; similarly work-shirk with cutoff $x^* = 1$ equals full work. To the contrary, we do allow for shirk-work cutoffs $x^* = 0$ or 1 because a shirk-work candidate equilibrium with cutoff $x^* = 0$ and $\widetilde{\eta}(x^*) = 0$, where the firm works at all reputation levels except at 0, is not just well-defined but has qualitatively different properties than a full work candidate equilibrium, see proof of Theorem 3(b).

2.2 Value Functions

To fix ideas and prepare for subsequent analysis, we now show that value functions are bounded and equilibrium value functions are monotone.

Lemma 2 *In any candidate equilibrium $\langle \eta, \widetilde{\eta} \rangle$, the value function of the firm $V_\theta(x)$ is bounded and takes values in $[-c/r, 1/r]$.*

Proof. At any time t profits are bounded: $x_t - c\eta_t \in [-c, 1]$. □

Lemma 3 *In any equilibrium $\langle \eta, \widetilde{\eta} \rangle$, the value function of the firm $V_\theta(x)$ is strictly increasing in reputation x.*

Proof. Fix initial reputations $x_0 < x_0'$ of a "low" and "high" firm with initial quality θ_0. Suppose the high firm chooses the non-Markovian strategy η' that mimics equilibrium investment of the low firm, i.e. if at time t after history h^{t-} the low firm has reputation $x_t = x_t(x_0, h_{t^-}, \widetilde{\eta})$ then $\eta_t' = \eta(\theta_t, x_t(x_0, h_{t^-}, \widetilde{\eta}))$. Adopting this strategy, the high firm's quality θ_t' is governed by the same process as the equilibrium quality θ_t of the low firm. Thus these firms face the same distribution of public histories and the reputation of the high firm never falls behind, i.e. $x_t' \geq x_t$ with strict inequality for t close to zero. Then the profit of the high firm with investment strategy η' always exceeds the equilibrium profit of the low firm, i.e. $x_t' - c\eta_t' \geq x_t - c\eta(x_t)$, because revenue is greater for the high firm by the above argument, and costs are equal by construction. This completes the proof because the equilibrium value of the high firm $V_\theta(x_0')$ is weakly higher than its value from the feasible strategy η'. □

Value functions are not necessarily continuous in reputation. At a shirk-work cutoff with divergent drift, future reputation is discontinuous as a function of current reputation and so are value functions. However, Lemma 8(a) in Appendix C.2 shows that value functions are continuous for work-shirk candidate equilibria.

2.3 Optimal Investment Choice

Fix a candidate equilibrium. The marginal benefit of investment over $[t, t+dt)$ is the probability of a technology shock hitting, λdt, times the difference in value functions $\Delta(x) := V_H(x) - V_L(x)$, which we call the *value of quality*. The marginal cost of investment is c, so equilibrium investment $\eta(\theta, x)$ must satisfy

$$\eta(\theta, x) = \begin{cases} 1 & \text{if } c < \lambda\Delta(x), \\ 0 & \text{if } c > \lambda\Delta(x). \end{cases} \tag{2.6}$$

Quality after the shock is independent of current quality, so the benefit of investment is independent of the firm's current quality. This implies that our results are not driven by the asymmetric information about product quality, but solely by the unobserved investment into future quality. Lemma 4 formalizes this intuition:

Lemma 4 *A candidate equilibrium* $\langle \eta, \widetilde{\eta} \rangle$ *is an equilibrium if and only if the 'bang-bang' equation (2.6) holds for all* (θ, x)*. Hence equilibrium investment is independent of quality as long as* $\lambda\Delta(x) \neq c$*.*

Proof. For a rigorous proof of the above intuition we expand the firm's current value into its profits over $[t, t+\delta)$ and its expected continuation value

$$V_{\theta_0}(x_0) = (x_0 - c\eta(x_0))\,\delta + \mathbb{E}_{x_0,\theta_0}\left[e^{-r\delta} V_{\theta_\delta}(x_\delta)\right] + o(\delta)$$

The continuation value in turn depends on $\eta(x_0)$ only in the case of a technology shock,

$$\mathbb{E}_{x_0,\theta_0}\left[e^{-r\delta} V_{\theta_\delta}(x_\delta)\right] = e^{-(r+\lambda)\delta} \underbrace{\mathbb{E}_{x_0,\theta_0}\left[V_{\theta_0}(x_\delta)\right]}_{\text{no } \lambda\text{-shock}} + \lambda\delta \underbrace{\left(\eta(x_0) V_H(x_0) + (1-\eta(x_0)) V_L(x_0)\right)}_{\lambda\text{-shock}} + o(\delta)\,.$$

Both steps use the fact the investment function is forward-continuous; the second step also uses that the value function is forward-continuous in expectation, i.e. $V_\theta(x_0) = \lim_{\delta\to 0} \mathbb{E}\left[V_\theta(x_\delta)\right]$. The marginal profit of investment over $[t, t+\delta)$ is thus given by $-c\delta + \lambda\delta\left(V_H(x_0) - V_L(x_0)\right)$ so the optimal investment η^* is characterized by (2.6). □

When marginal costs and benefits of investment coincide, i.e. when $c = \lambda\Delta(x)$, optimal investment is indeterminate and may in principle depend on the firm's quality. Consider such an equilibrium $\langle \eta, \widetilde{\eta} \rangle$ with $\eta(H, x) \neq \eta(L, x)$. By Lemma 4, the candidate equilibrium $\langle \overline{\eta}, \widetilde{\eta} \rangle$ with investment $\overline{\eta}(x) = \widetilde{\eta}(x)$ is also an equilibrium because $\langle \eta, \widetilde{\eta} \rangle$ and $\langle \overline{\eta}, \widetilde{\eta} \rangle$ give rise to identical value functions and investment incentives. As they also give rise to the same reputational dynamics we consider these equilibria as identical and henceforth restrict attention to candidate equilibria $\langle \eta, \widetilde{\eta} \rangle$ with $\eta(\theta, x) = \widetilde{\eta}(x)$ and denote them by η.

2.4 First-Best Solution

As a benchmark, suppose product quality is publicly observed so price equals quality. The benefit of investing equals the obsolescence rate λ, times the price differential 1, divided by the effective discount rate $r + \lambda$. Thus first-best investment is given by:

$$\eta = \begin{cases} 1 & \text{if } c < \frac{\lambda}{r+\lambda} \\ 0 & \text{if } c > \frac{\lambda}{r+\lambda} \end{cases}\,. \tag{2.7}$$

In our model, there is no equilibrium with positive investment if $c > \lambda/(r+\lambda)$: Investment decreases welfare and consumers receive zero utility in equilibrium, so firm profits must be negative.

The firm therefore prefers to shirk at all levels of reputation, thereby guaranteeing itself a non-negative payoff. Our results are therefore non-trivial only if $c < \lambda / (r + \lambda)$.

3 Value of Quality

In any candidate equilibrium, the firm's value $V_\theta(x)$ is a function of its reputation x and its quality θ. While reputation directly determines revenue (see Lemma 3), quality derives its value indirectly through its effect on reputation. More precisely, we show below (in Theorem 1) that the value of quality can be written as a present asset value of future reputational dividends.

To analyze the value of quality $\Delta(x) = V_H(x) - V_L(x)$, we expand the value functions into current profits and continuation values as in the proof of Lemma 4. Current profits cancel because both current revenue and costs depend on reputation but not on quality. The continuation values are discounted at both the interest rate r and the quality obsolescence rate λ,

$$\Delta(x_0) = e^{-(r+\lambda)dt} \left[\mathbb{E}_{\theta=H}[V_H(x_{dt})] - \mathbb{E}_{\theta=L}[V_L(x_{dt})] \right] \tag{3.1}$$

where $\mathbb{E}_{\theta=H}[\cdot]$ means that x_t evolves conditional on $\theta = H$. For a recursive formulation of $\Delta(x)$ we need to evaluate the value functions at the same levels of future reputation, and do so by adding and subtracting a term $\mathbb{E}_{\theta=L}[V_H(x_{dt})]$ to obtain

$$\Delta(x_0) = e^{-(r+\lambda)dt}\Big[\mathbb{E}_{\theta=H}[V_H(x_{dt})] - \mathbb{E}_{\theta=L}[V_H(x_{dt})]\Big] + e^{-(r+\lambda)dt}\Big[\mathbb{E}_{\theta=L}[V_H(x_{dt})] - \mathbb{E}_{\theta=L}[V_L(x_{dt})]\Big].$$

The first term is the *reputational dividend* that captures the immediate reputational benefit of high versus low quality; the second term is the continuation value. The dividend consists of the incremental probability $\mu = \mu_H - \mu_L$ of a signal due to high quality, times the value of the reputational jump $j(x) - x$. Hence:

$$\Delta(x_0) = e^{-(r+\lambda)dt} \underbrace{\mu \left[V_H(j(x_{dt})) - V_H(x_{dt}) \right]}_{\text{Rep. dividend}} dt + e^{-(r+\lambda)dt} \underbrace{\mathbb{E}_{\theta=L}[\Delta(x_{dt})]}_{\text{Cont. value}} \tag{3.2}$$

Integrating this equation yields equation (3.3) in Theorem 1, which expresses the asset value of quality as the discounted sum of future reputational dividends. Equivalently, equation (3.4) follows from the alternative decomposition of (3.1) when we add and subtract $\mathbb{E}_{\theta=H}[V_L(x_{dt})]$ instead of $\mathbb{E}_{\theta=L}[V_H(x_{dt})]$.

Theorem 1 *Fix any candidate equilibrium η. Then two closed-form expressions for the value of* *quality are given by:*

$$\Delta(x_0) = \mathbb{E}_{x_0,\theta^\infty=L}\left[\int_0^\infty e^{-(r+\lambda)t}\mu\left[V_H\left(j(x_t)\right)-V_H(x_t)\right]dt\right], \tag{3.3}$$

$$= \mathbb{E}_{x_0,\theta^\infty=H}\left[\int_0^\infty e^{-(r+\lambda)t}\mu\left[V_L\left(j(x_t)\right)-V_L(x_t)\right]dt\right], \tag{3.4}$$

where $\theta^\infty = L$ is short for $\theta_t = L$ for all $t \in [0,\infty)$.

Proof. See Appendix A.2. □

While standard reputation models incentivize effort through an immediate effect on the firm's reputation, investment in our model pays off through quality with a delay. Once quality is established, it is persistent and generates a stream of reputational dividends until it becomes obsolete. We must therefore evaluate the reputational incentives at future levels of reputation x_t, rather than just at the current level x_0.

4 Perfect Poisson Learning

We first consider Poisson processes where a signal perfectly reveals the firm's quality. Theorems 2 and 3 highlight how different learning processes lead to opposite investment incentives and reputational dynamics. These cases are highly tractable and help to build intuition for more general learning processes. The value functions can also be calculated explicitly, as shown in Appendix B, which may be useful in applications.

4.1 Perfect Good News

Assume that consumers learn about quality via product *breakthroughs* that reveal high quality $\theta_t = H$ with arrival rate μ. That is, $\mu_H = \mu$ and $\mu_L = 0$. When a breakthrough occurs, the reputation jumps to one. Absent a breakthrough, updating evolves deterministically according to

$$d\left(x_t\right) = \lambda\left(\widetilde{\eta}_t - x_t\right) - \mu x_t\left(1-x_t\right). \tag{4.1}$$

The reputational dividend is the value of having a high quality in the next instant. This equals the value of increasing the reputation from its current value to one, times the probability of a breakthrough, i.e. $\mu(V_H(1)-V_H(x))$. Using equation (3.3), the value of quality is given by

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu[V_H(1)-V_H(x_t^\varnothing)]dt, \tag{4.2}$$

where $x_t^\varnothing$ is the deterministic solution of the ODE $\dot{x}_t = d\left(x_t\right)$ with initial value x_0. We do not need to take an expectation in equation 4.2 because there are no signals conditional on low quality, so the reputational trajectory equals $x_t^{\varnothing}$.

In equilibrium, the reputational dividend $V_H(1) - V_H(x_t^{\varnothing})$ is decreasing in $x_t^{\varnothing}$ so that $\Delta(x_0)$ is decreasing in x_0. Intuitively, a breakthrough that boosts the firm's reputation to one is most valuable for a firm with a low reputation. Thus, investment incentives decrease in reputation and any equilibrium must be full shirk, full work, or work-shirk.

In a work-shirk equilibrium with cutoff x^* reputational dynamics converge to a cycle. Absent a breakthrough, the firm's reputation converges to a stationary point $\hat{x} = \min\{\lambda/\mu, x^*\}$ where the firm works with positive probability. At a breakthrough, the firm's reputation jumps to one. The firm is then believed to be shirking, so its reputation drifts down to $\hat{x}$, absent another breakthrough. In the long-run, the firm's reputation therefore cycles over the range $[\hat{x}, 1]$. In case that $\lambda \geq \mu$, the firm's reputation drifts up whenever it is believed to be working (see Figure 3(a)). Here, reputational drift is zero at $\hat{x} = x^*$, and the firm chooses to work with intensity $\eta(x^*) = x^* \left(1 + \frac{\mu}{\lambda}(1 - x^*)\right)$ at this point. In case that $\lambda < \mu$, a cutoff $x^* > \lambda/\mu$ is permeable and reputation drifts into the work region $[0, x^*]$ towards $\hat{x} = \lambda/\mu$.

Theorem 2 *Under perfect good news learning:*

(a) Every equilibrium is work-shirk or full shirk.

(b) An equilibrium exists.

(c) In any equilibrium reputational dynamics are ergodic.

(d) If $\lambda \geq \mu$, the equilibrium is unique.

Proof. Part (a). Fix an equilibrium. Reputation $x_t^{\varnothing}$ follows (4.1), so an increase in x_0 raises $x_t^{\varnothing}$ at each point in time. Lemma 3 states that $V_H(x)$ is strictly increasing in x, so equation (4.2) implies that $\Delta(x_0)$ is strictly decreasing in x_0.

We can rule out a full work equilibrium: If $\eta(x) = 1$ for all x, then $x_0 = 1$ implies $x_t^{\varnothing} = 1$ for all t by (4.1); thus $\Delta(1) = 0$ and a firm with perfect reputation prefers to shirk.

Part (b). For any $x^* \in [0, 1]$ let $\Delta_{x^*}(x)$ be the value of quality of a firm with reputation x in the candidate equilibrium with cutoff x^* (where $x^* = 0$ represents full shirk, and $x^* = 1$ full work).[10] Lemma 9 in Appendix C.2 shows that value functions, and thus the value of quality $\Delta_{x^*}(x^*)$ at the cutoff is continuous in x^*. If $\lambda\Delta_0(0) \leq c$, full shirk is an equilibrium. Otherwise, if $\lambda\Delta_0(0) \geq c$, there must be some x^* with $\lambda\Delta_{x^*}(x^*) = c$ by the intermediate value theorem since $\Delta_1(1) = 0$

Part (c). Follows from Lemma 1.

[10] Recall that investment at the work-shirk cutoff x^* is uniquely pinned down by condition (2.5).

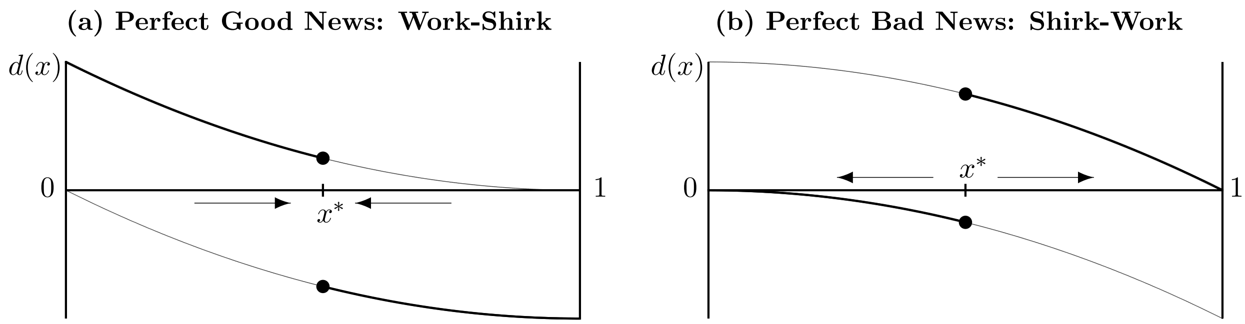

Figure 3: **Reputational drift in work-shirk and shirk-work equilibria.** This figure illustrates how the reputational drift $d(x)$ changes with the reputation of the firm, x. These pictures assume $\lambda = \mu$ for perfect good news, $\lambda = \mu_L$ for perfect bad news, and $x^* = 1/2$. The dark line shows equilibrium drift and the arrows show its direction.

Part (d). Given $\lambda \geq \mu$, the deterministic trajectory $x_t^{\varnothing}$ is stationary at x^* and equation (B.4) in Appendix B.1 delivers a closed-form expression for $\Delta_{x^*}(x^*)$ which is decreasing in x^*, implying uniqueness of equilibrium. □

To understand the uniqueness result, Theorem 2(d), consider a work-shirk candidate equilibrium with cutoff x^* and let x be such that $\lambda\Delta(x) = c$. An increase in x^* means the firm's reputation will not drift down as far, absent a breakthrough. This change benefits low-quality firms more than high-quality firms, reducing $\Delta(\cdot)$. As a result, $x(x^*)$ is decreasing in x^* and there is a unique fixed point where $x(x^*) = x^*$.

4.2 Perfect Bad News

Assume that x_t is generated by *breakdowns* that reveal low quality $\theta_t = L$ with arrival rate $\mu_L > 0$, while high quality products never suffer breakdowns, i.e. $\mu_H = 0$. When a breakdown occurs, the reputation drops to zero. Absent a breakdown, updating evolves deterministically according to

$$d(x_t) = \lambda(\widetilde{\eta}_t - x_t) + \mu_L x_t(1 - x_t). \tag{4.3}$$

The reputational dividend is the value of having a high quality in the next instant. Quality insures the firm against a breakdown, so the reputational dividend equals $\mu_L(V_L(x) - V_L(0))$. Using equation (3.4), the value of quality is:

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu_L[V_L(x_t^{\varnothing}) - V_L(0)]dt. \tag{4.4}$$

where $x_t^{\varnothing}$ is the deterministic solution of the ODE (4.3) with initial value x_0. We do not need to take an expectation in equation 4.4 because there are no signals conditional on high quality, so the reputational trajectory equals $x_t^{\varnothing}$.

In equilibrium, the reputational dividend $V_L(x_t^{\varnothing}) - V_L(0)$ is increasing in $x_t^{\varnothing}$, so that $\Delta(x_0)$ is increasing in x_0. Intuitively, a breakdown that destroys the firm's reputation is most damaging for a firm with a high reputation. Thus, investment incentives increase in reputation and any equilibrium must be full work, full shirk, or shirk-work.

Shirk-work beliefs imply that reputational dynamics diverge. Consider a shirk-work equilibrium where the firm shirks if its reputation is below x^*, and works above x^*. A firm that starts with reputation above x^* converges to reputation $x = 1$, absent a breakdown. If the firm is hit by such a breakdown while its product quality is still low, reputation drops to zero and is trapped there forever. A firm with reputation below x^* initially shirks and may have either rising or falling reputation, depending on parameters. In either case, its reputation will either end up at zero or one.

Investment incentives in any shirk-work candidate equilibrium are maximized at $x = 1$ and equal $\Delta(1) = \frac{\mu_L(1-c)}{r(r+\lambda+\mu_L)}$.[11] Thus, to obtain an equilibrium other than full shirk we have to assume:

$$c < \lambda \frac{\mu_L(1-c)}{r(r+\lambda+\mu_L)}. \tag{4.5}$$

Theorem 3 *Under perfect bad news learning:*

(a) Every equilibrium is shirk-work, full shirk or full work.

(b) An equilibrium exists.

(c) In any shirk-work equilibrium with cutoff $x^* \in (0,1)$ *reputational dynamics are not ergodic.*

(d) Assume (4.5) holds and $\lambda \geq \mu_L$. *There are* $a < b$ *such that every* $x^* \in [a,b]$ *is the cutoff of a shirk-work equilibrium.*

Proof. Part (a). Fix an equilibrium. Reputation $x_t^\varnothing$ follows (4.3), so an increase in x_0 raises $x_t^\varnothing$ at each point in time. Lemma 3 states that $V_L(x)$ is strictly increasing in x, so equation (4.4) implies that $\Delta(x_0)$ is increasing in x_0.

Part (b). By part (a) any equilibrium is defined by a shirk-work cutoff $x^* \in [0,1]$ and the investment at the cutoff $\eta(x^*)$. Let $\hat{x} = 1 - \lambda/\mu_L$ be the reputation where drift is zero when $\widetilde{\eta}(\hat{x}) = 0$. If $x^* \in (0,\hat{x})$ then the cutoff is permeable with positive drift and we must have $\eta(x^*) = 1$. If $x^* \geq \hat{x}$ then drift is divergent at the cutoff, so both $\eta(x^*) = 0$ and $\eta(x^*) = 1$ are possible.[12]

For $x^* \geq \hat{x}$ or $x^* = 0$, let $\Delta_{x^*,0}(\cdot)$ be the value of quality in the candidate equilibrium where the firm shirks at reputations $x \in [0,x^*]$ and works at reputations $x \in (x^*,1]$. For $x^* = 1$ this is the full shirk candidate equilibrium. Similarly, for any $x^* \in [0,1]$ define $\Delta_{x^*,1}(\cdot)$ when the firm shirks in $[0,x^*)$ and works in $[x^*,1]$. For $x^* = 0$ this is the full work candidate equilibrium.

