SINGLE MARKET NONPARAMETRIC IDENTIFICATION OF MULTI-ATTRIBUTE HEDONIC EQUILIBRIUM MODELS

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Abstract. This paper derives conditions under which preferences and technology are nonparametrically identified in hedonic equilibrium models, where products are differentiated along more than one dimension and agents are characterized by several dimensions of unobserved heterogeneity. With products differentiated along a quality index and agents characterized by scalar unobserved heterogeneity, single crossing conditions on preferences and technology provide identifying restrictions. We develop similar shape restrictions in the multi-attribute case. These shape restrictions, which are based on optimal transport theory and generalized convexity, allow us to identify preferences for goods differentiated along multiple dimensions, from the observation of a single market. We thereby extend identification results in Matzkin (2003) and Heckman, Matzkin, and Nesheim (2010) to accommodate multiple dimensions of unobserved heterogeneity. One of our results is a proof of absolute continuity of the distribution of endogenously traded qualities, which is of independent interest.

Keywords: Hedonic equilibrium, nonparametric identification, multidimensional unobserved heterogeneity, cyclical monotonicity, optimal transport.

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1. Introduction

Recent years have seen renewed interest in hedonic models, particularly their identification and estimation. Hedonic models were initially introduced to analyze price responses to quality parameters of differentiated goods. They allow to answer such questions as: (1) Given the fact that the amenities offered by cars constantly evolve over time, how can one construct a price index measuring the evolution of car prices and deflating improvements in amenities? (2) How can one explain price differentiation in wine, art, luxury goods, professional sports wages? (3) What does the correlation between the wage differentials and the level of risk associated to a given job reveal about individuals' valuation for their own life? (4) How can one analyze individual preferences for environmental features?

These questions gave rise to a vast literature, which aims at modeling implicit markets for quality differentiated products. There are two layers to this literature. The first layer is the literature on "hedonic regressions," i.e., regressions of prices against good attributes, with corrections for the standard endogeneity issue that consumers with greater taste for quality will consume more of it. The second layer has broader scope: the literature on "hedonic equilibrium models" incorporates a supply side with differentiated productivity over various quality parameters and studies the resulting equilibrium. This approach dates back at least as far as Tinbergen (1956); and Rosen (1974) provides a famous two-step procedure to estimate general hedonic models and thereby analyze general equilibrium effects of changes in buyer-seller compositions, preferences and technology on qualities traded at equilibrium and their price. Following the influential criticism of Rosen's strategy in Brown and Rosen (1982) and the invalidity of supply side observable characteristics as instruments in structural demand estimation as discussed in Epple (1987) and Bartik (1987), it was generally believed that identification in hedonic equilibrium models required data from multiple markets, as in Epple (1987), Kahn and Lang (1988) and, more recently, Bajari and Benkard (2005) and Bishop and Timmins (2011).

Ekeland, Heckman, and Nesheim (2004) show, however, that hedonic equilibrium models are in fact identified from single market data, under separability assumptions, as in Ekeland, Heckman, and Nesheim (2004), or shape restrictions, as in Heckman, Matzkin, and

Nesheim (2010). The common underlying framework is that of a perfectly competitive market with heterogeneous buyers and sellers and traded product quality bundles and prices that arise endogeneously in equilibrium. Preferences are quasi-linear in price and under mild semicontinuity assumptions, Ekeland (2010) and Chiappori, McCann, and Nesheim (2010) show that equilibria exist, in the form of a joint distribution of product and consumer types (who consumes what), a joint distribution of product and producer types (who produces what) and a price schedule such that markets clear for each endogenously traded product type. Equilibrium existence results are valid in hedonic markets for multi-attribute products, but existing single market identification strategies restrict attention to a single quality dimension and scalar unobserved heterogeneity in consumer preferences and production technology. Ekeland, Heckman, and Nesheim (2004) require marginal utility (resp. marginal product) to be additively separable in unobserved consumer (resp. producer) characteristic. Heckman, Matzkin, and Nesheim (2010) show that demand is nonparametrically identified under a single crossing condition and that various additional shape restrictions allow identification of preferences without additive separability.

The objective of these papers, which is also ours, is to recover structural preference and technology parameters from the observation of who trades what and at what price. In the identification exercise, price is assumed known, as are the distributions characterizing who produces and consumes which good. Since price is observed and the environment is perfectly competitive, identification of preferences and identification of technology can be treated independently and symmetrically. Take the consumer problem, for instance. Under a single crossing condition on the utility function (also known as Spence-Mirlees in the mechanism design literature), the first order condition of the consumer problem yields an increasing demand function, i.e., quality demanded by the consumer as an increasing function of her unobserved type, interpreted as unobserved taste for quality. Assortative matching guarantees uniqueness of demand, as the unique increasing function that maps the distribution of unobserved taste for quality, which is specified a priori, and the distribution of qualities, which is observed. Hence demand is identified as a quantile function, as in Matzkin (2003). Identification, therefore, is driven by a shape restriction on the utility function.

We show that similar shape restrictions on the utility function also yield identification conditions in the case of non scalar characteristics and unobserved heterogeneity. In the special case where marginal utility is additively separable in the unobservable taste vector, concavity yields nonparametric identification of the utility function, according to the celebrated Brenier Theorem of optimal transport theory (Theorem 8 in Appendix 6.1). More generally, a generalization of the single crossing condition known as the *Twist Condition* in optimal transport theory and a generalized convexity shape restriction yield identification of the utility function in hedonic equilibrium models with multiple quality dimensions. The distribution of unobserved heterogeneity is fully specified a priori and cannot be identified from single market data without additional separability conditions or exclusion restrictions. An important result we prove on hedonic equilibrium is a set of mild conditions on the primitives under which the endogenous distribution of qualities traded at equilibrium is absolutely continuous. The proof of absolute continuity of the distribution of qualities traded at equilibrium is based on an argument from Figalli and Juillet (2008), also applied in Kim and Pass (2014).