[11] This is because $V_H(1) = (1-c)/r$ and $V_L(1) = \frac{r+\lambda}{r+\lambda+\mu_L}(1-c)/r$.

[12] For brevity, we ignore the third possible value $\eta(x^*) = x^*(1-\mu_L(1-x^*)/\lambda)$ that would lead to $d(x^*) = 0$.

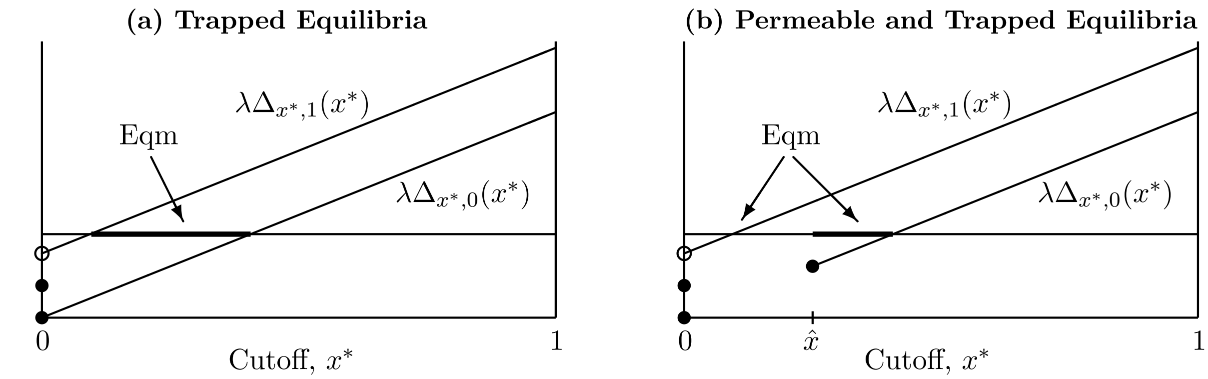

Figure 4: **Sets of equilibria under perfect bad news learning.** This picture shows the value of quality at the shirk-work cutoff $\Delta_{x^*,1}(x^*)$ and $\Delta_{x^*,0}(x^*)$ used in the proof of Theorem 3. On the **left-hand side** we have $\lambda \geq \mu_L$, so that $\hat{x} \leq 0$. The figure shows an interval of equilibria. On the **right-hand side** we have $\lambda < \mu_L$, so that $\hat{x} > 0$. In this case, $\Delta_{x^*,0}(x^*)$ is only defined for $x^* = 0$ and $x^* \geq \hat{x}$. The figure shows a single permeable equilibrium (the single point) and an interval of trapped equilibria.

Fix $x^* > 0$. In Appendix B.3 we show that a permeable cutoff defines an equilibrium if the firm is indifferent between working and shirking at the cutoff. That is, $x^* < \hat{x}$ defines an equilibrium if

$$\lambda\Delta_{x^*,1}\left(x^*\right) = c. \tag{4.6}$$

A cutoff with divergent drift defines an equilibrium if, at the cutoff, the firm prefers to work when believed to be working and prefers to shirk when believed to be shirking. That is, $x^* \geq \hat{x}$ defines an equilibrium if

$$\lambda\Delta_{x^*,0}(x^*) \leq c \leq \lambda\Delta_{x^*,1}(x^*). \tag{4.7}$$

To complete the existence proof, we consider three cases. If $\lambda\Delta_{x^*,1}(x^*) > c$ for all $x^* > 0$, there is an equilibrium where the firm works at all reputation levels except zero. If $\lambda\Delta_{x^*,1}(x^*) < c$ for all $x^* > 0$, then full shirk is an equilibrium. Otherwise, we show that $\Delta_{x^*,1}\left(x^*\right)$ is continuous in x^* so the intermediate value theorem implies that either (4.6) or (4.7) are satisfied.

Part (c). Both $x = 0$ and $x = 1$ are absorbing states in a shirk-work equilibrium. Hence, dynamics cannot be ergodic.

Part (d). If $\lambda \geq \mu_L$, then $\hat{x} \leq 0$ and $\Delta_{x^*,0}(x^*)$ is defined for all $x^* \in [0,1]$. Equation (4.7) is then satisfied for some x^*: for the upper bound, condition (4.5) implies $c\lambda\Delta_{x^*,1}(x^*) > c$ for $x^* = 1$; for the lower bound, $\lambda\Delta_{x^*,0}(x^*) = 0 < c$ for $x^* = 0$. In Appendix B.3 we then prove that, since the reputational drift at the cutoff is divergent, $\Delta_{x^*,\eta^*}(\cdot)$ is discontinuous at x^* and $\Delta_{x^*,0}(x^*) < \Delta_{x^*,1}(x^*)$. Hence (4.7) is satisfied for a continuum of x^*, as illustrated by Figure 4(a). □

When $\lambda \geq \mu_L$, adverse beliefs outweigh the absence of breakdowns and reputational drift is always negative in the shirk-region $[0, x^*)$, as shown in Figure 3(b). We call such an equilibrium

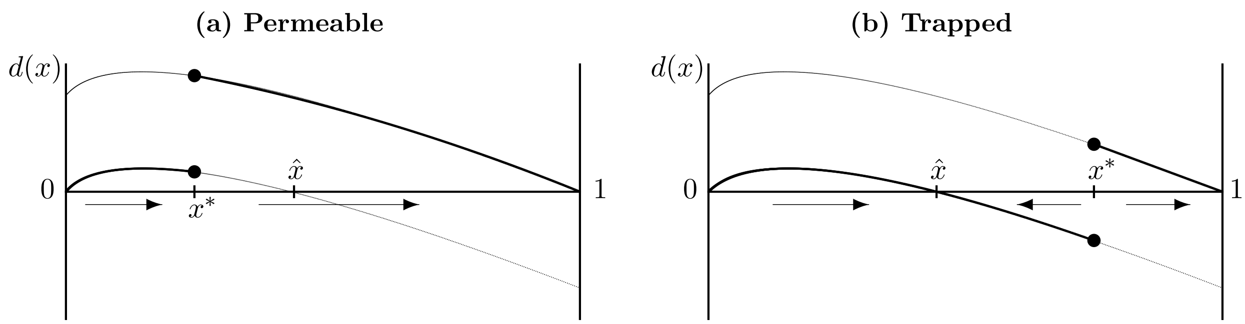

Figure 5: **Reputational drift in shirk-work equilibrium under bad news.** This figure illustrates how the reputational drift $d(x)$ changes with the reputation of the firm, x. These pictures assume $\lambda < \mu_L$ so that $\hat{x} \in (0, 1)$. The left picture has $x^* < \hat{x}$, giving rise to a permeable equilibrium. The right picture has $x^* > \hat{x}$, giving rise to a trapped equilibrium. The dark line shows equilibrium drift and the arrows show its direction.

a *trapped equilibrium* because the firm cannot escape from the shirk-region. When the firm's reputation is above the cutoff, favorable market beliefs contribute to an increasing reputation and the firm invests to insure itself against a product breakdown. At the cutoff, the firm works if it is believed to be working and shirks if it is believed to be shirking.

Theorem 3(d) shows that there is an interval of shirk-work cutoffs satisfying the equilibrium condition (4.7), as shown in Figure 4(a). The multiplicity is driven by a discontinuity in the value function at the shirk-work cutoff, caused by the divergent reputational dynamics. Intuitively, market beliefs become self-fulfilling. If the market believes the firm is shirking, it faces low future reputation and dividends and investment incentives are low. Conversely, if the market believes the firm is working, its reputation will rise and incentivizes the firm to invest in order to protect its appreciating reputation.

When $\lambda < \mu_L$ the reputational dynamics may have additional interesting features. Let $\hat{x} = 1-\lambda/\mu_L$ be the stationary point of the drift where adverse market beliefs exactly cancel the absence of breakdowns. In this case there may exist a qualitatively different *permeable equilibrium:* When $x^* < \hat{x}$, the reputational drift is strictly positive on $(0, 1)$ as the absence of breakdowns outweighs the adverse beliefs in the shirk-region $(0, x^*)$; this is illustrated in Figure 5(a). If x_t passes x^* before a breakdown hits, the firm starts to work and its reputation converges to one if the technology shock hits before the breakdown. Since the value functions are continuous at a permeable cutoff x^*, there is at most one permeable equilibrium.

There may also exist trapped equilibria in this case. When $x^* > \hat{x}$, reputation in the shirk-region drifts towards $\hat{x}$ but the shirk-region is absorbing, just like in the case $\lambda \geq \mu_L$, as shown in Figure 5(b). Since reputational drift is divergent at x^*, the value function is discontinuous and there is a continuum of such equilibria. Figure 4(b) illustrates that permeable and trapped equilibria co-exist for some parameter values.

4.3 Work-Shirk vs. Shirk-Work Equilibria

Investment incentives differ fundamentally between the work-shirk equilibria under perfect good news learning and the shirk-work equilibria under perfect bad news learning. In the former case investment is rewarded by reputational boosts; these boosts are temporary because adverse equilibrium beliefs at high reputations bring the reputation down again. In the latter case investment averts a reputational loss; this loss is permanent because adverse equilibrium beliefs at low reputations prevent a recovery. When the rate of quality obsolescence λ is high, the benefit of a reputational boost disappears quickly while a reputational loss is still permanent. In this sense, incentives in a shirk-work equilibrium are stronger than those in a work-shirk equilibrium.

Theorem 4 *There exists λ^* such that for all $\lambda > \lambda^*$:*

(a) Under perfect good news learning, full shirk is the unique equilibrium.

(b) Under perfect bad news learning, any cutoff $x^* \in (0,1]$ defines a shirk-work equilibrium, if condition (4.5) is satisfied.

Proof. Part (a). By Theorem 2(d) equilibrium is unique when $\lambda \geq \mu$ and we just need to show that full shirk is an equilibrium. Investment incentives are decreasing by the proof of Theorem 2(a), so it suffices to verify that $\lambda\Delta_0(0) \leq c$. By equation (3.4) the value of quality is bounded above by the perpetuity value of the maximal dividends:

$$\lambda\Delta_0(0) = \mathbb{E}_{\theta^\infty=H}\left[\lambda\int_0^\infty e^{-(r+\lambda)s}\mu\left[V_L(1) - V_L(x_s)\right]ds\right] \leq \frac{\lambda}{r+\lambda}\mu V_L(1).$$

The key step in the argument is that the dividend, and its upper bound $\mu V_L(1)$, vanish for high values of λ. The drift $d(x) \leq -\lambda x$ decreases reputation at an exponential rate, i.e. $x_t^{\varnothing} \leq e^{-\lambda t}x_0$, so for $x_0 = 1$ we have

$$\mu V_L(1) = \mu\int_0^\infty e^{-rt}x_t^{\varnothing}dt \leq \frac{\mu}{r+\lambda}.$$

For $\lambda > \mu/c$, this term is smaller than c and we get

$$\lambda\Delta_0(0) \leq \frac{\lambda}{r+\lambda}\frac{\mu}{r+\lambda} < c,$$

as required.

Part (b). Assume $\lambda \geq \mu$, pick any $x^* > 0$ and assume for convenience that $\eta(x^*) = 0$. If the firm starts in the shirk-region at $x_0 \leq x^*$ then its reputation will remain there forever and reputational dividends are bounded above by $\mu_L V_L(x^*)$. By the proof of part (a), this upper bound is less than the cost c for high values of λ, and the firm prefers to shirk.

If the firm starts in the work-region at $x_0 > x^*$ then the drift $d(x) > \lambda(1-x)$ decreases the probability of low quality at an exponential rate, i.e. $1 - x_t^{\varnothing} \le e^{-\lambda t}(1-x_0)$, and the value function of a high-quality firm approaches the first-best perpetuity value:

$$V_H(x_0) = \int_0^\infty e^{-rt}\left(x_t^{\varnothing} - c\right)dt \ge \int_0^\infty e^{-rt}\left(1 - e^{-\lambda t}(1-x_0) - c\right)dt = \frac{1-c}{r} - \frac{1-x_0}{r+\lambda}$$

Using equation (3.3) we write the value of quality as function of dividends $\mu_L V_H(x_t)$:

$$\lambda\Delta_{x^*}(x_0) = \mathbb{E}_{\theta^\infty = L}\left[\lambda\int_0^\infty e^{-(r+\lambda)t}\mu_L(V_H(x_t) - V_H(0))dt\right] = \lambda\int_0^\infty e^{-(r+\lambda)t}e^{-\mu_L t}\mu_L V_H\left(x_t^{\varnothing}\right)dt$$

where the second line uses $V_H(0) = 0$ and conditions on the absence of breakdowns $\mathbb{E}_{\theta^\infty = L}[V_H(x_t)] = e^{-\mu_L t}V_H\left(x_t^{\varnothing}\right)$.

Together with $V_H\left(x_t^{\varnothing}\right) \ge V_H(x_0)$, this yields a lower bound for investment incentives

$$\lambda\Delta_{x^*}(x_0) \ge \frac{\lambda\mu_L}{r+\lambda+\mu_L}\left(\frac{1-c}{r} - \frac{1-x_0}{r+\lambda}\right).$$

This bound approaches $\mu_L(1-c)/r$ for high values of λ. Assumption (4.5) implies that $\mu_L(1-c)/r > c$, so for sufficiently large λ, working is optimal for any x^* and all $x_0 > x^*$. $\square$

As $\lambda \to \infty$ our investment game approaches a repeated game where the firm chooses its quality at every instant. Abreu, Milgrom, and Pearce (1991) study a repeated prisoners' dilemma with imperfect monitoring that approaches the same limit game as the frequency of play increases. They find that only 'bad news' signals that indicate defection can sustain cooperation, while 'good news' signals that indicate cooperation are too noisy to deter defections without destroying all surplus by punishments on the equilibrium path. Thus sustained cooperation depends on the learning process in the same way as in our model. While the common limit already suggests this analogy, our model highlights an alternative mechanism that distinguishes the role of bad news signals in overcoming moral hazard, namely divergent reputational dynamics.

Theorem 4 has a surprising consequence: Providing more information about the firm's quality may be detrimental to equilibrium investment. Specifically, consider a shirk-work equilibrium under perfect bad news learning. Suppose we improve the learning process by introducing an additional perfect good news signal, so that low quality is revealed perfectly with intensity μ_b and high quality with intensity μ_g (the two signals are assumed to be independent conditional on current quality). The analysis in Sections 2 and 3 extends immediately to this learning process with two Poisson signals and the reputational dividend is given by:

$$\mu_g(V_\theta(1) - V_\theta(x)) + \mu_b(V_\theta(x) - V_\theta(0)) = (\mu_b - \mu_g)V_\theta(x) + \mu_g V_\theta(1) - \mu_b V_\theta(0).$$

When good news is more frequent than bad news, i.e. $\mu_g > \mu_b$, reputational dividends and value of quality are decreasing in reputation and any equilibrium must be work-shirk. If additionally λ is high enough, the proof of Theorem 4(a) extends to this learning process, implying that full shirk is the only equilibrium.

Thus for identical parameter values r, c, λ, there exist shirk-work equilibria if only the perfect bad news signal is available, while full shirk is the only equilibrium when a perfect good news signal is introduced additionally. Under perfect bad news learning a firm with a high reputation works because a breakdown permanently destroys its reputation. Additional good news signals grant the firm a second chance after a breakdown and undermine incentives to work hard in the first place.

5 Imperfect Poisson Learning

In this section we suppose consumers learn about product quality through imperfect signals with Poisson arrival rates $\mu_H > 0$ and $\mu_L > 0$. The analysis becomes more involved than in the case of perfect Poisson learning because reputational dividends tend to be hump-shaped due to the $x(1-x)$ dampening factor in the Bayesian updating formula (2.3), rather than being monotonic as with perfect learning. Integrals over such hump-shaped future dividends may take a complicated shape. Nevertheless, Theorem 1 implies three robust qualitative features of equilibria across all imperfect Poisson learning processes.

First, investment at the top cannot be sustained in equilibrium. If the firm is believed to be working at the top, the value of quality is zero at $x = 1$ since current dividends are zero and, as the firm's reputation stays at $x = 1$, future dividends are zero as well. Intuitively, a firm that is believed to be working at the top is almost certain to have a high reputation in the future, undermining incentives to actually invest. The same argument applies in case of perfect good news learning, while perfect bad news invalidates this argument as reputation drops to zero at a breakdown. Second, for intermediate levels of reputation, dividends and the value of quality are bounded below and the firm invests if the cost is low enough. Third, investment at the bottom can be sustained in equilibrium. If the firm is believed to be working at the bottom, incentives are high because the favorable beliefs push the firm's reputation to intermediate levels where dividends are high. In this case, the firm invests at low levels of reputation not because of the immediate reputational dividends, which are close to zero, but because of the higher future dividends when the firm's reputation is sensitive to actual quality. Thus, the time-lag of the investment process together with the reputational drift imply a fundamental asymmetry between incentives at the top and the bottom.

These three arguments suggest that a work-shirk equilibrium exists for small costs; Theorem 5 confirms this for a large class of imperfect Poisson learning processes.

Theorem 5 (Existence) *Assume either imperfect bad news learning, or imperfect good news learning and $\lambda < \mu$. Then there exists $\bar{c}$ such that for any costs $c \in (0, \bar{c})$:*

(a) There exists a work-shirk equilibrium with cutoff $x^* \in (0,1)$.

(b) Reputational dynamics in such an equilibrium are ergodic.

Proof. For part (a) see Appendix C. Part (b) follows from Lemma 1. □

We prove Theorem 5(a) by evaluating the reputational dividends that constitute the value of quality and incentivize investment. Let $\Delta_{x^*}(x)$ be the value of quality for a firm with reputation x in a work-shirk candidate equilibrium with cutoff x^*. In the limit case of full work, the value of quality $\Delta_1(x)$ is strictly positive on $[0,1)$ and monotonically decreasing on $[1-\varepsilon, 1]$ with limit $\Delta_1(1) = 0$, as illustrated in Figure 6. Thus for small c there exists x^* such that:

$$\lambda \Delta_1(x) \begin{cases} > c & \text{for } x < x^* \qquad \text{(Work at low reputations),} \\ = c & \text{for } x = x^* \qquad \text{(Indifference at cutoff } x^*\text{),} \\ < c & \text{for } x > x^* \qquad \text{(Shirk at high reputations).} \end{cases} \tag{5.1}$$

To prove existence we essentially want to replace $\Delta_1(\cdot)$ on the left-hand-side with $\Delta_{x^*}(\cdot)$. This step requires not only that $\Delta_{x^*}(\cdot)$ is close to $\Delta_1(\cdot)$, but also that $\Delta_{x^*}(\cdot)$ is decreasing at the cutoff so that the firm prefers to shirk at high reputations above cutoff x^*.[13] This is not immediate. Suppose that learning is via good news, that the work-shirk cutoff x^* is close to 1, and that the reputational drift at x^* is convergent. A reputational increment is valuable to the firm only as long as $x_t \neq x^*$: When $x_t = x^*$ the increment disappears because of the convergent drift at x^*. Therefore, the marginal value of reputation $V'_\theta(x)$ vanishes at the cutoff x^*, and the reputational dividend is increasing at x^*. Thus, we need to take seriously the possibility that $\Delta_{x^*}(\cdot)$ may be increasing at x^* as well.[14]

To show that the value of quality is decreasing at the work-shirk cutoff, we need a better understanding of reputational dynamics and marginal values $V'_\theta(x)$ for $x, x^* \approx 1$. Below the cutoff, dynamics are approximately governed by drift $d(x) \approx (\lambda - \mu)(1-x)$ and jumps of size $\mu(1-x)/\mu_H$ with arrival rate μ_θ. Above the cutoff, the drift $d(x) \approx -\lambda$ is so large compared to the size of the shirk-region $1 - x^*$ that reputational dynamics are essentially reflected at the cutoff.

The marginal value of reputation and dividends above the cutoff $x > x^*$ are then small in relation to those below the cutoff $x < x^*$. This is because a reputational increment essentially disappears when $x_t = x^*$, which happens much sooner for initial reputation $x_0 > x^*$ than for initial

[13] Recall that investment at the work-shirk cutoff x^* is uniquely pinned down by condition (2.5).

[14] Indeed, this is exactly what goes wrong for perfect good news learning and $\lambda > \mu$ as pointed out in Lemma 13(c). For such a learning process and low costs, any equilibrium has to involve an interval of reputations where the firm is indifferent and chooses an internal level of investment $\eta \in (0,1)$.

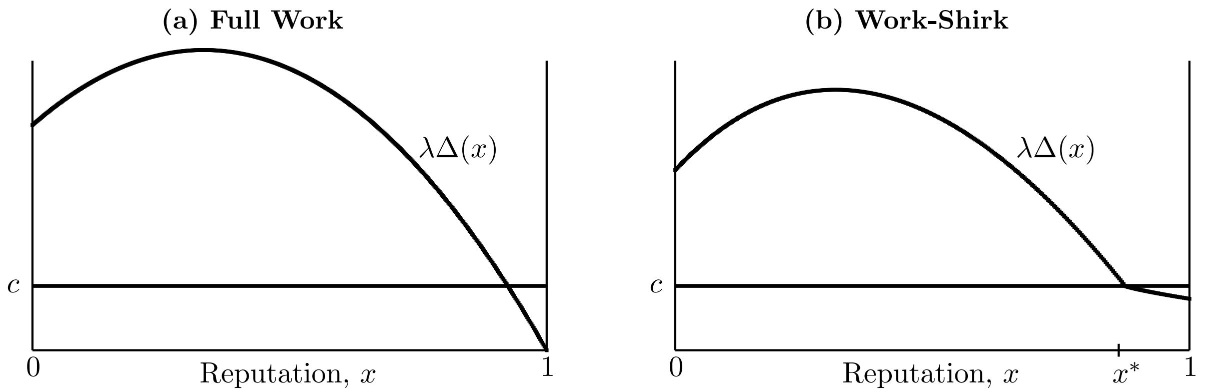

Figure 6: **Illustration of the value of quality under full work and in a work-shirk equilibrium.**

reputation $x_0 < x^*$. Therefore, the value of quality at the cutoff $\Delta_{x^*}(x^*)$ is largely determined by the dividends at $x < x^*$. For initial reputation above the cutoff $x_0 > x^*$ the value of quality $\Delta_{x^*}(x_0)$ is an average of low dividends while $x_t > x^*$, and a continuation value $\Delta_{x^*}(x^*)$ when x_t hits x^*. This average comes to less than $\Delta_{x^*}(x^*)$, so $\Delta_{x^*}(\cdot)$ is decreasing at the cutoff.

Reputational dynamics in this work-shirk equilibrium are ergodic by Theorem 5(b). With bad news learning and x^* close enough to 1, the reputational drift is negative in the shirk region above x^*, so the firm's reputation eventually cycles over $[0, x^*]$.[15] In the work-region below x^*, the firm's reputation drifts up towards x^*. While bad signals can impose set-backs to the firm's reputational ascent, favorable equilibrium beliefs make reputation a sub-martingale so it eventually reaches the work-shirk cutoff x^*. At the cutoff, the firm invests at intensity $\eta(x^*) \in (0, 1)$, and reputation remains constant until the next signal arrives, whereupon reputation drops and the firm resumes to work.

With good news learning, $\mu > \lambda$, and x^* close enough to 1, the reputational drift in the work region below λ/μ is positive, so reputation eventually cycles over $[\lambda/\mu, 1]$. Above λ/μ, reputational drift is negative both in the work-region $[\lambda/\mu, x^*]$ and the shirk-region $[x^*, 1]$ and evolves in a pattern of downward drift and upward jumps. Below x^*, reputation is a sub-martingale and signals eventually take the firm's reputation into the shirk-region above the cutoff. At this point, the strong negative drift above $x^* \approx 1$ quickly takes the reputation back through the cutoff into the work-region.

Slow learning at $x \approx 0$ and $x \approx 1$ suggests another, *shirk-work-shirk* type of equilibrium with a shirk-work cutoff x_1^* and a work-shirk cutoff x_2^*. The existence of such a shirk-work-shirk equilibrium hinges on the following condition. A learning process satisfies (HOPE) when a firm with some initial reputation x_0 has a chance of experiencing a higher reputation in the future,

[15] Specifically, the condition is that $-\lambda x^* + (\mu_L - \mu_H) x^* (1 - x^*) \leq 0$ or $x^* \geq 1 - \lambda / (\mu_L - \mu_H)$.

even if it is believed to be shirking

$$\Pr\left[x_t > x_0 | x_0, \tilde{\eta} = 0\right] > 0 \qquad \text{for some } x_0 \text{ and } t > 0. \tag{HOPE}$$

This condition is satisfied for any good news learning process since the firm's reputation rises at a signal arrival. For bad news learning it is satisfied if $\mu_L - \mu_H > \lambda$; then the absence of breakdowns dominates the adverse equilibrium beliefs and reputational drift $d(x) \approx (\mu_L - \mu_H - \lambda)\, x$ is positive at low levels of reputation. It is not satisfied for bad news learning with $\mu_L - \mu_H \leq \lambda$ where both drift and jumps are negative if the firm is believed to be shirking.

Theorem 6 (Uniqueness) *Fix any imperfect Poisson learning process:*

(a) If (HOPE) is satisfied, then for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for any cost $c \in (0, c_\varepsilon)$ and any equilibrium, the firm works at all reputation levels $x \in (0, 1-\varepsilon)$.

(b) If (HOPE) is not satisfied, then there exists $\bar{c} > 0$ such that for any $c < \bar{c}$ there exists $\varepsilon_c > 0$ such that for any $x_1^* < \varepsilon_c$, there exists a shirk-work-shirk equilibrium with shirk-work cutoff x_1^* and work-shirk cutoff x_2^* close to 1.

Proof. See Appendix D. □

Theorem 6(a) states that with (HOPE) and small costs, the firm works at all low and intermediate levels of reputation. Any such equilibrium gives rise to similar value functions and reputational dynamics; we therefore consider such equilibria to be essentially identical and interpret Theorem 6(a) as essential uniqueness of equilibrium under (HOPE).

For an intuition, first note that reputational dividends and the value of quality are bounded below on any interval $[\varepsilon, 1-\varepsilon]$, so when costs are small the firm prefers to invest at all intermediate levels of reputation. If the firm is believed to work for all reputations above some $x_1^* \leq \varepsilon$, these beliefs induce a strong positive reputational drift at low levels of reputation and investment incentives are strongly influenced by the high reputational dividends around $x \approx 1/2$, even if the firm's current reputation is at x_1^*. Under (HOPE) a firm with reputation just below x_1^* has a non-zero chance of seeing its reputation increase above x_1^*, so its investment incentives are also bounded below by these high dividends.

If (HOPE) is violated, then Theorem 6(b) states that the work-shirk equilibrium coexists with a continuum of shirk-work-shirk equilibria. In such an equilibrium, a firm with a low reputation is trapped in the shirk-region $[0, x_1^*]$ from which it cannot escape because adverse equilibrium beliefs dominate the weak effects of market learning. A firm with reputation above x_1^* to the contrary expects the favorable equilibrium beliefs to increase its reputation to intermediate levels where dividends are high. Divergent reputational drift at the shirk-work cutoff creates a discontinuity in the value function that incentivizes investment above the cutoff but not in the shirk-region just below. Investment incentives are then greatest just above the shirk-work cutoff where the discontinuity in the value functions leads to large reputational dividends. Such a shirk-work-shirk equilibrium captures the idea that a reputable firm (with reputation at or above the high, work-shirk cutoff) has low investment incentives and becomes complacent; when it is hit by bad news signals (and its reputation drops towards the low, shirk-work cutoff) it is put in the 'hot-seat' where one more breakdown would finish it off. In such an equilibrium a firm that fails once fights for its survival, but a firm that fails repeatedly gives up.

To formally construct shirk-work-shirk equilibria, we first choose the lower, shirk-work cutoff low enough so as to discourage work in the shirk-region $[0, x_1^*]$ and then reapply the arguments in the proof of Theorem 5(a) to prove existence of the upper, work-shirk cutoff with the required properties.[16]

The case of perfect bad news learning is the limit of imperfect bad news learning processes as $\mu_H \to 0$. Despite the apparent conflict between shirk-work equilibria in Theorem 3 and work-shirk equilibria in Theorem 5 the equilibria under perfect bad news are approximated by equilibria under imperfect bad news. The main difference is that perfect bad news learning allows for work at the very top, while equilibrium under imperfect learning requires shirking at the top as market learning and dividends disappear at $x = 1$. However, as $\mu_H \to 0$ and imperfect learning becomes perfect, dividends just below $x = 1$ increase and the shirk-region at the top vanishes continuously.

At the bottom, the results for perfect and imperfect bad news learning are virtually identical. For perfect bad news, condition (HOPE) is equivalent to $\mu_L > \lambda$. If (HOPE) fails and costs are small, there is a multiplicity of equilibria; this is established by Theorem 6(b) for imperfect signals, and by Theorem 3(d) for perfect signals. If (HOPE) holds and costs are small, Theorem 6(a) states that the firm works on $(0, 1 - \varepsilon)$ under imperfect learning; Figure 4(b) together with equation (4.7) implies that the firm works on $(0, 1]$ under perfect learning.[17]

6 Conclusion

This paper studies the moral hazard problem of a firm that produces experience goods and controls quality through its investment choice. Investment is incentivized by consumers' learning about product quality which feeds into the firm's reputation and future revenue.

The key feature distinguishing our paper from classical models of reputation and repeated games is that we model product quality as a function of past investments rather than current

[16] The above analysis relies on the assumption of low costs c to ensure work for intermediate reputations $x \in [\varepsilon, 1 - \varepsilon]$. Shirk-work-shirk equilibria may also exist when costs are higher.

[17] However we would not say that the full work equilibrium under perfect bad news is 'essentially unique' because there is also an equilibrium where the firm shirks at $x = 0$. Behavior at 0 does not matter for imperfect learning because the reputation never reaches 0, but it does matter in the case of perfect bad news learning.

effort. This capital-theoretic model of persistent quality seems realistic: The current state of General Motors is a function of its past hiring policies, investment decisions and reorganisations, all of which are endogenous and have lasting effects on quality. The model also yields new economic insights: When the market learns quality via breakthroughs of high quality products, a high-reputation firm runs down its quality and reputation, while a low-reputation firm keeps investing to achieve a breakthrough. Conversely, when the market learns quality via breakdowns of low quality products, a low-reputation firm has weak incentives to invest, while a high reputation firm keeps investing to protect its reputation.

There are many interesting ways to extend this model to capture additional important aspects of firm reputation. In the working paper version (Board and Meyer-ter-Vehn (2010b)) of this paper, we study more general imperfect learning processes that can contain a finite number of imperfect Poisson signals and a Brownian signal, where firm quality determines the drift of the process. Theorems 1, 5, and 6 extend to this more general class of learning processes and the condition for uniqueness (HOPE) is satisfied if there exists at least one good news Poisson signal or a non-trivial Brownian signal. In a companion paper (Board and Meyer-ter-Vehn (2010a)), we suppose the firm faces a cost of remaining in the industry and goes out of business when its continuation value drops to zero. We then investigate the investment incentives of a firm that is about to exit. We find that a firm that is ignorant of its own quality stops investing and coasts into liquidation when its life-expectancy is short, while a firm that does know its own quality may fight until the bitter end.

Beyond firm reputation, we hope that our model will prove useful in other fields. In corporate or international finance, where default signals bad news about a borrower, the shirk-work equilibria generate endogenous credit-traps. In political economy, where a scandal is bad news about a politician, the divergent dynamics imply that a politician who is caught will cheat even more, whereas a lucky politician will become more honest. And in personnel economics, our model predicts that in 'superstar markets', where agents are judged by their successes, performance tends to be mean-reverting.

A Proofs from Sections 2 and 3

A.1 Proof of Lemma 1

We first consider a work-shirk equilibrium and assume $\lambda \geq \mu$; this implies that reputational drift is convergent at the work-shirk cutoff x^*, and there exists a time $T > 0$ after which reputation has evolved to x^* with positive probability from any initial reputation $x_0 \in [0,1]$, i.e. $\Pr_{x_0}(x_T = x^*) = \alpha > 0$. Let $F_{n,\nu}(\cdot)$ be the cdf. of x_{nT} conditional on $x_{\nu T} = x^*$ and let $F^n(x) = \alpha \sum_{\nu=0}^{n} (1-\alpha)^\nu F_{n,\nu}(x)$ be a finite geometric sum of these distributions of x_{nT}, with the largest weight α on the distribution $F_{n,0}$ conditioning on $x_0 = x^*$ and the smallest weight $\alpha(1-\alpha)^n$ on the distribution $F_{n,n} = \mathbb{I}_{x^*}$ which is just an atom of size one at x^*.

Fix any $x_0 \in [0,1]$ and let $t \geq (n+1)T$. We argue that for any $x \in [0,1]$

$$|\Pr(x_t \leq x) - F^n(x)| \leq (1-\alpha)^{n+1}. \tag{A.1}$$

To do so, we first exploit the definition of α and T to write the distribution of x_{t-nT} as a convex combination $\alpha \mathbb{I}_{x^*} + (1-\alpha) H_0$ of an atom of size α at x^* and some residual distribution H_0. Rolling time forward by T, we write the distribution of $x_{t-(n-1)T}$ as $\alpha F_{1,0} + (1-\alpha)(\alpha F_{1,1} + (1-\alpha) H_1)$, where the residual distribution H_1 has evolved into an atom $\alpha F_{1,1} = \alpha \mathbb{I}_{x^*}$ and a residual distribution H_1. By induction, we write the distribution of x_t as

$$\alpha \sum_{\nu=0}^{n} (1-\alpha)^\nu F_{n,\nu} + (1-\alpha)^{n+1} H_n = F^n + (1-\alpha)^{n+1} H_n$$

for some residual distribution H_n, implying (A.1). Thus for any $n < m$ the distribution of x_{mT} is $(1-\alpha)^{n+1}$-close to F^n and $(1-\alpha)^{m+1}$-close to F^m, so the sequence $(F^n)_{n\in\mathbb{N}}$ is Cauchy in the complete metric space of increasing functions from $[0,1]$ to $[0,1]$ equipped with the sup-norm; thus $(F^n)_{n\in\mathbb{N}}$ converges to some $F = \lim F^n$. By the above argument, reputation x_t converges to F in distribution for any initial x_0. This completes the proof for work-shirk equilibria with $\lambda \geq \mu$.

For work-shirk equilibria when $\lambda < \mu$ as well as full work and full shirk equilibria, reputation x_t never reaches the stationary point $\hat{x}$, i.e. where $d(\hat{x}) = 0$. Thus, for any time T there is no single reputation level x that is attained at time T with positive probability. To extend our proof to these cases, we remark that it is immaterial to our argument that the distribution $\mathbb{I}_{x^*}$, which is reached after time T with probability α, is a point distribution. Even if the stationary point $\hat{x}$ of the reputational drift is never reached, there exists a cdf G on $[0,1]$ together with a time T and a probability $\alpha > 0$, such that the distribution of x_T conditional on any x_0 can be written as $\alpha G + (1-\alpha) H$ for some residual distribution H. We then replace the point distribution $\mathbb{I}_{x^*}$ with the general distribution G in the above construction of F to conclude.

A.2 Proof of Theorem 1

Fix any candidate equilibrium η. For $\delta > 0$ small but finite, we expand firm value into current profits and the continuation value

$$\begin{aligned} V_\theta(x_0) &= \mathbb{E}_{x_0,\theta_0=\theta}\left[\int_0^\infty e^{-rt}\left(x_t - c\eta(x_t)\right)dt\right] \\ &= \mathbb{E}_{x_0,\theta_0=\theta}\left[\int_0^\delta e^{-rt}\left(x_t - c\eta(x_t)\right)dt\right] + \mathbb{E}_{x_0,\theta_0=\theta}\left[e^{-r\delta}V_{\theta_\delta}(x_\delta)\right] \\ &= \underbrace{\int_0^\delta \left(x_t^\varnothing - c\eta\left(x_t^\varnothing\right)\right)dt}_{\text{profit in } [0,\delta]} + e^{-r\delta}\underbrace{\int_0^\delta \lambda e^{-\lambda(\delta-t)}\left(\eta\left(x_t^\varnothing\right)V_H\left(x_\delta^\varnothing\right) + \left(1-\eta\left(x_t^\varnothing\right)\right)V_L\left(x_\delta^\varnothing\right)\right)dt}_{\text{cont. value after } \lambda\text{-shock}} \\ &\quad + e^{-r\delta}e^{-\lambda\delta}\underbrace{\mathbb{E}_{x_0,\theta^\delta=\theta}\left[V_\theta(x_\delta)\right]}_{\text{cont. value w.o. } \lambda\text{-shock}} + O\left(\delta^2\right), \end{aligned}$$

where the $O\left(\delta^2\right)$-term captures the $O(\delta)$-possibility of a signal in the first two terms which are themselves of order δ.

Taking differences $V_H(x_0) - V_L(x_0)$, current profits and continuation values after a technology shock cancel, and the value of quality can be written as a (discrete) reputational dividend plus its continuation value:

$$\begin{aligned} \Delta(x_0) &= V_H(x_0) - V_L(x_0) \\ &= e^{-(r+\lambda)\delta}\left(\mathbb{E}_{x_0,\theta^\delta=H}\left[V_H(x_\delta)\right] - \mathbb{E}_{x_0,\theta^\delta=L}\left[V_L(x_\delta)\right]\right) + O\left(\delta^2\right) \\ &= e^{-(r+\lambda)\delta}\left(\mathbb{E}_{x_0,\theta^\delta=H}\left[V_H(x_\delta)\right] - \mathbb{E}_{x_0,\theta^\delta=L}\left[V_H(x_\delta)\right] + \mathbb{E}_{x_0,\theta^\delta=L}\left[V_H(x_\delta)\right] - \mathbb{E}_{x_0,\theta^\delta=L}\left[V_L(x_\delta)\right]\right) + O\left(\delta^2\right) \\ &= e^{-(r+\lambda)\delta}D_{H,\delta}(x_0) + \mathbb{E}_{x_0,\theta^\delta=L}\left[\Delta(x_\delta)\right] + O\left(\delta^2\right). \end{aligned}$$

where $D_{H,\delta}(x_0) = \mathbb{E}_{x_0,\theta^\delta=H}\left[V_H(x_\delta)\right] - \mathbb{E}_{x_0,\theta^\delta=L}\left[V_H(x_\delta)\right]$ is the reputational dividend, and $\Delta(x_\delta) = V_H(x_\delta) - V_L(x_\delta)$ the continuation value of quality.

Iterating this calculation for $\Delta(x_\delta)$ we get:

$$\Delta(x_0) = \sum_{n=0}^{\infty} e^{-(r+\lambda)(n+1)\delta}\mathbb{E}_{x_0,\theta^{n\delta}=L}\left[D_{H,\delta}(x_{n\delta}) + O\left(\delta^2\right)\right]. \tag{A.2}$$

To compute the reputational dividend, we neglect the $O\left(\delta^2\right)$-possibility of a double signal in $[0,\delta]$ and let x_δ^t be reputation at time δ conditional on a signal at time $t \in [0,\delta]$. Then either term in the dividend is given by:

$$\begin{aligned}\mathbb{E}_{x_0,\theta^\delta=\theta}\left[V_H\left(x_\delta\right)\right] &= e^{-\mu_\theta\delta}V_H(x_\delta^{\varnothing})+\int_0^\delta e^{-\mu_\theta t}\mu_\theta V_H\left(x_\delta^t\right)dt+O\left(\delta^2\right)\\ &= V_H(x_\delta^{\varnothing})+\mu_\theta\int_0^\delta\left(V_H\left(x_\delta^t\right)-V_H(x_\delta^{\varnothing})\right)dt+O\left(\delta^2\right)\\ &= V_H(x_\delta^{\varnothing})+\mu_\theta\int_0^\delta\left(V_H\left(j\left(x_t^{\varnothing}\right)\right)-V_H(x_t^{\varnothing})\right)dt+O\left(\delta^2\right).\end{aligned}$$

In this calculation, the second line uses the approximations $e^{-\mu_\theta\delta}=1-\mu_\theta\delta+O\left(\delta^2\right)$ and $e^{-\mu_\theta t}=1+O\left(\delta\right)$. The third line uses the approximations $V_H\left(x_\delta^t\right)=V_H\left(j\left(x_t^{\varnothing}\right)\right)+O\left(\delta\right)$ and $V_H(x_\delta^{\varnothing})=V_H(x_t^{\varnothing})+O\left(\delta\right)$.

Now the difference is easy to compute

$$\begin{aligned}D_{H,\delta}\left(x_0\right) &= \mathbb{E}_{x_0,\theta^\delta=H}\left[V_H\left(x_\delta\right)\right]-\mathbb{E}_{x_0,\theta^\delta=L}\left[V_H\left(x_\delta\right)\right]\\ &= \int_0^\delta\mu\left(V_H\left(j\left(x_t^{\varnothing}\right)\right)-V_H(x_t^{\varnothing})\right)dt+O\left(\delta^2\right).\end{aligned}$$

We reintroduce the $O\left(\delta^2\right)$-possibility of double signals in $[0,\delta]$:

$$D_{H,\delta}\left(x_0\right)=\mathbb{E}_{x_0,\theta^\delta=L}\left[\int_{t=0}^\delta\mu\left(V_H\left(j\left(x_t\right)\right)-V_H(x_t)\right)dt\right]+O\left(\delta^2\right).$$

The same holds for the reputational dividend evaluated at later times:

$$e^{-(r+\lambda)(n+1)\delta}D_{H,\delta}\left(x_{n\delta}\right)=\mathbb{E}_{x_{n\delta},\theta^{(n+1)\delta}=L}\left[\int_{t=n\delta}^{(n+1)\delta}e^{-(r+\lambda)t}\mu\left(V_H\left(j\left(x_t\right)\right)-V_H(x_t)\right)dt\right]+O\left(\delta^2\right),$$

so that plugging back into equation (A.2), and taking the limit $\delta\to 0$, we get

$$\begin{aligned}\Delta\left(x_0\right) &= \sum_{n=0}^{\infty}\mathbb{E}_{x_0,\theta^{n\delta}=L}\left[\mathbb{E}_{x_{n\delta},\theta^{(n+1)\delta}=L}\left[\int_{t=n\delta}^{(n+1)\delta}e^{-(r+\lambda)t}\mu\left(V_H\left(j\left(x_t\right)\right)-V_H(x_t)\right)dt\right]+O\left(\delta^2\right)\right]\\ &= \mathbb{E}_{x_0,\theta^\infty=L}\left[\int_{t=0}^\infty e^{-(r+\lambda)t}\mu\left(V_H\left(j\left(x_t\right)\right)-V_H(x_t)\right)dt\right],\end{aligned}$$

as required.

B Perfect Poisson Learning

In this appendix we solve the perfect learning specifications of Section 4 explicitly by calculating equilibrium value functions in closed form. This approach highlights the analytic tractability of these learning specifications and delivers a more explicit understanding of the value functions and the value of quality. Some of the derived expressions are also used in the proofs of Section 4.

We assume throughout that $\lambda \geq |\mu|$, so that the direction of the reputational drift is determined by market beliefs.

B.1 Perfect Good News

Shirk-region, above the cutoff $x \geq x^*$. Suppose $x_0 = 1$ and let $x_t^{\varnothing}$ solve the law of motion given by the drift equation (4.1). For $x > x^*$, the firm strictly prefers to shirk; for $x = x^*$ the firm is indifferent, and we assume it shirks. As a result, $x_t^{\varnothing}$ is strictly decreasing until it reaches x^* and stays there. Conditional on low quality, reputational dynamics are deterministic and firm value is given by:

$$V_L(x_s^{\varnothing}) = \int_{t=0}^{\infty} e^{-rt} x_{t+s}^{\varnothing} dt \tag{B.1}$$

With a high quality product dynamics are more complicated, because the reputation jumps to one at a breakthrough and quality disappears at a technology shock:

$$\begin{aligned} V_H(x_s^{\varnothing}) &= \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} [x_{t+s}^{\varnothing} + \lambda V_L(x_{t+s}^{\varnothing}) + \mu V_H(1)] dt \\ &= \int_{t=0}^{\infty} x_{t+s}^{\varnothing} e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\mu+\lambda)t} \right] dt + \frac{\mu}{r+\lambda+\mu} V_H(1), \end{aligned} \tag{B.2}$$

where we rewrote the $\lambda V_L(x_{t+s}^{\varnothing})$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda V_L(x_{t+s}^{\varnothing}) dt = \frac{\lambda}{\lambda+\mu} \int_{t=0}^{\infty} x_{t+s}^{\varnothing} e^{-rt} [1 - e^{-(\mu+\lambda)t}] dt.$$

We evaluate (B.2) at $x_s^{\varnothing} = 1$, and rearrange

$$V_H(1) = \frac{r+\lambda+\mu}{r+\lambda} \int_{t=0}^{\infty} x_t^{\varnothing} e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\mu+\lambda)t} \right] dt.$$

The value of quality is the difference between the value functions (B.2) and (B.1)[18]

$$\Delta(x_s^\varnothing) = \frac{\mu}{r+\lambda}\int_{t=0}^{\infty} x_t^\varnothing e^{-rt}\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t}\right]dt - \frac{\mu}{\lambda+\mu}\int_{t=0}^{\infty} x_{t+s}^\varnothing e^{-rt}\left[1 - e^{-(\mu+\lambda)t}\right]dt. \tag{B.3}$$

When $x_s^\varnothing = x^*$, we get

$$\Delta(x^*) = \frac{\mu}{r+\lambda}\int_{t=0}^{\infty}(x_t^\varnothing - x^*)e^{-rt}\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t}\right]dt. \tag{B.4}$$

Quality at x^* is valuable because of the possibility that reputation jumps from x^* to $x = 1$. The discounted probability of this event is captured by the $\mu/(r+\lambda)$-term, while the terms in brackets capture the possibilities of technology shocks and breakthroughs as $x_t^\varnothing$ descends from 1 to x^*.

Work-region, below the cutoff $x \leq x^*$. For this case suppose the reputational trajectory $x_t^\varnothing$ starts at $x_0 = 0$. The firm weakly prefers to work and we assume it always does, so $x_t^\varnothing$ is strictly increasing until it reaches x^*. With a high quality product, the firm's reputation drifts up until $x_t^\varnothing = x^*$, or a breakthrough hits:

$$V_H(x_s^\varnothing) = \int_{t=0}^{\infty} e^{-(r+\mu)t}[(x_{t+s}^\varnothing - c) + \mu V_H(1)]dt. \tag{B.5}$$

With a low quality product, the firm's reputation drifts up until $x_t^\varnothing = x^*$, or a λ-shock hits:

$$\begin{aligned} V_L(x_s^\varnothing) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t}[(x_{t+s}^\varnothing - c) + \lambda V_H(x_{t+s}^\varnothing)]dt \\ &= \int_{t=0}^{\infty}(x_{t+s}^\varnothing - c)\left[\frac{\lambda}{\lambda-\mu}e^{-(r+\mu)t} - \frac{\mu}{\lambda-\mu}e^{-(r+\lambda)t}\right]dt + \frac{\lambda}{r+\lambda}\frac{\mu}{r+\mu}V_H(1), \end{aligned} \tag{B.6}$$

where we rewrote the $\lambda V_H(x_{t+s}^\varnothing)$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda)t}\lambda V_H(x_{t+s}^\varnothing)dt = \frac{\lambda}{\lambda-\mu}\int_{t=0}^{\infty}(x_{t+s}^\varnothing - c)e^{-rt}(e^{-\mu t} - e^{-\lambda t})dt + \frac{\lambda}{r+\lambda}\frac{\mu}{r+\mu}V_H(1).$$

The value of quality is the difference between the value functions (B.5) and (B.6):

$$\Delta(x_s^\varnothing) = \frac{r}{r+\lambda}\frac{\mu}{r+\mu}V_H(1) - \frac{\mu}{\lambda-\mu}\int_{t=0}^{\infty}(x_{t+s}^\varnothing - c)e^{-rt}(e^{-\mu t} - e^{-\lambda t})dt$$

The first term captures firm value after the breakthroughs and the second term captures the opportunity cost of the breakthroughs, i.e. firm value absent a breakthrough.

[18] Equivalently, one could compute the reputational dividend from the value functions and substitute it into the dividend formula for the value of quality (3.3).

B.2 Perfect Bad News

Work-region, above the cutoff $x > x^*$. First assume that $x^* > 0$, so that $V_L(0) = 0$, and assume $\eta(x^*) = 1$. Define the reputational trajectory $x_t^\varnothing$ by the initial condition $x_0 = x^*$ the drift equation (4.3). Then, $x_t^\varnothing$ is strictly increasing and converges to 1. Conditional on high quality, reputational dynamics are deterministic and firm value equals

$$V_H(x_s^\varnothing) = \int_{t=0}^{\infty} e^{-rt}(x_{t+s}^\varnothing - c)\, dt. \tag{B.7}$$

With a low quality product, dynamics are more complicated because the reputation jumps to 0 at a breakdown and quality improves at a λ-shock,

$$\begin{aligned} V_L(x_s^\varnothing) &= \int_{t=0}^{\infty} e^{-(r+\lambda+\mu_L)t}[(x_{t+s}^\varnothing - c) + \lambda V_H(x_{t+s}^\varnothing) + \mu_L V_L(0)]\, dt. \\ &= \int_{t=0}^{\infty} e^{-rt}(x_{t+s}^\varnothing - c)\left[\frac{\lambda}{\lambda+\mu_L} + \frac{\mu_L}{\lambda+\mu_L} e^{-(\lambda+\mu_L)t}\right] dt, \end{aligned} \tag{B.8}$$

where we rewrote the $\lambda V_H(x_{t+s}^\varnothing)$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda+\mu_L)t} \lambda V_H(x_{t+s}^\varnothing)\, dt = \frac{\lambda}{\mu_L+\lambda}\int_{t=0}^{\infty} e^{-rt}(x_{t+s}^\varnothing - c)[1 - e^{-(\lambda+\mu_L)t}]\, dt.$$

The value of quality is the difference between the value functions (B.7) and (B.8),

$$\Delta(x_s^\varnothing) = \frac{\mu_L}{\lambda+\mu_L}\int_{t=0}^{\infty} e^{-rt}(x_{t+s}^\varnothing - c)(1 - e^{-(\lambda+\mu_L)t})\, dt. \tag{B.9}$$

Quality insures the firm against the loss of reputation when the breakdown hits before the λ-shock.

Shirk-region, below the cutoff $x < x^*$. Next, assume $\eta(x^*) = 0$ and again define $x_t^\varnothing$ by the initial condition $x_0 = x^*$ the drift equation (4.3) with the difference that now $x_t^\varnothing$ is strictly decreasing and converges to zero. With a low quality product, reputation is deterministic and firm value equals

$$V_L(x_s^\varnothing) = \int_{t=0}^{\infty} e^{-(r+\mu_L)t} x_{t+s}^\varnothing\, dt. \tag{B.10}$$

With a high quality product, quality disappears at a λ-shock and the firm's value function is

$$\begin{aligned} V_H(x_s^\varnothing) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t}[x_{t+s}^\varnothing + \lambda V_L(x_{t+s}^\varnothing)]\, dt. \\ &= \int_{t=0}^{\infty} x_{t+s}^\varnothing \left[\frac{\lambda}{\lambda-\mu_L} e^{-(r+\mu_L)t} - \frac{\mu_L}{\lambda-\mu_L} e^{-(r+\lambda)t}\right] dt, \end{aligned} \tag{B.11}$$

where we rewrote the $\lambda V_L(x^{\varnothing}_{t+s})$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda)t}\lambda V_L(x^{\varnothing}_{t+s}) = \frac{\lambda}{\lambda-\mu_L}\int_{t=0}^{\infty} e^{-rt}x^{\varnothing}_{t+s}(e^{-\mu_L t}-e^{-\lambda t})dt.$$

The value of quality is the difference of the value functions (B.11) and (B.10):

$$\Delta(x^{\varnothing}_s) = \frac{\mu_L}{\lambda-\mu_L}\int_{t=0}^{\infty} e^{-rt}x^{\varnothing}_{t+s}(e^{-\mu_L t}-e^{-\lambda t})\,ds. \tag{B.12}$$

Again, quality insures the firm against the loss of reputation when the breakdown hits before the λ-shock.

Full work. Suppose the firm always works. Let $x^{\varnothing}_t$ solve the drift equation with initial condition $x_0 = 0$. The value function for the high quality firm is given by (B.7). The value function of the low quality firm becomes

$$V_L(x^{\varnothing}_s) = \int_{t=0}^{\infty} e^{-rt}(x^{\varnothing}_{t+s}-c)\left[\frac{\lambda}{\lambda+\mu_L}+\frac{\mu_L}{\lambda+\mu_L}e^{-(\lambda+\mu_L)t}\right]dt + \frac{\mu_L}{r+\lambda+\mu_L}V_L(0). \tag{B.13}$$

Setting $s = 0$, we obtain

$$V_L(0) = \frac{r+\lambda+\mu_L}{r+\lambda}\int_{t=0}^{\infty} e^{-rt}(x^{\varnothing}_t-c)\left[\frac{\lambda}{\lambda+\mu_L}+\frac{\mu_L}{\lambda+\mu_L}e^{-(\lambda+\mu_L)t}\right]dt.$$

The value of quality is the difference between (B.7) and (B.13):

$$\Delta(x^{\varnothing}_s) = \frac{\mu_L}{\lambda+\mu_L}\int_{t=0}^{\infty} x^{\varnothing}_{t+s}e^{-rt}\left[1-e^{-(\lambda+\mu_L)t}\right]dt - \frac{\mu_L}{r+\lambda}\int_{t=0}^{\infty} x^{\varnothing}_t e^{-rt}\left[\frac{\lambda}{\lambda+\mu_L}+\frac{\mu_L}{\lambda+\mu_L}e^{-(\mu_L+\lambda)t}\right]dt.$$

where cost terms cancel since both high- and low quality firms always work. This equation parallels equation (B.3) in the good-news case. When $s = 0$, this becomes

$$\Delta(0) = \frac{\mu_L}{\lambda+\mu_L}\int_{t=0}^{\infty} e^{-rt}x^{\varnothing}_t\left[\frac{r}{r+\lambda}-\frac{r+\mu_L+\lambda}{r+\lambda}e^{-(\lambda+\mu_L)t}\right]dt. \tag{B.14}$$

Here, the value of quality realizes when a breakdown hits before the first λ-shock.

B.3 Proof of Theorem 3(b) and (d)

We now formally establish equilibrium existence and multiplicity. This addresses the gaps left open by the arguments in the proof sketch of Theorem 3(b) and (d).

Lemma 5 *Fix $x^* > 0$. If $\lambda\Delta_{x^*,1}(x^*) \geq c$, then:*

(M_1) $\Delta_{x^*,1}(\cdot)$ is strictly increasing on $[0, 1]$.

If additionally $x^* \in [\hat{x}, 1]$ *so that* $\Delta_{x^*,0}(x^*)$ *is well-defined, then:*

(M$_0$*)* $\Delta_{x^*,0}(\cdot)$ *is strictly increasing on* $[0, 1]$.

(R) $\Delta_{x^*,0}(x^*) < \Delta_{x^*,1}(x^*)$.

Proof. We first show that value functions are strictly increasing when $\lambda\Delta_{x^*,1}(x^*) \geq c$. For $x_0 \in [x^*, 1]$ value functions $V_\theta(x_0) = \mathbb{E}\left[\int e^{-rt}(x_t - c\eta_t)\, dt\right]$ are strictly increasing in x_0 because revenue x_t is increasing in x_0 and cost $c\eta_t$ is independent of x_0. Then also $\Delta_{x^*,\eta^*}(x_0) = \int e^{-(r+\lambda)t}\mu_L V_L\left(x_t^\varnothing\right) dt$ is strictly increasing in x_0 on $[x^*, 1]$ and the firm prefers to work at all $x \in [x^*, 1]$.

To show that value functions are increasing on all of $[0, 1]$, we can now re-use the proof of Lemma 3 that establishes strict monotonicity for equilibrium value functions. Consider a 'high' firm with initial reputation x_0' and a 'low' firm that starts at $x_0 < x_0'$. The high firm works weakly more than the low firm and prefers to do so by the argument above. Thus the high firm's value in the candidate equilibrium is greater than the value it could get from mimicking the low firm, which in turn is greater than the low firm's value. Therefore value functions are increasing in reputation.

Then, equation (4.4) implies that the value of quality $\Delta_{x^*,\eta^*}(\cdot)$ is strictly increasing in x, and that $\Delta_{x^*,0}(x^*) < \Delta_{x^*,1}(x^*)$. □

Lemma 6 *(C*$_1$*)* $\Delta_{x^*,1}(x^*)$ *is continuous in* x^* *on* $(0, 1]$ *(but may be discontinuous at* $x^* = 0$*).*

(C$_0$*)* $\Delta_{x^*,0}(x^*)$ *is continuous in* x^* *on* $[\hat{x}, 1]$.

Proof. Part (C$_1$). We first show that future reputation x_t is continuous in the cutoff. Let $x_t = x_t(x^*, h_t)$ be reputation at time t when $x_0 = x^*$, the public history is given by h_t and market beliefs are shirk-work with cutoff x^*. Absent a breakdown, i.e. for the empty history, x_t drifts from x^* towards 1 if $\eta^* = 1$, and drifts towards $\hat{x}$ if $\eta^* = 0$. At a breakdown, reputation x_t drops to zero and stays there. In both cases $x_t(x^*, h_t)$ is continuous in x^*.

Firm value at the cutoff $V_{\theta,x^*,1}(x^*) = \mathbb{E}\left[\int e^{-rt}(x_t - c\eta_t)\, dt\right]$ and the value of quality $\Delta_{x^*,1}(x^*)$ are then continuous in x^* as well: revenue x_t is continuous in x^* and cost $c\eta_t$ is independent of x^*. At $x^* = 0$, future reputation, firm value and the value of quality are generally not continuous: After a breakdown, reputation is trapped at zero for all $x^* > 0$, but not when $x^* = 0$ and $\eta^* = 1$.

Part (C$_0$) follows by the same argument; with $\eta(x^*) = 0$, future reputation, firm value, and the value of quality at the cutoff are also continuous at $x^* = 0$ as reputation after a breakdown is trapped at zero even if $x^* = 0$. □

Sufficiency on indifference conditions (4.6) and (4.7). Lemmas 5 and 6 imply that these conditions are actually sufficient for a shirk-work equilibrium with cutoff x^* and $\eta(x^*) = 1$. Condition (4.6) states that the firm is indifferent at the cutoff, i.e. if $\lambda\Delta_{x^*,1}(x^*) = c$. Then by (M$_1$)

investment incentives are strictly increasing and the firm indeed prefers to shirk below the cutoff and to work above.

Condition (4.7) states that at the shirk-work cutoff, the firm prefers to shirk if it is believed to be shirking and it prefers to work if it is believed to be working, i.e. $\lambda\Delta_{x^*,0}(x^*) \leq c \leq \lambda\Delta_{x^*,1}(x^*)$ for some $x^* \geq \widehat{x}$ with $x^* > 0$. Then by (M_1) investment incentives are strictly increasing and the firm indeed prefers to work above x^*. To show that the firm prefers to shirk below x^* note that $\Delta_{x^*,0}(\cdot) = \Delta_{x^*,1}(\cdot)$ on $[0, x^*)$ since the firm's reputation is trapped in $[0, x^*)$ and never reaches x^*. As $\lambda\Delta_{x^*,0}(x^*) \leq c$ and $\Delta_{x^*,0}(\cdot)$ is increasing by (M_0), we get $\lambda\Delta_{x^*,1}(x) < \lambda\Delta_{x^*,0}(x^*) < c$ for all $x \in [0, x^*)$ as desired.

Equilibrium existence, Theorem 3(b). Property (C_1) implies that one of the following must be true:

$$\lambda\Delta_{x^*,1}(x^*) \begin{cases} > c & \text{for all } x^* \in (0,1], \\ = c & \text{for some } x^* \in (0,1], \\ < c & \text{for all } x^* \in (0,1]. \end{cases} \tag{B.15}$$

In the first case, shirk-work with cutoff $x^* = 0$ and $\eta(x^*) = 0$ is an equilibrium. With initial reputation $x_0 = 0$, reputation is trapped at 0 and investment incentives vanish, i.e. $\lambda\Delta_{0,0}(0) = 0 < c$, so the firm prefers to shirk at $x = 0$. With initial reputation $x_0 > 0$ in the work-region, reputation drifts up and x_t is contained in $\{0\} \cup [x_0, 1]$. Therefore, we have $\lambda\Delta_{0,0}(x) = \lambda\Delta_{x,1}(x)$ for all $x > 0$. By the first line of (B.15) we have $\lambda\Delta_{x,1}(x) > c$ for all $x > 0$, and the firm prefers to work at all $x > 0$.

In the second case, $\lambda\Delta_{x^*,1}(x^*) = c$ implies that (4.6) is satisfied, and shirk-work with cutoff x^* and $\eta(x^*) = 1$ is an equilibrium.

In the third case, we have $\lambda\Delta_{x^*,1}(x^*) < c$ for $x^* = 1$, so full shirk is an equilibrium. To formally prove this, fix $x^* = 1$ and consider $c' < c$ that solves $\lambda\Delta_{x^*,1}(x^*) = \lambda\frac{\mu_L(1-c')}{r(r+\lambda+\mu_L)} = c'$. If cost was equal to c', then by (M_1) we would have $\lambda\Delta_{x^*,0}(x^*) < \lambda\Delta_{x^*,1}(x^*) = c' < c$. By definition of full shirk, value functions and $\Delta_{x^*,0}(x^*)$ do not depend on c, so the firm prefers to shirk; thus full shirk is an equilibrium.

Equilibrium multiplicity, Theorem 3(d). If $\lambda \geq \mu_L$, then $\hat{x} \leq 0$ and we need to show that (4.7) is satisfied for a continuum of x^*. Reconsider the three cases listed in (B.15). Assumption (4.5) rules out the third case. In the second case, (4.5) together with (C_1) imply that we have $\lambda\Delta_{x^*,1}(x^*) = c$ for some $x^* \in (0,1)$. Then property (R) implies $\lambda\Delta_{x^*,0}(x^*) < c$. As $\Delta_{x^*,1}(x^*)$ is increasing in x^*,[19] and $\Delta_{x^*,0}(x^*)$ is continuous in x^* by (C_0), condition (4.7) holds for for all x in an interval $[x^*, x^* + \varepsilon]$. In the first case of (B.15) we have $\lambda\Delta_{x^*,1}(x^*) > c$ for all $x^* \in (0,1]$. As

[19]To formally show this, consider $\Delta_{x^*,1}(x^*) = \int e^{-(r+\lambda)t}\mu_L V_L(x_t^{\varnothing})\,dt$ and note that $V_L(x_t^{\varnothing})$ is increasing on $[x^*, 1]$ and independent of x^* as long as $x^* < x_t^{\varnothing}$.

$\Delta_{x^*,0}(x^*) = 0 < c$ for $x^* = 0$, and property (C_0) ensures continuity of $\Delta_{x^*,0}(x^*)$, condition (4.7) holds for for all x^* in an interval $(0, \varepsilon]$.

C Imperfect Poisson Learning: Proof of Theorem 5(a)

In Section C.1 we perform a change of variables by writing reputation as the log-likelihood ratio of high quality ℓ. In Section C.2 we provide some preliminary results about work-shirk candidate equilibria. We then show that for sufficiently small c there exists a work-shirk candidate equilibrium with cutoff ℓ^* such that:

(a) At the cutoff the firm is indifferent: $\lambda\Delta_{\ell^*}(\ell^*) = c$ (Section C.3, Lemma 10),

(b) Below the cutoff the firm prefers to work: $\lambda\Delta_{\ell^*}(\ell) > c$ for $\ell < \ell^*$ (Section C.4, Lemma 13),

(c) Above the cutoff the firm prefers to shirk: $\lambda\Delta_{\ell^*}(\ell) < c$ for $\ell > \ell^*$ (Section C.5, Lemma 14).

C.1 Log-likelihood Ratio Transformation

For most of the technical proofs in the appendix we represent reputation not by the probability of high quality $x = \Pr(\theta = H)$, but by its log-likelihood ratio $\ell(x) = \log(x/(1-x)) \in \mathbb{R} \cup \{-\infty, \infty\}$. The relevant transformation functions are:

$$\ell(x) = \log \frac{x}{1-x} \qquad\qquad x(\ell) = \frac{e^\ell}{1+e^\ell} \qquad\qquad \frac{dx}{d\ell} = \frac{e^\ell}{(1+e^\ell)^2} = x(1-x)$$

We adopt a physics approach to notation by simply writing $V_\theta(\ell)$ for the value of the firm in ℓ-space, and similarly for all other functions of reputation.

Reputational Dynamics: The advantage of this transformation is that Bayesian updating, equations (2.2) and (2.3), are linear in ℓ-space. At a signal, reputation jumps from ℓ_{t^-} to $\ell_t = j(\ell_{t^-}) = \ell_{t^-} + \log(\mu_H/\mu_L)$. Absent a signal, the reputational drift consists of a constant term $-\mu$ induced by Bayesian learning and a term induced by equlibrium beliefs that now includes the derivative $d\ell/dx$:

$$d(\ell) = \frac{d\ell}{dx}\lambda(\widetilde{\eta} - x) - \mu = \lambda\frac{(1+e^\ell)^2}{e^\ell}\left(\widetilde{\eta} - \frac{e^\ell}{1+e^\ell}\right) - \mu = \begin{cases} \lambda(1+e^{-\ell}) - \mu & \text{for } \widetilde{\eta} = 1, \\ -\lambda(1+e^\ell) - \mu & \text{for } \widetilde{\eta} = 0. \end{cases} \tag{C.1}$$

In a work-shirk candidate equilibrium with high cutoff $\ell^* \gg 0$, the drift at high reputations $\ell \gg 0$ is approximately constant equal to $\lambda - \mu$ below the cutoff and approximately $-\infty$ above the cutoff.

C.2 Work-Shirk Candidate Equilibria

We now provide some preliminary results about value functions and reputational dynamics in work-shirk candidate equilibria that we later draw upon to prove existence of a work-shirk equilibrium.

As a first step we study how reputation $\ell_t = \ell_t(\ell_0, h_t, \widetilde{\eta})$ at time t depends on initial reputation ℓ_0, given history h_t and believed investment $\widetilde{\eta}$.[20]

Lemma 7 *In any work-shirk candidate equilibrium with cutoff ℓ^*:*

(a) Reputational increments are decreasing:

$$\frac{\partial \ell_t}{\partial \ell_0}(\ell_0, h_t, \widetilde{\eta}) < 1 \text{ for all } t > 0.$$

(b) Suppose the drift is convergent around cutoff ℓ^*. Reputational increments disappear at ℓ^*: If $\ell_T = \ell^*$ for some T then:

$$\frac{\partial \ell_t}{\partial \ell_0}(\ell_0, h_t, \widetilde{\eta}) = 0 \text{ for all } t > T$$

(c) Suppose the drift is negative around cutoff ℓ^*. Reputational increments shrink at ℓ^*: If $\ell_T = \ell^*$ for some T then:

$$\frac{\partial \ell_t}{\partial \ell_0}(\ell_0, h_t, \widetilde{\eta}) < e^{-\ell^*}\mu/\lambda \text{ for all } t > T$$

Proof. Consider the reputational trajectories ℓ_t, ℓ_t' originating at $\ell_0 < \ell_0'$. Signals shift ℓ_t and ℓ_t' by the same amount, while the drift (C.1) shrinks $\ell_t' - \ell_t$ at rate $\lambda(1 + e^{-\ell_t'}) - \lambda\left(1 + e^{-\ell_t}\right) \approx -\lambda e^{-\ell_t}(\ell_t' - \ell_t) < 0$ in the work-region and similarly at rate $-\lambda e^{\ell_t}(\ell_t' - \ell_t) < 0$ in the shirk-region. This implies that the partial derivative exists and proves part (a). Part (b) follows because the drift at the cutoff equals 0. Part (c) follows because the reputational drift decelerates from $\left|-\lambda\left(1 + e^{\ell^*}\right) - \mu\right| > \lambda e^{\ell^*}$ to $\left|\lambda\left(1 + e^{-\ell^*}\right) - \mu\right| < \mu$ at the cutoff. So when the trajectory ℓ_t hits ℓ^* the reputational increment decreases by the above factor and by part (a) it never grows. $\square$

Next, we use these facts about reputational dynamics to provide some regularity results about value functions in a work-shirk candidate equilibrium and an explicit formula for the marginal value of reputation.

Lemma 8 *In any work-shirk candidate equilibrium with cutoff ℓ^*:*

(a) Value $V_\theta(\ell)$ is continuous in reputation ℓ.

(b) At the cutoff the value of quality $\Delta_{\ell^*}(\ell^*)$ is strictly positive.

(c) If the firm is indifferent at the cutoff ℓ^*, i.e. $\lambda\Delta_{\ell^*}(\ell^*) = c$, value functions are differentiable with derivative:

$$V_\theta'(\ell) = \mathbb{E}_{\theta_0=\theta}\left[\int_{t=0}^{\infty} e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}(\ell, h_t, \widetilde{\eta})dt\right] > 0 \text{ for all } \ell \in \mathbb{R}. \quad \text{(C.2)}$$

[20] Actual investment η and quality θ^t affect reputation ℓ_t only through the history h_t.

Lemma 8(c) has the flavor of the envelope theorem: when the firm's first-order condition holds at the cutoff, then a change in initial reputation only affects its payoff through the reputational evolution. A firm with a lower initial reputation works more, leading to a gain of $\Delta(\ell)$ when a technology shock hits. This gain is exactly offset by the extra cost born by the firm. The marginal value of reputation $V_\theta'(\ell)$ is thus determined solely by the 'durability' of the reputational increment $\ell_t' - \ell_t$.

Proof. Like in Lemma 3 fix initial reputations $\ell < \ell'$ of a 'low' and a 'high' firm and let $\ell_t = \ell_t(\ell, h_t, \widetilde{\eta})$ and $\ell_t' = \ell_t(\ell', h_t, \widetilde{\eta})$ be as in Lemma 7. Let $\eta_t' = \eta_t'(h_t) = \eta(\ell_t(\ell', h_t, \widetilde{\eta}))$ be investment of the high firm at time t expressed in a non-Markovian manner directly as a function of the public history h_t. We decompose the incremental value of reputation as follows:

$$V_\theta(\ell') - V_\theta(\ell) = [V_\theta(\ell') - V_{\theta,\eta'}(\ell)] + [V_{\theta,\eta'}(\ell) - V_\theta(\ell)] \tag{C.3}$$

where $V_{\theta,\eta'}(\ell)$ is the value of the low firm that mimics the high firm by adopting the non-Markovian investment strategy η'.

The first term in (C.3) is the benefit of the high firm's reputational advantage when the low firm mimics the high firm. It is determined by the derivative of future reputation with respect to current reputation:

$$V_\theta(\ell') - V_{\theta,\eta'}(\ell) = \mathbb{E}_{\theta_0=\theta,\eta'}\left[\int e^{-rt}\left(x\left(\ell_t(\ell', h_t, \widetilde{\eta})\right) - x\left(\ell_t(\ell, h_t, \widetilde{\eta})\right)\right) dt\right]$$

This term is always positive. Taking the limit $\ell' \to \ell$ and applying the chain rule gives rise to equation (C.2).

The problem with analyzing the second term in (C.3) is that quality and reputation drift apart due to the different investment plans. To overcome this problem, we first write the low firm's value $V_\theta(\ell)$ as an expectation of future profits conditional on the firm's quality being controlled by investment plan η'.

$$\begin{aligned}
V_\theta(\ell) &= \mathbb{E}_{\eta,\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(x(\ell_t) - c\eta(\ell_t)\right) dt\right] \\
&= \mathbb{E}_{\eta',\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(x(\ell_t) - c\eta(\ell_t) + \left(\eta(\ell_t) - \eta(\ell_t')\right)\lambda\Delta(\ell_t)\right) dt\right] \\
&= \mathbb{E}_{\eta',\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(x(\ell_t) - c\eta(\ell_t') + \left(\eta(\ell_t) - \eta(\ell_t')\right)(\lambda\Delta(\ell_t) - c)\right) dt\right] \\
&= V_{\theta,\eta'}(\ell) + \mathbb{E}_{\eta',\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\eta(\ell_t) - \eta(\ell_t')\right)(\lambda\Delta(\ell_t) - c)\, dt\right]
\end{aligned}$$

where $\ell_t = \ell_t(\ell, h_t, \widetilde{\eta})$ and $\ell_t' = \ell_t(\ell', h_t, \widetilde{\eta})$. The key step is line 2: Whenever the low firm invests according to η, i.e. $\eta(\ell_t) = 1$, its continuation value increases by $\lambda\Delta(\ell_t)$ relative to the low firm that mimics the high firm and shirks, i.e. $\eta(\ell'_t) = 0$. When writing the low firm's value as future profits conditional on the less favorable distribution of technology shocks induced by η', we must compensate the low firm by capitalizing these forgone benefits. Line 3 rearranges terms, and line 4 highlights that the difference in values is due to the profitability of investing, $\lambda\Delta(\ell_t) - c$, when the low firm does and the high firm does not, i.e. $\eta(\ell_t) - \eta(\ell'_t) = 1$.

We now show that this difference term is of order $O(\ell' - \ell)$:

$$\begin{aligned}
\left|V_{\theta,\eta'}(\ell) - V_\theta(\ell)\right| &\leq \mathbb{E}_{\eta',\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\eta(\ell_t) - \eta\left(\ell'_t\right)\right)\left|\lambda\Delta(\ell_t) - c\right| dt\right] \\
&\leq \mathbb{E}_{\eta',\widetilde{\eta}}\left[\int_{t:\ell_t\leq\ell^*\leq\ell'_t} e^{-rt}\left|\lambda\Delta(\ell_t) - c\right| dt\right] \\
&\leq \max_{\ell''\in[\ell^*-(\ell'-\ell),\ell^*]}\left|\lambda\Delta(\ell'') - c\right|\left(\ell' - \ell\right)/\lambda
\end{aligned}$$

The final inequality uses that $\ell'_t - \ell_t$ is decreasing in t by Lemma 7(a), and the rate of decrease is at least λ whenever $\ell_t < \ell^* \leq \ell'_t$ or $\ell_t \leq \ell^* < \ell'_t$. This proves that value functions are continuous, i.e. part (a).

For part (c), we assume $\lambda\Delta(\ell^*) = c$. By part (a) Δ is continuous so that $\max_{\ell''\in[\ell^*-(\ell'-\ell),\ell^*]}|\lambda\Delta(\ell'') - c|$ converges to 0 as $\ell' \to \ell$. Thus, the second term $V_{\theta,\eta'}(\ell) - V_\theta(\ell)$ in equation C.3 is of order $o(\ell' - \ell)$.

To prove (b) assume by contradiction that the value of quality is non-positive at the cutoff $\Delta_{\ell^*}(\ell^*) \leq 0$. Then $\lambda\Delta_{\ell^*}(\ell) < c$ for ℓ close to the cutoff ℓ^*, and the second term $V_{\theta,\eta'}(\ell) - V_\theta(\ell)$ is positive. The first term $V_\theta(\ell') - V_{\theta,\eta'}(\ell)$ is positive by construction, so value is increasing in reputation. This implies that reputational dividends are strictly positive and by Theorem 1, $\Delta_{\ell^*}(\cdot)$ is strictly positive as well. □

Remark 1 *If the reputational drift is convergent at cutoff ℓ^*, we can truncate the integral (C.2) at time T when the reputational evolution hits ℓ^* by Lemma 7(b). When the reputational drift is permeable at cutoff ℓ^* and $\ell^* \gg 0$, we can essentially do the same by Lemma 7(c).*

Finally we establish a slightly different continuity result when the work-shirk cutoff is altered at the same time as the firm's reputation. This result is essential for finding work-shirk candidate equilibria with indifference at the cutoff. Let $V_{\theta,\ell^*}(\ell)$ be the value of a firm with initial quality θ and reputation ℓ in the work-shirk candidate equilibrium with cutoff ℓ^*.

Lemma 9 *Across work-shirk candidate equilibria, firm value at the cutoff $V_{\theta,\ell^*}(\ell^*)$ and value of quality at the cutoff $\Delta_{\ell^*}(\ell^*)$ are continuous in ℓ^*.*

Proof. Consider a 'low' firm who has cutoff ℓ^* and reputation ℓ_t starting at $\ell_0 = \ell^*$. Compare this to a 'high' firm who has cutoff $\ell^* + \varepsilon$ and reputation ℓ'_t starting at $\ell'_0 = \ell^* + \varepsilon$.

Let τ be the first time that quality differs for these firms. We first show that $0 \leq \ell'_t - \ell_t \leq \varepsilon$ for $t < \tau$. The first inequality holds for $t = 0$ and then ℓ'_t remains above ℓ_t because it enjoys more favorable beliefs while the jump at a signal does not change $\ell'_t - \ell_t$. The second inequality is due to the fact that whenever $\ell'_t - \ell_t = \varepsilon$ either both firms work or both firms shirk. In either case the proof of Lemma 7 shows that the reputational increment $\ell'_t - \ell_t$ decreases. Hence the two firms' reputations always converge when they are ε apart, as required.

We next show that until time τ the investment decisions of the two firms are similar. Formally, we claim that for any $\delta > 0$ and $T < \infty$, there is a sufficiently small $\varepsilon = \ell'_0 - \ell_0$ such that

$$r\mathbb{E}\left[\int_0^{\min\{\tau,T\}} e^{-rt}\left|\eta'_t - \eta_t\right| dt\right] < \delta, \tag{C.4}$$

where η'_t and η_t are investment of the high and low firm at time t. We know from the above that $\ell'_t - \ell_t \in [0,\varepsilon]$ for $t < \tau$, and for $\ell'_t - \ell_t = \varepsilon$ we have $\eta'_t = \eta_t$, unless $\ell'_t = \ell^* + \varepsilon$ and $\ell_t = \ell^*$ in which case $|\eta_t - \eta'_t|$ is of order ε. For $\ell'_t - \ell_t < \varepsilon$, we have $\eta'_t \geq \eta_t$ with equality unless $\ell_t, \ell'_t \in [\ell^*, \ell^* + \varepsilon]$. The key step of the argument is that the set of times $\Omega = \{t : \ell_t, \ell'_t \in [\ell^*, \ell^*+\varepsilon], \eta'_t \geq \eta_t + \delta\}$ where investment levels differ by more than δ consists of a finite number of short time intervals. The reason is that for $t \in \Omega$ the reputational drift differs by $d(\ell'_t) - d(\ell_t) = \lambda\delta(1+e^{\ell_t})^2/e^{\ell_t}$ and thus the length of an interval of such t is bounded above by $\varepsilon/\left(d(\ell'_t) - d(\ell_t)\right)$. Any two such intervals must be separated by a signal and in expectation there is only a finite number of signals in $[0,T]$. This implies equation (C.4).

As $\varepsilon \to 0$, the high and the low firm invest at almost the same intensity by equation (C.4), so for any T and δ, we can choose $\varepsilon > 0$ such that $\Pr(\tau > T) > 1 - \delta$. Putting this all together,

$$\begin{aligned}
|V_{\theta,\ell^*+\varepsilon}(\ell^*+\varepsilon) - V_{\theta,\ell^*}(\ell^*)| &= \left|\mathbb{E}\left[\int_0^\infty e^{-rt}[(x(\ell'_t) - \eta'_t c) - (x(\ell_t) - \eta_t c)]dt\right]\right| \\
&\leq \mathbb{E}\left[\int_0^T e^{-rt}\left|(x(\ell'_t) - \eta'_t c) - (x(\ell_t) - \eta_t c)\right| dt\right] + e^{-rT}\frac{(1+c)}{r} \\
&\leq \mathbb{E}\left[\int_0^{\min\{\tau,T\}} e^{-rt}\left|(x(\ell'_t) - \eta'_t c) - (x(\ell_t) - \eta_t c)\right|\right] dt + [e^{-rT} + \delta]\frac{(1+c)}{r} \\
&\leq \mathbb{E}\left[\int_0^{\min\{\tau,T\}} e^{-rt}\left|x(\ell'_t) - x(\ell_t)\right| dt\right] + [e^{-rT} + 2\delta]\frac{(1+c)}{r} \\
&\leq \left[\frac{\varepsilon}{4} + e^{-rT} + 2\delta\right]\frac{(1+c)}{r}
\end{aligned}$$

The first inequality truncates the integral; the second uses the fact that $\Pr(\tau > T) > 1 - \delta$; the third uses (C.4); the last line uses $dx/dl = x(1-x) < 1/4$. Hence the difference between the value functions shrinks as $\varepsilon \to 0$ and $V_{\theta,\ell^*}(\ell^*)$ is continuous in ℓ^*. Thus $\Delta_{\ell^*}(\ell^*) = V_{H,\ell^*}(\ell^*) - V_{L,\ell^*}(\ell^*)$ is continuous in ℓ^*, as required. $\square$

C.3 Indifference at the Cutoff

We now show that for small costs there exists a high cutoff ℓ^* that satisfies the indifference condition. Since Δ and V depend on c, we subscript them with c where useful.

Lemma 10 *For every $\ell \in \mathbb{R}$ there exists $c(\ell) > 0$ such that for all $c^* < c(\ell)$ there exists $\ell^* > \ell$ such that in the work-shirk candidate equilibrium with cutoff ℓ^*, we have $\lambda\Delta_{\ell^*,c^*}(\ell^*) = c^*$.*

Proof. Fix $\ell \in \mathbb{R}$ and consider $\Delta_{\ell,c}(\ell)$ as a function of $c \in [0, \lambda/(r+\lambda)]$. By Lemma 8(b) we have $\Delta_{\ell,c}(\ell) > 0$ for all c. Since $\Delta_{\ell,c}(\ell)$ is continuous in c, it takes on its minimum $\Delta_{\ell,c'}(\ell) > 0$ at some c'.

Let $c(\ell) = \lambda\Delta_{\ell,c'}(\ell)$ and fix any $c^* \in (0, c(\ell))$. Using the definitions of c' and c^*,

$$c^* < c(\ell) = \lambda\Delta_{\ell,c'}(\ell) \leq \lambda\Delta_{\ell,c^*}(\ell),$$

so the firm prefers to work. On the other hand the value of quality vanishes at the top in the full-work candidate equilibrium

$$\lambda\Delta_{\infty,c^*}(\infty) = 0 < c^*,$$

so by continuity of $\Delta_{\ell',c^*}(\ell')$ as a function of $\ell' \in [\ell, \infty]$, Lemma 9, there exists $\ell^* \in (\ell, \infty)$ with $c^* = \lambda\Delta_{\ell^*,c^*}(\ell^*)$. $\square$

The daunting array of quantifiers in the statement of this lemma guarantees that we can assume ℓ^* with $c^* = \lambda\Delta_{\ell^*,c^*}(\ell^*)$ is as large as needed in the upcoming arguments.

C.4 Work at Low Reputations

We show that the firm prefers to work below the cutoff ℓ^* by studying the marginal value of reputation $V_\theta'(\ell)$ in Lemma 11, reputational dividends below the cutoff in Lemma 12, and the value of quality below the cutoff in Lemma 13.

Lemma 11 *Fix any $\alpha > 0$, ℓ^* sufficiently large and assume that in the work-shirk candidate equilibrium with cutoff ℓ^* the firm is indifferent at the cutoff, i.e. $\lambda\Delta_{\ell^*}(\ell^*) = c$. Then the marginal value of reputation $V_\theta'(\ell)$ 'diminishes' to the right of ℓ^* :*

$$\frac{V_\theta'(\ell'')}{V_\theta'(\ell')} \in O(e^{-\ell^*}) \text{ for all } \ell' \in [0, \ell^* - \alpha] \text{ and } \ell'' \in [\ell^*, \infty).$$

Intuitively, incremental reputation above ℓ^* is less 'durable' because it disappears when reputation ℓ_t hits the cutoff ℓ^* as explained in Remark 1. To formalize this it is useful to define the *cutoff time* $T(\ell_0) = \min\{t | \ell_t = \ell^*\}$: This is the first time that the reputational dynamics starting at ℓ reaches the cutoff ℓ^*.

Proof. We argue that $V_\theta'(\ell')$ is bounded below by a term of order $e^{-\ell^*}$ while $V_\theta'(\ell'')$ is of order $e^{-2\ell^*}$. The key equation is

$$V_\theta'(\ell_0) = \mathbb{E}_{\ell_0}\left[\int_0^\infty e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}dt\right] \approx \mathbb{E}_{\ell_0}\left[\int_0^{T(\ell_0)} e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}dt\right]$$

from Lemma 8(c), where we can essentially truncate the integral at the cutoff time $T(\ell_0)$ by Lemma 7(b) and (c).

For initial reputation below the cutoff $\ell_0 = \ell' < \ell^* - \alpha$ we truncate the integral at $T(\ell')$ to obtain a lower bound $f(\ell^* - \ell')e^{\ell^*}/(1+e^{\ell^*})^2$ for $V_\theta'(\ell')$, where the factor $f(\ell^* - \ell')$ measures the expected cutoff time. If reputational drift below the cutoff is positive we can choose f to be a linear and positive function of $\ell^* - \ell$; if reputational drift is negative we can choose $f(\ell^* - \ell)$ greater than a positive constant for all $\ell^* - \ell$ because reputation is drifting away from the cutoff and can only reach it after a good news signal arrives.

For initial reputation above the cutoff $\ell_0 = \ell'' > \ell^*$ the cutoff time satisfies $T(\ell'') \le e^{-\ell^*}/\lambda$ if no signal arrives. To see this, we revert into x-space where the negative drift in the shirk region is approximately $-\lambda$ and the distance from the initial reputation to the cutoff is no more than $1 - x^* = 1/\left(1+e^{\ell^*}\right)$. To get an upper bound for $V_\theta'(\ell'')$ we expand the integral until a signal arrives or the work-shirk cutoff is reached:

$$V_\theta'(\ell'') \le \int_0^{e^{-\ell^*}/\lambda} e^{-rt}\frac{e^{\ell_t^\varnothing}}{(1+e^{\ell_t^\varnothing})^2}\frac{\partial \ell_t^\varnothing}{\partial \ell_0}dt + \frac{\mu e^{-\ell^*}}{\lambda}V_\theta'(\ell^*) + \mu_\theta\frac{e^{-\ell^*}}{\lambda}\max_{\ell > j(\ell^*)}\left\{V_\theta'(\ell)\right\}.$$

The first term is the value of the reputational increment while the reputation $\ell_t^\varnothing$ is drifting from $\ell_0 = \ell''$ to ℓ^*. The second term is an upper bound for the continuation value of the reputational increment once the reputation hits ℓ^* and the increment decreases by a factor $\mu e^{-\ell^*}/\lambda$ if the drift around the cutoff is negative, Lemma 7(c) (if the drift at the cutoff is convergent the increment disappears entirely by Lemma 7(b)). The last term captures the probability of a signal arrival before $e^{-\ell^*}/\lambda$, times the continuation value in case of the arrival. The continuation values are of order $e^{-\ell^*}$ so that all three terms are of order $e^{-2\ell^*}$. $\square$

Lemma 12 *Fix any $\alpha > 0$, ℓ_D sufficiently large, $\ell^* > \ell_D$ sufficiently large and suppose that in the work-shirk candidate equilibrium the firm is indifferent at the cutoff, i.e. $\lambda\Delta_{\ell^*}(\ell^*) = c$. Then the reputational dividend $D_\theta(\ell) = \mu(V_\theta(j(\ell)) - V_\theta(\ell))$ is strictly decreasing on $[\ell_D, \ell^* - \alpha]$ in the good news case, and strictly decreasing on $[\ell_D, \ell^*]$ in the bad news case.*

Proof. In the good news case we have $\mu > 0$ and $j(\ell) > \ell$ and need to show $D'_\theta(\ell) = \mu(V'_\theta(j(\ell)) - V'_\theta(\ell)) < 0$ for all $\ell \in [\ell_D, \ell^* - \alpha]$. For ℓ with $j(\ell) \geq \ell^*$ this follows from Lemma 11.

For ℓ with $j(\ell) < \ell^*$ we again evaluate the marginal value of reputation with equation (C.2):

$$V'_\theta(\ell_0) = \mathbb{E}_{\ell_0}\left[\int_0^\infty e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}dt\right] \approx \mathbb{E}_{\ell_0}\left[\int_0^{T(\ell_0)} e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}dt\right].$$

A reputational increment essentially disappears when the trajectory hits the cutoff. The higher trajectory starting at $j(\ell)$ hits the cutoff before the lower trajectory starting at ℓ. Thus, we can restrict attention to $t < T(\ell_0)$ and $\ell_t < \ell^*$ in the integral.

As long as $\ell_D \gg 0$, the two trajectories $\ell_t(\ell, h_t, \widetilde{\eta})$ and $\ell_t(j(\ell), h_t, \widetilde{\eta})$ evolve essentially in parallel at a constant distance of $j(\ell) - \ell$ because $\partial \ell_t / \partial \ell_0 \approx 1$ by the proof of Lemma 7(a). Then $V'_\theta(\ell) > V'_\theta(j(\ell))$ follows because $e^\ell/(1+e^\ell)^2 \approx e^{-\ell}$ is decreasing in ℓ. This argument actually shows that the rate of decrease $D'_\theta(\ell)$ is of order $(j(\ell) - \ell)\, e^{-\ell}$.

In the bad news case we have $\mu < 0$ and $j(\ell) < \ell$ and need to show $D'_\theta(j(\ell)) = \mu(V'_\theta(j(\ell)) - V'_\theta(\ell)) < 0$ for all $\ell \in [\ell_D, \ell^*]$. This follows by the same argument as in the good news case. □

Lemma 13 shows that firms with low reputations work. For reputations $\ell \in [\ell_\Delta, \ell^*]$ for some ℓ_Δ defined below, we prove that the firm prefers to work by showing that $\Delta(\ell)$ is decreasing on $[\ell_\Delta, \ell^*]$. For reputations $\ell < \ell_\Delta$ the result follows from the closeness of $\Delta_{\ell^*}(\cdot)$ and $\Delta_\infty(\cdot)$.

Lemma 13 *Assume that c is sufficiently small, ℓ^* is sufficiently large, and that in a work-shirk candidate equilibrium with cutoff ℓ^* the firm is indifferent at the cutoff, i.e. $\lambda\Delta_{\ell^*}(\ell^*) = c$.*

(a) If learning is bad news, then $\lambda\Delta_{\ell^*}(\ell) > c$ for all $\ell < \ell^*$.

(b) If learning is good news and $\lambda < \mu$, then $\lambda\Delta_{\ell^*}(\ell) > c$ for all $\ell < \ell^*$.

Proof. We first show for both cases that for any ℓ_Δ there exists ℓ^* large and c small, such that $\lambda\Delta_{\ell^*}(\ell) < c$ for all $\ell < \ell_\Delta$: As $\ell^* \to \infty$, $\Delta_{\ell^*}(\ell)$ converges pointwise to $\Delta_\infty(\ell)$ for all ℓ. Let $\ell^* \gg \ell_\Delta$. For any $\ell < \ell_\Delta$, the marginal value of reputation, reputational dividends, and thus the value of quality, depend on the cutoff ℓ^* only on trajectories ℓ_t that reach ℓ^*. The weight of these trajectories converges to 0 as $\ell^* \to \infty$, so the convergence is uniform for $\ell < \ell_\Delta$.

The function $\Delta_\infty(\cdot)$ is bounded away from 0 on $[-\infty, \ell_\Delta]$, and so is $\Delta_{\ell^*}(\ell)$. For small costs c, we get

$$\lambda\Delta_{\ell^*}(\ell) > c \quad \text{for } \ell \in [-\infty, \ell_\Delta],$$

as required.

Next we show that $\Delta_{\ell^*}(\cdot)$ is monotonically decreasing on $[\ell_\Delta, \ell^*]$. To do so, we apply Theorem 1 to write the value of quality as integral over future reputational dividends

$$\Delta_{\ell^*}(\ell) = \mathbb{E}_{\theta^\infty = L}\left[\int_0^\infty e^{-(r+\lambda)t} D_H(\ell_t)\, dt\right]$$

Part (a): Fix ℓ_D from Lemma 12 such that the dividend $D_H(\ell)$ is decreasing on $[\ell_D, \ell^*]$. Choosing ℓ_Δ large enough and $\ell_0 \in [\ell_\Delta, \ell^*]$, the probability that $\ell_t \in [\ell_D, \ell^*]$ is close to one, and the claim follows by Lemma 12.

Part (b): Fix α and ℓ_D from Lemma 12 such that the dividend $D_H(\ell)$ is decreasing on $[\ell_D, \ell^* - \alpha]$. When $\lambda < \mu$ and ℓ^* is sufficiently large, then at the cutoff ℓ^* reputation drifts into the work-region. Choosing ℓ_Δ large enough and $\ell_0 \in [\ell_\Delta, \ell^*]$, the probability that $\ell_t \in [\ell_D, \ell^* - \alpha]$ is close to one, and the claim follows by Lemma 12. □

If learning is good news and $\lambda \geq \mu$, then Lemma 13 fails. This case is different because the reputational drift in the work-region below the cutoff is positive, $d(\ell) = \lambda(1-x) - \mu x(1-x) > 0$, and the shirk-region $[\ell^*, \infty]$ is absorbing. Thus, the high reputational dividends $D_\theta(\ell)$ for reputation below the cutoff $\ell < \ell^* - \alpha$, which are the dominating term in Lemma 13(a) and (b), are irrelevant in this case. We can further show that actually $\lambda\Delta_{\ell^*}(\ell^* - \varepsilon) < c$ for all sufficiently small $\varepsilon > 0$. Thus, for good news learning and $\lambda \geq \mu$ there is no work-shirk equilibrium when costs are low, and any equilibrium must involve mixing $\eta(\ell) \in (0,1)$ for some interval of reputations.

C.5 Shirk at High Reputations

Lemma 14 *Assume that c is sufficiently small, ℓ^* is sufficiently large, and that in a work-shirk candidate equilibrium with cutoff ℓ^* the firm is indifferent at the cutoff, i.e. $\lambda\Delta_{\ell^*}(\ell^*) = c$.*

(a) If learning is bad news, then $\lambda\Delta_{\ell^*}(\ell) < c$ for all $\ell > \ell^*$.

(b) If learning is good news and $\lambda < \mu$, then $\lambda\Delta_{\ell^*}(\ell) < c$ for all $\ell > \ell^*$.

Proof. Part (a): The idea of the proof is to write the value of quality as a short stream of dividends and a continuation value and to show that both the dividends and the continuation value are higher for $\ell'_0 = \ell^*$ than for $\ell_0 > \ell^*$. We terminate the dividend expansion at time $T = \min\{t : \ell_t \leq \ell^*\}$ when the reputation starting at ℓ_0 first reaches ℓ^* or jumps over ℓ^* at the arrival of a signal.

$$\begin{aligned}
\Delta_{\ell^*}(\ell_0) &= \mathbb{E}_{\theta^\infty = L}\left[\int_0^T e^{-(r+\lambda)t} D_H(\ell_t)\, dt + e^{-(r+\lambda)T}\Delta_{\ell^*}(\ell_T)\right], \\
\Delta_{\ell^*}(\ell^*) &= \mathbb{E}_{\theta^\infty = L}\left[\int_0^T e^{-(r+\lambda)t} D_H(\ell'_t)\, dt + e^{-(r+\lambda)T}\Delta_{\ell^*}(\ell'_T)\right],
\end{aligned}$$

By definition of T, we have $\ell'_t \le \ell^* < \ell_t$ and by Lemma 11 we get $D_H(\ell'_t) > D_H(\ell_t)$. The continuation value is also smaller in the first expression by Lemma 13(a): If the dividend expansion ends at a signal, then $\ell'_T < \ell_T \le \ell^*$; otherwise we have $\ell'_T \le \ell^* = \ell_T$.

Part (b): In this case the idea is to expand $\Delta_{\ell^*}(\ell)$ until cutoff time $T(\ell)$ and compare the dividends $D(\ell_t)$ to the annuity value of $\Delta_{\ell^*}(\ell^*)$:

$$\begin{aligned}\Delta_{\ell^*}(\ell) - \Delta_{\ell^*}(\ell^*) &= \mathbb{E}\left[\int_0^{T(\ell)} e^{-(r+\lambda)t} D_H(\ell_t)\,dt + e^{-(r+\lambda)T(\ell)}\Delta_{\ell^*}(\ell^*)\right] - \Delta_{\ell^*}(\ell^*) \\ &= \mathbb{E}\left[\int_0^{T(\ell)} \left(e^{-(r+\lambda)t} D_H(\ell_t) - (r+\lambda)\Delta_{\ell^*}(\ell^*)\right) dt\right] \qquad \text{(C.5)}\end{aligned}$$

To show that the integrand is negative, we now expand $(r+\lambda)\Delta_{\ell^*}(\ell^*)$ into reputational dividends $D_H(\ell^*_t)$, that exceed $D_H(\ell_t)$ on average:

$$\begin{aligned}(r+\lambda)\Delta(\ell^*) &= (r+\lambda)\int_0^\infty e^{-(r+\lambda)t}\mathbb{E}\left[D_H(\ell^*_t)\right] dt \\ &\ge (r+\lambda)\int_0^\infty e^{-(r+\lambda)t}\Pr\left(\ell^*_t \in [\ell_D, \ell^* - \alpha]\right) \inf_{\ell \in [\ell_D, \ell^*-\alpha]} \{D_H(\ell)\}\, dt \\ &\ge \sup_{\ell' \ge \ell^*} \{D_H(\ell)\} \\ &\ge \sup_{t \le T(\ell)} \{D_H(\ell_t)\}\end{aligned}$$

The third line uses that for $\alpha > 0$ sufficiently small, and ℓ^* sufficiently large we get $\Pr(\ell^*_t \in [\ell_D, \ell^* - \alpha])$ close to 1, while by choosing ℓ^* large enough, we get $\inf_{\ell \in [\ell_D, \ell^* - \alpha]}\{D_H(\ell)\} / \sup_{\ell > \ell^*}\{D_H(\ell)\}$ as large as necessary by Lemma 11. □

D Imperfect Poisson Learning: Proof of Theorem 6(a) and (b)

D.1 Proof of Theorem 6(a)

We prove Theorem 6(a) in two steps. Lemma 15 shows that for low costs, the firm prefers to invest at intermediate reputations in any candidate equilibrium. Lemma 16 shows that if market learning ensures (HOPE) and the firm invests at intermediate reputations, it also prefers to invest when its reputation is low. Intuitively, a firm with reputation just below a tentative shirk-work cutoff hopes to achieve an intermediate reputation in the future and thus prefers to work.

Lemma 15 *On any bounded interval $\left[-\overline{\ell},\overline{\ell}\right]$ the value of quality $\Delta\left(\ell\right)$ is bounded away from zero, uniformly across cost c and candidate equilibria η.*

Proof. Lemma 3 states that equilibrium value functions are strictly monotone. Its proof actually implies the slightly stronger result that $V_\theta\left(\ell'\right)-V_\theta\left(\ell\right)$ for $\ell'>\ell$ is bounded below by some strictly positive function $\gamma\left(\ell,\ell'\right)>0$ that is independent of c and the equilibrium η. This is because the reputational increment $\ell'_t-\ell_t$ diminishes at most at the finite rate $\lambda\left(1+e^{\ell}\right)+\lambda\left(1+e^{-\ell}\right)$. By Theorem 1 this uniform lower bound on the value of incremental reputation implies a uniform lower bound on the value of quality on any compact interval $\left[-\overline{\ell},\overline{\ell}\right]$. □

Lemma 16 *Fix any $\overline{\ell}>0$. If market learning ensures (HOPE) then there exists $c_{\overline{\ell}}$ such that for all $c\in\left(0,c_{\overline{\ell}}\right)$ a firm with reputation $\ell\in(-\infty,\overline{\ell})$ works in equilibrium.*

Proof. From Lemma 15 we know that the value of quality is uniformly bounded below on $\left[-\overline{\ell},\overline{\ell}\right]$. By contradiction, consider a candidate equilibrium with a shirk-region in $(-\infty,-\overline{\ell}]$ and let $\ell_*<-\overline{\ell}$ be the highest shirk-work cutoff. We expand $\Delta(\ell_*-\varepsilon)$ until the first time T when $\ell_t\geq-\overline{\ell}$. The value of quality at $\ell_*-\varepsilon$ must exceed its continuation value in the contingency that T is reached:

$$\Delta(\ell_*-\varepsilon)\geq\mathbb{E}[e^{-(r+\lambda)T}\Delta(\ell_T)]. \tag{D.1}$$

As $\Delta(\ell_T)$ is bounded below we just need to show that $\mathbb{E}[e^{-(r+\lambda)T}]$ is bounded below. By the assumption that market learning ensures (HOPE), and by choosing $-\overline{\ell}$, and thus ℓ_*, low enough, the firm's initial reputation $\ell_*-\varepsilon$ will rise above ℓ_* with positive probability. Once $\ell_t>\ell_*$, equilibrium beliefs $\widetilde{\eta}=1$ will push reputation to $-\overline{\ell}$ in finite time with positive probability.

Thus the right-hand-side of (D.1) is uniformly bounded below, and by choosing c small enough we get $\lambda\Delta(\ell_*-\varepsilon)>c$. Therefore, the candidate equilibrium with shirk-work cutoff ℓ_* is not an equilibrium. □

D.2 Proof of Theorem 6(b)

We need to show that there is a continuum of shirk-work-shirk equilibria if condition (HOPE) is violated. A shirk-work-shirk candidate equilibrium with shirk-work cutoff [21] x_1^* and work-shirk cutoff x_2^* is an equilibrium if

$$\lambda\Delta(x) = \begin{cases} \leq c & \text{for } x \in [-\infty, x_1^*) \\ \geq c & \text{for } x \in [x_1^*, x_2^*] \\ \leq c & \text{for } x \in [x_2^*, \infty] \end{cases}$$

To satisfy the first condition, it suffices to choose any $x_1^* < rc/\lambda$. This guarantees $V_\theta(x) < c/\lambda$ for all $x < x_1^*$ because reputational drift and jumps are negative below x_1^*. Thus $\lambda\Delta(x) < c$ as required.

To satisfy the second and third condition we reapply the proof of Theorem 5: In the shirk-work candidate equilibrium with cutoff x_1^*, the value of quality in the work-region is bounded below for intermediate reputations and it disappears for high reputations. Again, for low values of c, there exists a work-shirk cutoff x_2^* close to 1, such that the firm prefers to work below this cutoff and shirk above the cutoff.

[21] Suppose for simplicity that the firm works at the shirk-work cutoff, i.e. $\eta(\ell_1^*) = 1$.

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A Reputational Theory of Firm Dynamics*

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Abstract

We study the lifecycle of a firm who sells a product of uncertain quality, characterizing the optimal investment and exit decisions and the resulting firm dynamics. We investigate two model variations. If the firm shares the market's uncertainty, it learns about its product quality through past actions and public signals. This learning generates a level of self-esteem which coincides with its public reputation only on the equilibrium path. We show that the firm is incentivized to invest by the marginal value of self-esteem, and that the firm stops investing when its reputation approaches the exit threshold and its life-expectancy vanishes. In contrast, when the firm knows its product quality perfectly, both high- and low-quality firms invest at the threshold where low-quality firms exit the market. While the life-expectancy of a low-quality firm vanishes, investment remains profitable because investment success boosts the firm's quality and averts exit.

1 Introduction

Maintaining a good reputation is essential for survival in many industries. Professionals (e.g. consultants, lawyers, academics) invest in their skills to develop a reputation for solving problems, but may quit and change occupation if they do not succeed. Similarly, restaurants try to build a reputation for high quality food and service, but many fail with 25% of young restaurants exiting each year (Parsa et al. (2005)). This paper develops a simple model to study the lifecycle of such a firm. We characterize the optimal exit decision, and analyze how this impacts the firm's investment incentives.

In the model, illustrated in Figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a stochastic function of

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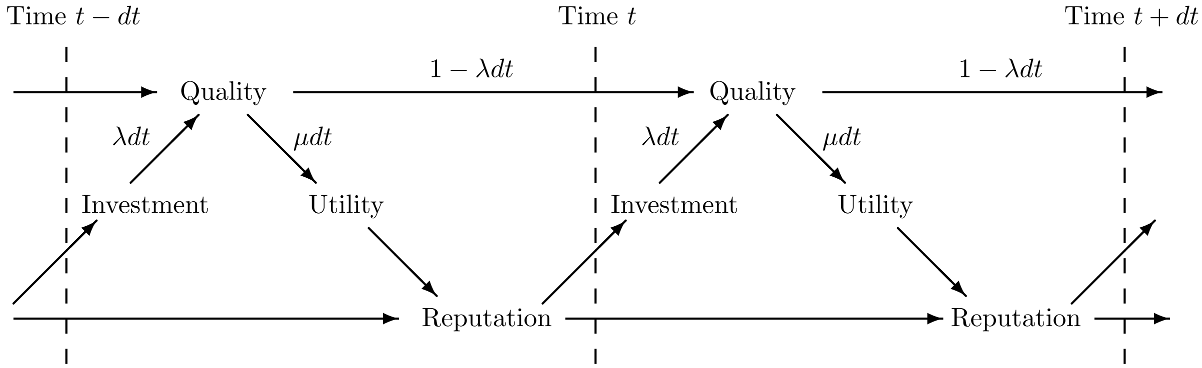

Figure 1: **Gameform.** Quality is a function of the firm's investment and its past quality. Consumer utility is an imperfect signal of quality that the market uses to update the firm's reputation.

the firm's past investments. Consumers observe neither quality nor investment directly and learn about the product quality through breakthroughs that can only be produced by a high-quality product. At each point in time, consumers' willingness to pay is determined by the market belief that the quality is high, x_t, which we call the reputation of the firm. This reputation changes over time as a function of (a) the equilibrium beliefs of the firm's investments, and (b) market learning via product breakthroughs. The firm can exit the market at any time, and does so when its reputation falls below some threshold.

Our analysis and the form of the results depend on whether or not the firm knows its own quality. Which case is more relevant will depend on the application at hand. For example, an academic who obtains breakthroughs by publishing papers, knows her past publishing success and how much she invests in her skills, but shares the profession's uncertainty of her current ability and future success. In contrast, a restaurateur, who obtains breakthroughs through newspaper reviews, can learn directly from customer feedback whether it has a potentially successful concept, a signal not available to the market as a whole.

We first suppose the firm does not know its own quality. In equilibrium, the state of the game is summarized by the firm's reputation. Off the equilibrium path the firm's belief about its quality, its self-esteem, diverges from its reputation because reputation is governed by believed investment while self-esteem is governed by actual investment. The firm's value is thus a function of both its reputation and its self-esteem, and investment incentives are determined by the marginal value of self-esteem. In our first major result, we show this marginal value of self-esteem can be written as a present value of future reputational dividends, which capture the immediate marginal benefit of self-esteem.

In equilibrium, the firm's self-esteem and reputation coincide. When this reputation is very low, the firm's losses exceed the option value of staying in the market, and it exits the market. For a firm above the exit threshold, investment incentives are hump-shaped in the firm's reputation. For low reputations near the exit threshold, the firm's life expectancy is very short and the firm stops investing. In equilibrium the market anticipates this effect and adverse market beliefs further accelerate the firm's demise. For intermediate reputation levels, the marginal value of self-esteem is bounded below and the firm invests when costs are sufficiently low. Finally, when the firm's reputation is close to 1, the firm cannot work in equilibrium. If the firm was believed to work at such a reputation, the lack of any breakthrough would be attributed to bad luck, undermining the incentive to actually invest.

Next, we suppose the firm knows its own quality. Here the firm's value is a function of its reputation and its quality, and investment is incentivized by the difference in value between a high and low quality firm. As above, we express this value of quality as the net present value of future reputational dividends, and use this expression to characterize incentives.

In equilibrium, the firm's quality affects its exit decision but not its investment. Specifically, there is a threshold where a low-quality firm exits, a high quality firm remains in the market, and the exit rate of the low-quality firm keeps the reputation of surviving firms at this threshold. In equilibrium, the firm's investment incentives are decreasing in its reputation and are maximized at the exit threshold, so the firm fights until the bitter end. While the low-quality firm's life expectancy vanishes as it approaches the exit threshold, investment success is observable and averts exit, resulting in investment incentives that are of first order. In contrast, with unknown quality, investment success still needs to be learned by the firm and investment incentives are of second order.

While we derive our main results for a perfect good news specification of market learning, many of these effects are robust to more general stationary learning structures with imperfect Poisson and Brownian signals. We also discuss how to model entry into the market in order complete the firm's lifecycle.

1.1 Literature

Our model is based on Board and Meyer-ter-Vehn (2010), which bridges classic models of reputation (e.g. Holmström (1999), Mailath and Samuelson (2001)) and models of repeated games (e.g. Fudenberg et al. (1990)). Our earlier paper characterizes firms' investment problems without considering entry or exit. As equilibrium investment does not depend on quality the distinction between the known and unknown quality cases is moot, and the issue of self-esteem does not arise.

Bar-Isaac (2003) analyses the optimal exit decision of a firm with fixed quality. He lays the foundation for our paper, introducing the distinction between known and unknown quality. He also shows that threshold exit rules are optimal and that a firm that knows it has high quality never exits because exit of low types bounds the reputational evolution from below. We build on Bar-Isaac's paper by analyzing how these exit decisions impact the firm's investment decisions at different stages of its lifecycle.

Kovrijnykh (2007) introduces exit into the career concerns model of Holmström (1999). This leads to the same issues of inner- and outer-reputation as in the present paper. Because of tractability problems with the normal-linear model, this paper only considers a three-period model, which limits the scope of the results.[1]

There are many other models of firm dynamics. In complete information models (e.g. Jovanovic (1982), Hopenhayn (1992), Ericson and Pakes (1995)), firms differ in their capabilities, and choose when to enter, exit and invest. In comparison, we allow quality to be imperfectly observed, introducing a role for reputation that affects the firm dynamics and investment incentives. In contrast to repeated games models (e.g. Gale and Rosenthal (1994), Rob and Fishman (2005)), our firm has a state variable, enabling us to impose more discipline on equilibria by focusing on Markovian equilbria.

2 Model

Overview: There is one firm and a continuum of consumers. Time $t \in [0, \infty)$ is continuous and infinite; the common interest rate is $r \in (0, \infty)$. At time t the firm produces one unit of a product that can have high or low quality, $\theta_t \in \{L = 0, H = 1\}$. The expected instantaneous value of the product to a consumer equals $\theta_t dt$. The market belief about product quality $x_t = \Pr(\theta_t = H)$ is called the firm's *reputation*. The firm chooses investment $\eta_t \in [0, 1]$ at cost $(c\eta_t + k)\, dt$, where c is the cost of investment and k is the operating cost; the firm can exit the market at any time.

Technology: Product quality θ_t is a function of past investments $(\eta_s)_{0 \le s \le t}$ via a Poisson process with arrival rate λ that models quality obsolescence. Absent a shock quality is constant, $\theta_{t+dt} = \theta_t$; when a shock occurs previous quality becomes obsolete and is determined by the level of investment, $Pr(\theta_{t+dt} = H) = \eta_t$.[2] Quality at time t is then a geometric sum of past investments,

$$\Pr(\theta_t = H) = \int_0^t \lambda e^{\lambda(s-t)} \eta_s ds + e^{-\lambda t} \Pr(\theta_0 = H). \tag{2.1}$$

Information: Consumers observe neither investment η nor product quality θ. We analyse both the case were the firm does not know its own quality (Section 3) and where it does (Section 4). Consumers (and the firm) learn about quality through product *breakthroughs* that arrive to high-quality firms at Poisson rate μ. The quality obsolescence process and the breakthrough process are statistically independent. The breakthroughs can be related to consumers' utility by assuming that $dU_t = 0$ almost always, with each breakthrough yielding utility $1/\mu$.

[1] This distinction between private and public beliefs is also present in Bonatti and Horner's (2010) model on strategic experimentation.

[2] This formulation provides a tractable way to model product quality as a function of past investments. For example, one can interpret investment as the choice of absorptive capacity, determining the ability of a firm to recognise new external information and apply it to commercial ends (Cohen and Levinthal (1990)).

Reputation Updating: The reputation increment $dx_t = x_{t+dt} - x_t$ is governed by product breakthroughs, their absence, and market beliefs about investment $\widetilde{\eta}$. A breakthrough reveals high quality, so the firm's reputation immediately jumps to one, $x_{t+dt} = 1$. Absent a breakthrough, dx is deterministic and by independence it can be decomposed additively:

$$dx = \lambda(\widetilde{\eta} - x)dt - \mu x(1 - x)dt. \tag{2.2}$$

The first term is the differential version of equation (2.1); if expected quality after a technology shock $\widetilde{\eta}$ exceeds current expected quality x then reputation drifts up. The second term is the standard Bayesian increment in the absence of a breakthrough.

Profit and Consumer Surplus: The firm and consumers are risk-neutral. At time t the firm sets price equal to the expected value x_t, so consumers' expected utility is 0. The firm's flow profit is $(x_t - c\eta_t - k)dt$ and its discounted present value is thus given by:

$$V = \mathbb{E}\left[\int_{t=0}^{T} e^{-rt}(x_t - c\eta_t - k)dt\right] \tag{2.3}$$

Markov-Perfect-Equilibrium: We assume Markovian beliefs $\widetilde{\eta} = \widetilde{\eta}(x)$ and define Markov-Perfect-Equilibria in Sections 3 and 4 respectively.

3 Unknown Quality

If the firm does not know its own quality, its investment and exit decisions will depend on its public reputation $x_t = \Pr(\theta_t = H | U_s, s \in [0,t])$ and its *self-esteem* $z_t = \Pr(\theta_t = H | U_s, \eta_s, s \in [0,t])$. At time 0, we assume that $z_0 = x_0$. Subsequently, the dynamics of self-esteem are determined by

$$dz = \lambda\left(\eta\left(x,z\right) - z\right)dt - \mu z\left(1 - z\right)dt \tag{3.1}$$

absent a breakthrough, and $z_{t+dt} = 1$ after a breakthrough. This differs from (2.2) in that self-esteem depends on actual investment η, whereas reputation depends on believed investment $\widetilde{\eta}$. In equilibrium these coincide, but investment incentives are determined by off-equilibrium considerations.

In a Markovian equilibrium we can write the firm's value as function of its reputation and self-esteem, $V(x,z)$. A *Markov-Perfect-Equilibrium* $\langle \eta, \widetilde{\eta} \rangle$ then consists of an investment function $\eta : [0,1]^2 \to [0,1]$, exit-region $R \subseteq [0,1]^2$, and market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that: (1) Investment maximizes firm value, $V(x,z)$; (2) The firm exits when value is negative, $V(x,z) \leq 0$;

and (3) Market beliefs are correct, $\widetilde{\eta}(x) = \eta(x, x)$.[3]

The value function has a number of basic properties. $V(x,z)$ is increasing in reputation x, since this leads directly to higher revenue. $V(x,z)$ is increasing in self-esteem z, since a higher quality ultimately leads to higher reputation. Finally, $V(x,z)$ is convex in self-esteem z, since information about quality is valuable to the firm.[4]

3.1 Optimal Investment and Exit Decisions

Lemma 1 shows that investment incentives are determined by the marginal value of self-esteem, $\partial_z V(x,z)$. This is because investment directly controls quality and thus the firm's belief about its quality, i.e. its self-esteem.

Lemma 1. *Equilibrium investment satisfies*

$$\eta(x,z) = \begin{cases} 1 & \text{if } \lambda \partial_z V(x,z) > c \\ 0 & \text{if } \lambda \partial_z V(x,z) < c \end{cases}$$

Proof. Using (3.1), investment over $[t, t+dt]$ increases self-esteem by λdt, and therefore yields the firm $\lambda \partial_z V(x,z)$. □

The firm exits the industry when its value is negative. Since reputation declines continuously (or jumps up) and the firm's value is increasing in its reputation and self-esteem, the firm exits when its reputation falls to zero, $V(x,z) = 0$. On the equilibrium path, the firm exits at x^e defined by

$$V(x^e, x^e) = 0.$$

3.2 Marginal Value of Self-Esteem

In order to understand the firm's investment incentives we decompose the value of incremental self-esteem, $V(x,z') - V(x,z)$ into (a) its immediate benefit, called the *reputational dividend (of self-esteem)*, and (b) its continuation value. First we develop the value of a firm with self-esteem z:

$$V(x,z) = \underbrace{rdt\,(x - \eta c - k)}_{\text{Today's Payoff}} + (1 - rdt)\underbrace{V(x+dx, z+dz)}_{\text{No Breakthrough}} + z\mu dt\underbrace{(V(1,1) - V(x+dx, z+dz))}_{\text{Breakthrough}}$$

[3]If the firm does not exit when its value become negative, the market interprets this as a mistake and updates based on market learning and $\widetilde{\eta} = \widetilde{\eta}(x^e)$, where x^e is the exit point.

[4]To prove monotonicity suppose the firm with the higher reputation (self-esteem) mimics the firm with the lower reputation (self-esteem). To prove convexity, suppose $\overline{z}$ is a convex combination of z' and z'', and let firms z' and z'' mimic $\overline{z}$.

where the increments dx and dz are conditional on no breakthroughs, as given by (2.2) and (3.1), and are thus deterministic. If we instead start with self-esteem z' we can write a similar expression. Adding and subtracting $V(x+dx, z+dz)$ then yields,

$$V\left(x, z'\right) = rdt\left(x - \eta c - k\right) + \left(1 - rdt\right) V\left(x + dx, z' + dz'\right) + z'\mu dt\left(V\left(1,1\right) - V\left(x + dx, z + dz\right)\right) + z'\mu dt\left(V\left(x + dx, z + dz\right) - V\left(x + dx, z' + dz'\right)\right).$$

Having higher self-esteem does not affect revenue today, but alters the evolution of future reputation and therefore future revenue. In particular, from these two equations, the value of the increment $z' - z$ is given by[5]

$$\begin{aligned} V\left(x, z'\right) - V\left(x, z\right) &= \left(z' - z\right) \underbrace{\mu\left(V\left(1,1\right) - V\left(x, z\right)\right)}_{\text{Reputational Dividend}} dt \\ &+ \left(1 - \left(r + z\mu\right) dt\right) \underbrace{\left(V\left(x + dx, z' + dz'\right) - V\left(x + dx, z + dz\right)\right)}_{\text{Continuation Value}}. \end{aligned}$$

The first term is the reputational dividend: the immediate benefit of incremental self-esteem. The second term is the continuation value depreciated by interest rate r and rate of breakthroughs μz, that render incremental self-esteem obsolete. Integrating, and dividing by $(z_0' - z_0)$:

$$\begin{aligned} \frac{V\left(x_0, z_0'\right) - V\left(x_0, z_0\right)}{z_0' - z_0} &= \int_0^T \exp\left(-\int_0^t r + \mu z_s ds\right) \frac{z_t' - z_t}{z_0' - z_0} \mu\left(V\left(1,1\right) - V\left(x_t, z_t\right)\right) dt \\ &+ \exp\left(-\int_0^T r + \mu z_s ds\right) \frac{V\left(x_T, z_T'\right) - V\left(x_T, z_T\right)}{z_0' - z_0} \end{aligned}$$

where $T = T(x_0, z_0)$ is the first time that the trajectory (x_t, z_t) hits the exit region. The second term vanishes in the limit because optimal exits implies $V(x_T, z) = 0$ for $z \leq z_T$, and smooth-pasting of the value function then implies $V(x_T, z_T') - V(x_T, z_T) \in o(z_T' - z_T)$.

By the updating equation of self-esteem (3.1), the increment decreases at rate $d\ln(z_t' - z_t)/dt = 1 - (\lambda + \mu(1 - 2z_t))$. It follows that

$$\partial_z V\left(x_0, z_0\right) = \int_0^T e^{-\int_0^t r + \lambda + \mu(1 - z_s) ds} \mu\left(V\left(1,1\right) - V\left(x_t, z_t\right)\right) dt \tag{3.2}$$

Setting $z_t = x_t$, we conclude:

[5] When we cancel current cashflows we assume investment is identical on the two trajectories, $\eta(x, z) = \eta(x, z')$. This approximation is justified by the consideration that the amount of time at which $\eta(x_t, z_t) \neq \eta(x_t, z_t')$ is of order $z_0' - z_0$, and that the joint effect of the approximation at these times is of order $c - \partial_z V(x, z)$, which in equilibrium converges to 0 as $z_0' - z_0$ becomes small.

Proposition 1. *In equilibrium the marginal value of self-esteem is given by*

$$\Gamma(x_0) := \partial_z V(x_0, x_0) = \int_0^T e^{-\int_0^t r+\lambda+\mu(1-x_s)ds} D(x_t)dt \tag{3.3}$$

where $D(x) := \mu (V(1,1) - V(x,x))$ *is the equilibrium reputational dividend.*

Proposition 1 expresses the marginal value of self-esteem as the discounted sum of future reputational dividends. Since quality is persistent, investment does not pay off immediately but rather through a stream of dividends whose value depends on the future evolution of the firm's reputation. In equation (3.3), the dividends are discounted by the interest rate r, and the rate at which incremental self-esteem vanishes. The latter consists of two terms: if there is no breakthrough the gap closes at rate $-(\lambda + \mu(1-2z))\,dt$; if there is a breakthrough the gap completely closes, leading to a $-\mu z dt$ term.[6]

3.3 Shirk-Work-Shirk Equilibrium

In this Section we show that, when λ and c are sufficiently low, the firm shirks when its reputation is either low or high, and works for intermediate reputations. The intuition is as follows. For low reputations $x \approx x^e$, the firm is almost certain to go out of business soon, undermining its incentives to further invest into its product. In equilibrium the market anticipates that the firm gives up and the adverse beliefs accelerate the firm's demise. For intermediate reputations, incentives are bounded from below and the firm invests when the investment costs are sufficiently low. Finally, for high reputations $x \approx 1$, the firm cannot keep investing in equilibrium. If it did, its reputation would stay close to 1, because the market learns little from the absence of breakthroughs when it is sufficiently convinced of the firm's quality. This dynamic undermines the firm's incentive to actually invest.

While these effects are robust, their occurrence in the good news model relies on two restrictions on the model parameters. First, we assume that a firm's reputation declines in the absence of a breakthrough, even if the firm is believed to be investing. Using equation (2.2), this means that $x^e > \lambda/\mu$. Formally, we assume that

$$r(\lambda/\mu - k) + \lambda(1-k) < 0 \tag{A-λ}$$

which is satisfied if λ is sufficiently low, limiting the role of market beliefs. Equation (A-λ) says that the (negative) profits earned at $x = \lambda/\mu$ are lower than the option value of receiving a breakthrough. Second, we assume c is sufficiently low so that firms with intermediate reputations choose to invest.

[6] The astute reader will notice that we ignored the possibility that z' may have a breakthrough but not z, increasing $(dz' - dz)/(z' - z)$ at rate $\mu(1-z)dt$. However this term is captured by the reputational dividend.

Proposition 2. *Suppose (A-λ) holds and c is sufficiently low. In any equilibrium:*
(a) The optimal investment is characterized by cutoffs $\lambda/\mu < x^e < \underline{x} < \overline{x} < 1$ *such that the firm*

$$\begin{array}{ll} \textit{exits} & \textit{if } x \in [0, x^e] \\ \textit{shirks} & \textit{if } x \in [x^e, \underline{x}] \\ \textit{works} & \textit{if } x \in [\underline{x}, \overline{x}] \\ \textit{shirks} & \textit{if } x \in [\overline{x}, 1] \end{array}$$

(b) The optimal exit threshold x^e *satisfies*

$$(x^e - k) + x^e \mu V(1,1) = 0. \tag{3.4}$$

Proof. See Appendix A.1. □

The arguments leading up to the Proposition already show that every equilibrium must have (1) some shirking at the bottom, (2) some working in the middle, and (3) some shirking at the top. The proof, which is based on Proposition 1, strengthens these arguments to show that any equilibrium must be characterized by intervals.

Condition (3.4) follows from the indifference of a firm at reputation x^e to exit or stay. At this point, the firm's negative instantaneous profits $x^e - k$ are balanced by the option value $x^e \mu V(1,1)$ of staying in the market.

While our analysis focuses on learning through perfectly revealing good news signals, the spirit of these results extend to more general learning structures. Following Board and Meyer-ter-Vehn (2010), suppose the market learns through a signal Z_t, that is generated by a Brownian motion and finite number of Poisson processes. The Brownian component is given by $dU_{B,t} = \mu_B \theta_t dt + dW_t$, where W_t is a Wiener process. A Poisson process has a signal arrive at rate μ_θ. Such a signal is *good news* if the net arrival rate $\mu := \mu_H - \mu_L$ is positive, *perfect good news* if $\mu_L = 0$, *bad news* if $\mu < 0$, and *perfect bad news* if $\mu_H = 0$.

As shown in Appendix A.2, we can generalize equation (3.2) to express the marginal value of self-esteem as

$$\partial_z V(x_0, z_0) = \mathbb{E}\left[\int_0^T e^{-rt} \frac{\partial z(z_0, t)}{\partial z_0} D(x_t, z_t)\, dt\right] \tag{3.5}$$

where the reputational dividend is

$$D(x_t, z_t) = \left(\mathbb{E}_H\left[V(x_{t+dt}, z_{t+dt})\right] - \mathbb{E}_L\left[V(x_{t+dt}, z_{t+dt})\right]\right)/dt.$$

and $\mathbb{E}_\theta\left[V(x_{t+dt}, z_{t+dt})\right]$ conditions the evolution of x and z on θ. In the Appendix, we explicitly calculate the term $\partial z(z_0, t)/\partial z_0$ as a function of the learning process.

Using (3.5) one can extend our results to general learning structures. For low reputations

$x \approx x^e$, investment incentives disappear and the firm shirks if exit becomes imminent, $T \to 0$ a.s., as $x_0 \to x^e$. This condition holds under our good news specification or if there is any Brownian motion. For high reputations $x \approx 1$, investment incentives disappear and there will be some shirking at the top, as long as there is no perfect bad news signal. Under these circumstances, if the firm was thought to be working at $x \approx 1$, then x_t stays close to 1 with probability one for all t, yielding dividends $D(x_t, x_t) \approx 0$ forever.

3.4 Entry

We have characterized the firm's optimal investment and exit decisions. To complete the firm's lifecycle, suppose there is measure dt of potential entrants into the market over $[t, t+dt]$. Potential entrants have a public reputation x_0, and therefore only enter if $x_0 > x^e$. Once a firm enters the market he plays the game we have studied above, choosing his investment and exit decisions.[7]

As an application, consider the labor market for academics. When an agent enters the industry her type is unknown to her and the market, but her GPA is common knowledge and determines her initial reputation x_0. Agents with low GPAs choose not to enter the industry. Agents with high GPAs enter, invest in their skills over time and are free to exit at any point; the market then learns about their skills via their breakthroughs (e.g. publications). Proposition 2 predicts that an agent will stop investing in her skills shortly before she exits or after she has had a breakthrough.

4 Known Quality

We now turn to the case were the firm knows its quality, θ_t. In a Markovian equilibrium we can write the firm's value as function of its reputation and quality, $V_\theta(x)$. A *Markov-Perfect-Equilibrium* $\langle \eta, \widetilde{\eta} \rangle$ then consists of an investment function $\eta : [0,1] \times \{L, H\} \to [0,1]$, a exit region $R \subseteq [0,1] \times \{L, H\}$, and market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that: (1) Investment maximizes firm value $V_\theta(x)$; (2) The firm exits when its value is negative $V_\theta(x) \leq 0$; and (3) Market beliefs are correct, $\widetilde{\eta}(x) = (1-x)\eta_L(x) + x\eta_H(x)$.

It is straightforward to show that the value function $V_\theta(x)$ is increasing in reputation x, since this leads directly to higher revenue. In addition, $V_\theta(x)$ is increasing in quality θ, since this will ultimately lead to higher reputation.[8]

[7] This analysis implicitly assumes that the firm can only invest if they have paid the operating cost k to be in the market. For example, an academic can only invest in her skills if she no other job, where k is the opportunity cost.

[8] To prove, suppose the firm with the higher reputation (self-esteem) mimics the firm with the lower reputation (self-esteem).

4.1 Optimal Investment and Exit Decisions

The benefit of investment over $[t, t+dt]$ is the probability a technology shock hits, λdt, times the difference in value functions, $\Delta(x) := V_H(x) - V_L(x)$, which we call the *value of quality.* It follows that:

Lemma 2. *Optimal investment $\eta(x)$ is independent of quality θ and given by*

$$\eta(x) = \begin{cases} 1 & \text{if } \lambda\Delta(x) > c \\ 0 & \text{if } \lambda\Delta(x) < c \end{cases}$$

where $\Delta(x) := V_H(x) - V_L(x)$ is the value of quality.

The firm exits the industry when its value is negative. Since reputation declines continuously (or jumps up), the firm exits when its value falls to zero $V_\theta(x) = 0$. As quality is a valuable asset, the low-quality firm exits when the high-quality firm's value is strictly positive. This exit process of low-quality firms prevents a further decline in reputation, so a high quality firm never exits, as in Bar-Isaac (2001). As a result:

Lemma 3. *Define x^e by $V_L(x^e) = 0$.*
(a) The high-quality firm never exits, while
(b) The low-quality firm exits if $x_t \leq x^e$ and, if so, exits so that $x_{t+dt} = x^e$. At the cutoff x^e, the rate of exit is

$$q = \left[\mu - \frac{\lambda(\eta(x^e) - x^e)}{x^e(1-x^e)}\right] dt. \tag{4.1}$$

As a result, $x_t \in [x^e, 1]$ for $t > 0$.

Proof. Firm L quits to keep $x_{t+dt} = x^e$ when $x_t \leq x^e$. Hence $V_L(x) = 0$ and $V_H(x) > 0$ for $x \leq x^e$, and the high-quality firm never exits. When $x_t = x^e$, the low firm's quit probability can be calculated using Bayes rule:

$$q = 1 - \frac{1-x^e}{x^e} \times \frac{x^e + dx}{1-(x^e+dx)} \approx -\frac{dx}{x^e(1-x^e)}$$

so equation (2.2) yields (4.1). □

4.2 Value of Quality

In order to characterise investment incentives we need to evaluate the value of quality $\Delta(x) = V_H(x) - V_L(x)$. Following Board and Meyer-ter-Vehn (2010, Theorem 1), we develop the value functions into current profits and continuation values. Current profits cancel because both current revenue and costs depend on reputation but not on quality, yielding

$$\Delta(x) = (1 - rdt)(1 - \lambda dt)\mathbb{E}\left[V_H(x + d_H x) - V_L(x + d_L x)\right].$$

Adding and subtracting $V_H(x+d_L x)$, we can express the value of quality in terms of a reputational dividend and continuation value:

$$\Delta(x) = (1 - rdt - \lambda dt)\left(\hat{D}(x) + \mathbb{E}\left[\Delta(x + d_L x)\right]\right).$$

where

$$\hat{D}(x) := [V_H(x + d_H x) - V_H(x + d_L x)]/dt \tag{4.2}$$

is the *reputational dividend for known quality.* Evaluating (4.2) and integrating up, we express the value of quality as the discounted sum of future reputational dividends:

Proposition 3. *In equilibrium, the marginal value of quality is given by*

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\hat{D}(x_t)dt. \tag{4.3}$$

where $\hat{D}(x) = \mu(V_H(1) - V_H(x))$.

This representation of investment incentives with known quality differs from the unknown quality case in (3.3) in two respects. The substantial difference is that in (4.3) we integrate over $[0, \infty]$, whereas whereas in (3.3) the integral is over $[0, T]$. With known quality, a firm never exits with certainty because exit by low quality firms bounds reputational below at x^e and leaves even low quality firms indifferent about exiting. In contrast, with unknown quality the firm strictly prefers to exit after the exit time T. This difference can be reflected in the algebra by rewriting (4.3) as as $\Delta(x_0) = \int_0^{T^e} e^{-(r+\lambda)t}\hat{D}(x_t)dt + e^{-(r+\lambda)T^e}\Delta\left(x^e\right)$, where T^e is the time x_t hits x^e. In the known quality case the continuation value of the discrete quality increment $e^{-(r+\lambda)T^e}\Delta\left(x^e\right)$ is strictly positive, whereas in the unknown quality case, the continuation value of incremental self-esteem $V\left(x_T, z_T'\right) - V\left(x_T, z_T\right)$ is of second order because of smooth pasting.[9]

A second, more expositional, difference is that we choose simpler, if less canonical, functional forms in (4.3) than we did in (3.3). Specifically, we choose a constant discount rate $r + \lambda$ here. In Appendix A.3 we show that this is exactly compensated by replacing the reputational dividend $D\left(x\right) = V\left(1, 1\right) - V\left(x, x\right)$ with $\hat{D}(x) = \mu(V_H(1) - V_H(x))$.

[9] At the exit threshold, firm z_T receives zero profits when accounting for the option value of staying in the market. Hence firm z_T' receives small profits for a small period of time, which is of second order.

4.3 Work-Shirk Equilibrium

The reputational dividend equals the value of increasing the firm's reputation from x to 1, times the probability of a breakthrough. This dividend is decreasing in x, so equation (4.3) implies that the value of quality $\Delta(x_0)$ is decreasing in x_0. As a result, the firm's investment incentives are decreasing in its reputation:

Proposition 4. *Suppose (A-λ) holds and c is sufficiently low. In any equilibrium:*
(a) The optimal investment is characterized by cutoffs $\lambda/\mu < x^e < x^* < 1$ such that the firm[10]

$$\begin{array}{ll} \textit{exits} & \textit{if } x \in [0, x^e] \textit{ and quality is low} \\ \textit{works} & \textit{if } x \in [x^e, x^*] \\ \textit{shirks} & \textit{if } x \in [x^*, 1] \end{array}$$

(b) The optimal exit threshold x^e is characterized by

$$(x^e - c - k) + \lambda V_H(x^e) = 0 \tag{4.4}$$

at which point the low-quality firm exits at rate $q = \left[\mu - \frac{\lambda}{x^e}\right]$.

Proof. Assumption (A-λ) ensures that $x^e \geq \lambda/\mu$ and therefore x_t, as determined by (2.2), is decreasing in t. To see this suppose, by contradiction, that $x^e < \lambda/\mu$. Since the dynamics are stationary at $\hat{x} := \lambda/\mu$,

$$V_L(\hat{x}) = \frac{1}{r+\lambda}[r(\hat{x} - c\eta - k) + \lambda V_H(\hat{x})].$$

Observing that $V_H(\hat{x}) \leq 1 - k$, assumption (A-λ) implies that $V_L(\hat{x}) < 0$ as required.

The dividend $\hat{D}(x)$ is strictly decreasing in x, and x_t is decreasing in x_0, so $\Delta(x)$ is strictly decreasing in x. It remains to be shown that none of the intervals is trivial, i.e. that all the inequalities $\lambda/\mu < x^e < x^* < 1$ are strict. Assumption (A-λ) implies that the exit region $[0, x^e]$ exists. The cost is sufficiently small, so the work-region $[x^e, x^*]$ exists. Finally, the upper shirk-region $[x^*, 1]$ exists because $\Delta(x^*) = 0$ if $x^* = 1$.

Condition (4.4) is the indifference of a working firm with low quality to stay or exit when its reputation equals x^e. At this point, the negative instantaneous profits $x^e - c - k$ are balanced by the option value $\lambda V_H(x^e)$ of staying in the market. The exit rate follows from (4.1). □

Proposition 4 shows that, unlike the unknown quality case, the firm invests even as its reputation drops very low, and exit becomes imminent. This is because a technology shock increases the firm's product quality by a discrete amount and averts exit. Consider a firm who is just about

[10] Recall exit is probabilistic, as established in Lemma 3.

to exit: with known quality investment pays off if there is a technology shock, which occurs with probability λdt; with unknown quality investment pays off if the firm's quality increases and it achieves a breakthrough, which happens with probability $\lambda dt \times \mu dt$.

In our model, a firm's knowledge of its own quality is not directly relevant for its investment decision, but does enable it to make better exit decisions. This change in exit behavior, in turn, affects the firm's investment incentives. While our analysis does not lend itself to compare equilibria across the two cases, our results suggest that, at low reputations x an informed firm has higher incentives to invest than an ignorant firm because it can condition its exit decision on the outcome of the investment. On the other hand, at higher levels of reputation where this value of knowing one's quality is lower, one may suspect that the ignorant firm has higher incentives to invest in order to stay away from the low reputation region.

Finally, Proposition 4 can be extended to more general learning structures using the general expression for reputational dividends (4.2). For low reputations $x \approx x^e$, the dividend and value of quality are strictly positive, so the firm invests if the cost is sufficiently low. For high reputations $x \approx 1$, investment incentives disappear as long as there is no perfect bad news signal. This will lead to a shirk region at the top.

4.4 Entry

We can now complete the firm's lifecycle by introducing entry into the model.[11] Suppose potential entrants are endowed with quality $\theta_0 \in \{L, H\}$. Then all high types will enter the market, while low types enter until the reputation of an entrant falls to x^e. If there is a large enough pool of low-quality entrants, then firms enter with reputation $x_0 = x^e$. At this point, all entrants invest in their quality, and low-quality firms immediately start exiting the market.

As an application, consider the lifecycle of a restaurant. There are some potential entrants with a good concept, and many others who have no great idea, but are hopeful. Once a restaurant enters the market, it chooses to invest in food, decor and service. A high quality restaurant may then achieve a breakthrough in term of a good review, while a low quality restaurant may exit. The model predicts that many new restaurants will exit rapidly, but will invest even when exit is imminent. This is consistent with evidence that 25% of new restaurants close each year, and that these restaurants work very hard to stay afloat (Parsa et al. (2005)).

5 Conclusion

This paper has studied the lifecycle of a firm that sells a product of unknown quality. The firm chooses whether to enter the industry and, after entering, can invest in its quality and ultimately exit. We showed that the reputational dynamics depend on whether the firm knows its quality.

[11] We assume entry is observable. This may not be the case: see Tadelis (1999) and Mailath and Samuelson (2001).

When the firm is uninformed, it exits when its reputation falls too low, and shirks when exit is imminent. When the firm is informed, only low-quality firms exit and such firms work no matter how low their reputation.

We have studied the lifecycle of a single firm, ignoring firm interaction by assuming that industry demand is perfectly elastic. As an extension, one could embed the model into a competitive industry, assuming consumers have heterogenous demand for quality. Since our model allows for entry and exit, the steady state would exhibit turnover related to the speed of learning μ and the rate of technological change λ.[12]

[12] There are a couple of papers along these lines. Vial (2008) introduces perfect competition into Mailath and Samuelson (2001) but has no exit. Atkeson, Hellwig, and Ordonez (2010) studies entry and exit in a monopolistically competitive industry, but has no investment.

A Omitted Material

A.1 Proof of Proposition 2

To prove Proposition 2 we first establish a lower bound on the marginal value of reputation.

Lemma 4. *Fix model parameters λ, μ, k, r. There exists $\delta > 0$ such that for all c, all equilibria η,*

$$\partial_x V(x,x) \geq \delta \quad \textit{for all } x \in [k,1].$$

A fortiori, the marginal reputational dividend has the upper bound:

$$D'(x) = -\partial_x V(x,x) - \partial_z V(x,x) \leq -\delta \quad \textit{for all } x \in [k,1].$$

Proof. In equilibrium we have $x^e < k$ because the firm exits only when cash flows are negative. Hence any firm with a reputation exceeding k stays in the market for a period of time that is bounded below.

Consider two firms with different reputations but the same self-esteem: (x_0, x_0) and (x_0', x_0), where $x_0' > x_0$. If the high firm mimics the investment strategy of the low firm, its reputation x_t', and thus its profits $x_t' - c\eta_t - k$, exceed those of the low firm at all times t. While the reputational increment $x_t' - x_t$ may decrease over time, it does so at a finite rate by (2.2). Thus the high firm can guarantee itself an incremental value $(V_{\text{mimic}} - V(x_0, x_0))$ that is bounded below by a linear function of $(x_0' - x_0)$. In equilibrium the high firm achieves a weakly higher value $V(x_0', x_0) \geq V_{\text{mimic}}$, finishing the proof. □

Proof of Proposition 2. Fix model parameters λ, μ, k, r. Assumption (A-λ) ensures that $x^e \geq \lambda/\mu$ and therefore x_t, as determined by (2.2), is decreasing in t. To see this suppose, by contradiction, that $x^e < \lambda/\mu$. Since the dynamics are stationary at $\hat{x} := \lambda/\mu$,

$$V(\hat{x}, \hat{x}) = \frac{1}{r + \hat{x}\mu}\left[r(\hat{x} - c\eta - k) + \hat{x}\mu V(1,1)\right]$$

Using $\hat{x} = \lambda/\mu$ and $V(1,1) \leq 1 - k$, assumption (A-λ) implies that $V(\hat{x}, \hat{x}) < 0$ as required.

We now show that there exists $\varepsilon > 0$, and $c > 0$, such that for any equilibrium η

1. $\lambda\Gamma$ is strictly increasing on $[x^e, x^e + \varepsilon]$ with $\lambda\Gamma(x^e) < c$.

2. $\lambda\Gamma$ is greater than c on $[x^e + \varepsilon, 1 - \varepsilon]$.

3. $\lambda\Gamma(x)$ crosses c once and from above on $[1 - \varepsilon, 1]$ with $\lambda\Gamma(1) < c$.

Part (1). Differentiating (3.3), the marginal value of self-esteem obeys the following differential equation:

$$\frac{d}{dt}\Gamma(x_t) = (r + \lambda + \mu(1-x))\,\Gamma(x_t) - D(x_t) \tag{A.1}$$

Since $\Gamma(x^e) = 0$, $d\Gamma(x_t)/dt < 0$ for $x_t \in [x^e, x^e + \varepsilon]$. Since $dx_t/dt < 0$, $\Gamma(x)$ is increasing in x.

Part (2). The dividend $D(x)$ is bounded below for $x \in [x^e, 1-\varepsilon]$. The time to exit T is bounded below for $x_0 \in [x^e + \varepsilon, 1-\varepsilon]$. Therefore, we can choose c low enough such that $c < \lambda\Gamma(x)$ for all $x \in [x^e + \varepsilon, 1-\varepsilon]$

Part (3). By part (2) we know that $\lambda\Gamma(1-\varepsilon) > c$. We also know that $\lambda\Gamma(1) \leq c$; otherwise the firm would invest at $x = 1$ which would imply that $dx = 0$ and $\Gamma(1) = 0$, yielding a contradiction.

Thus, $\lambda\Gamma(x)$ crosses c at least once from above. Suppose, by contradiction, that $\lambda\Gamma(x)$ crosses c at more than one point. Then there exist $x_1, x_2 \in [1-\varepsilon, 1]$ with $x_1 < x_2$, such that Γ has a local minimum at x_1 and a local maximum at x_2 with $\Gamma'(x_1) = \Gamma'(x_2) = 0$ and $\Gamma(x_1) \leq \Gamma(x_2)$.[13] Equation (A.1) implies that

$$\Gamma(x) = \frac{\mu D(x)}{r + \lambda + \mu(1-x)}$$

for $x = x_1, x_2$. We will now show that the RHS is strictly decreasing on $[1-\varepsilon, 1]$; this contradicts $\Gamma(x_1) \leq \Gamma(x_2)$ and finishes the proof. Differentiating the logarithm of the RHS yields

$$\frac{D'(x)}{D(x)} - \frac{-\mu}{r + \lambda + \mu(1-x)}.$$

The second term is bounded, while the first term is unboundedly negative for $x \approx 1$ because $D(x) \approx 0$ and $D'(x) \leq -\delta$ by Lemma 4. Hence the derivative of the RHS is negative, as required.

Finally, part (b) of the Proposition is explained in the text. □

A.2 General Market Learning: Derivation of (3.5)

For any general payoff function $f(x)$, the value function is given by

$$V(x, z) = rdtf(x) + (1 - rdt)\,\mathbb{E}_z[V(x_{dt}, z_{dt})] + O\left(dt^2\right)$$

where (x_{dt}, z_{dt}) are the stochastic values of reputation and self-esteem after dt, and

$$\mathbb{E}_z[V(x_{dt}, z_{dt})] := z\mathbb{E}_H[V(x_{dt}, z_{dt})] + (1-z)\mathbb{E}_L[V(x_{dt}, z_{dt})].$$

[13]For this to be true, it is sufficient that Γ is differentiable in the interior of work-regions and shirk-regions but it does not matter that Γ has kinks on the cutoffs.

The marginal value of self-esteem is then

$$\begin{aligned} V\left(x, z'\right) - V(x,z) &= (1 - rdt)\left(\mathbb{E}_{z'}\left[V\left(x_{dt}, z'_{dt}\right)\right] - \mathbb{E}_z\left[V\left(x_{dt}, z_{dt}\right)\right]\right) \\ &= \left(z' - z\right)\underbrace{\left(\mathbb{E}_H\left[V\left(x_{dt}, z_{dt}\right)\right] - \mathbb{E}_L\left[V\left(x_{dt}, z_{dt}\right)\right]\right)}_{\text{Dividend } D(x,z)dt} + (1 - rdt)\,\mathbb{E}_z\left[V\left(x_{dt}, z'_{dt}\right) - V\left(x_{dt}, z_{dt}\right)\right] \end{aligned}$$

Define the reputational dividend of self-esteem by

$$D(x,z) = \left(\mathbb{E}_H\left[V\left(x_{dt}, z_{dt}\right)\right] - \mathbb{E}_L\left[V\left(x_{dt}, z_{dt}\right)\right]\right)/dt$$

The rental value of marginal self-esteem then equals the dividend plus the appreciation:

$$rdt\left(V\left(x, z'\right) - V(x,z)\right) = \left(z' - z\right)D(x,z)\,dt + \mathbb{E}_z\left[d\left(V\left(x, z'\right) - V(x,z)\right)\right].$$

If we integrate and divide by $(z'_0 - z_0)$

$$\frac{V\left(x_0, z'_0\right) - V\left(x_0, z_0\right)}{z'_0 - z_0} = \mathbb{E}\left[\int_0^T e^{-rt}\frac{z'_t - z_t}{z'_0 - z_0}D\left(x_t, z_t\right)dt\right].$$

In the limit, this yields

$$\partial_z V\left(x_0, z_0\right) = \mathbb{E}\left[\int_0^T e^{-rt}\frac{\partial z\left(z_0, t\right)}{\partial z_0}D\left(x_t, z_t\right)dt\right]$$

as in equation (3.5).

We can further develop the $\partial z(z_0, t)/\partial z_0$ term for Brownian motion and Poisson shocks $y \in Y$.

$$\frac{z'_{dt} - z_{dt}}{z' - z} - 1 = \begin{cases} -\lambda dt & \text{equilibrium beliefs} \\ -(1 - 2z)\sum_y \mu_y dt & \text{absence of Poisson shocks} \\ (1 - 2z)\,\mu_B dW & \text{Brownian motion} \\ \mu_y \frac{(1-2z)\left(z\mu_{y,H} + (1-z)\mu_{y,L}\right) - \mu_y z(1-z)}{\left(z\mu_{y,H} + (1-z)\mu_{y,L}\right)^2} & \text{at Poisson shock } y \end{cases}$$

Taking the limit

$$\begin{aligned} \partial z\left(z_0, t\right)/\partial z_0 = &\exp\left(\int_0^t -\lambda + \sum_y \mu_y\left(1 - 2z_s\right)ds\right) && \text{(Drift)} \\ &\times \exp\left(-\int_0^t\left(\mu_B^2\left(1 - 2z_s\right)^2/2\right)ds + \int_0^t\left(1 - 2z_s\right)\mu_B dW_s\right) && \text{(Brownian)} \\ &\times \prod_{y\in Y, t_y \in P_y}\left(1 + \mu_y\frac{(1 - 2z)\left(z\mu_{y,H} + (1 - z)\,\mu_{y,L}\right) - \mu_y z(1 - z)}{\left(z\mu_{y,H} + (1 - z)\,\mu_{y,L}\right)^2}\right) && \text{(Poisson)} \end{aligned}$$

The first term in the second line is the Ito-term accounting for the fact that $\exp(f(W)\,dW) = 1 + f(W)\,dW + f(W)^2\,dt/2 + o(dt)$. In the third line, $P_y \subseteq [0,t]$ is the finite number of times that Poisson shock y hits.

A.3 The Value of Quality: An Alternative Expression

We can now apply Proposition 1 to provide an alternative derivation of the firm's investment incentives with known quality. Define

$$V(x,z) := zV_H(x) + (1-z)\,V_L(x).$$

Note that $V(x,z)$ is linear in z, whereas it is convex in the unknown quality case. Repeating the analysis in Section 3 yields:

$$\Delta(x_0) = \int_0^\infty e^{-\int_0^t (r+\lambda+\mu(1-x_s))ds}\,D(x_t)\,dt \tag{A.2}$$

where $D(x) = \mu(V(1,1) - V(x,x))$ is the reputational dividend.

To see that (A.2) and (4.3) are the same, we differentiate them and obtain:

$$(r+\lambda+\mu(1-x_t))\Delta(x_t) = \mu[V(1,1) - V(x,x)] + \frac{d}{dt}[\Delta(x_t)]$$
$$(r+\lambda)\Delta(x_t) = \mu[V_H(1) - V_H(x)] + \frac{d}{dt}[\Delta(x_t)]$$

which coincide since $\mu(1-x_t)\Delta(x_t) = \mu[V_H(x) - V(x,x)]$.

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