Related work. Beyond Ekeland, Heckman, and Nesheim (2004), Heckman, Matzkin, and Nesheim (2010) and other contributions cited so far, this paper is closely related to the growing literature on identification and estimation of nonlinear econometric models with multivariate unobserved heterogeneity on the one hand, and to the empirical literature on matching models where agents match along multiple dimensions on the other hand. The quantile identification strategy of Matzkin (2003) was recently extended to non scalar unobserved heterogeneity using the Rosenblatt (1952)-Knothe (1957) sequential multivariate quantile transform for nonlinear simultaneous equations models in Matzkin (2013) and bivariate hedonic models in Nesheim (2013). Chiappori, McCann, and Nesheim (2010) derive a matching formulation of hedonic models and thereby highlight the close relation between empirical strategies in matching markets and in hedonic markets. Galichon and Salanié (2012) extend the work of Choo and Siow (2006) and identify preferences in marriage markets, where agents match on discrete characteristics, as the unique solution of an optimal transport problem, but unlike the present paper, they are restricted to the case with a discrete quality space. The strategy is extended to the set-valued case by Chiong, Galichon, and Shum (2013), who use subdifferential calculus for identification of dynamic discrete choice problems. Dupuy, Galichon, and Henry (2014) use network flow techniques to identify discrete hedonic models.

Notation. Throughout the paper, we use the following notational conventions. Let $f(x, y)$ be a real-valued function on $\mathbb{R}^d \times \mathbb{R}^d$. When f is sufficiently smooth, the gradient of f with respect to x is denoted $\nabla_x f$, the matrix of second order derivatives with respect to x and y is denoted $D^2_{xy} f$. When f is not smooth, $\partial_x f$ refers to the subdifferential with respect to x, from Definition 5, and $\nabla_{ap,x} f$ refers to the approximate gradient with respect to x, from Definition 7. The set of all Borel probability distributions on a set Z is denoted $\Delta(Z)$. A random vector ε with probability distribution P is denoted $\varepsilon \sim P$, and $X \sim Y$ means that the random vectors X and Y have the same distribution. The product of two probability distributions μ and ν is denoted $\mu \otimes \nu$ and for a map $f : X \mapsto Y$ and $\mu \in \Delta(X)$, $\nu := f\#\mu$ is the probability distribution on Y defined for each Borel subset A of Y by $\nu(A) = \mu(f^{-1}(A))$. For instance, if T is a map from X to Y and ν a probability distribution on X, then $\mu := (id, T)\#\nu$ defines the probability distribution on $X \times Y$ by $\mu(A) = \int_X 1_A(x, T(x)) d\mu(x)$ for all measurable subset A of $X \times Y$. Given two probability distributions μ and ν on X and Y respectively, $\mathcal{M}(\mu, \nu)$ will denote the subset of $\Delta(X \times Y)$ containing all probability distributions with marginals μ and ν. We denote the inner product of two vectors x and y by $x'y$. The Euclidean norm is denoted $\| \cdot \|$. The notation $|a|$ refers to the absolute value of the real number a, whereas $|A|$ refers to the Lebesgue measure of set A. The set of all continuous real valued functions on Z is denoted $C^0(Z)$ and $B_r(x)$ is the open ball of radius r centered at x. Finally, V^* and V^{**} denote the convex conjugate and double conjugate (also called *convex envelope*) of a function V, as defined in (2) and below, and V^ζ and $V^{\zeta\zeta}$ denote the ζ-convex conjugate and double conjugate, as defined in (2) and (7).

Organization of the paper. The remainder of the paper is organized as follows. Section 2 sets the hedonic equilibrium framework out. Section 3 gives an account of the main results on nonparametric identification of preferences in single attribute hedonic models, mostly drawn from Ekeland, Heckman, and Nesheim (2004) and Heckman, Matzkin, and Nesheim (2010). Section 4 is the main section of the paper and shows how these results and the shape restrictions that drive them can be extended to the case of multiple attribute hedonic equilibrium markets. The last section discusses future research directions. Proofs of the main results are relegated to the appendix, as are necessary background results on optimal transport theory and hedonic equilibrium theory. We classify assumptions according to their type and function. Assumptions EC (equilibrium concept) and H (unobserved heterogeneity) refer to the model structure. Assumptions R1, R2 and R3 are nested regularity assumptions on the primitives of the model. Assumptions S1, S2 and S3 are shape restrictions on the utility function, and Assumptions C2 and C3 are shape restrictions on the consumer potential, from Definition 1.

2. Hedonic equilibrium and the identification problem

We consider a competitive environment, where consumers and producers trade a good or contract, fully characterized by its type or quality z. The set of feasible qualities $Z \subseteq \mathbb{R}^{d_z}$ is assumed compact and given a priori, but the distribution of the qualities actually traded arise endogenously in the hedonic market equilibrium, as does their price schedule $p(z)$. Producers are characterized by their type $\tilde{y} \in \tilde{Y} \subseteq \mathbb{R}^{d_{\tilde{y}}}$ and consumers by their type $\tilde{x} \in \tilde{X} \subseteq \mathbb{R}^{d_{\tilde{x}}}$. Type distributions $P_{\tilde{x}}$ on $\tilde{X}$ and $P_{\tilde{y}}$ on $\tilde{Y}$ are given exogenously, so that entry and exit are not modelled. Consumers and producers are price takers and maximize quasi-linear utility $U(\tilde{x}, z) - p(z)$ and profit $p(z) - C(\tilde{y}, z)$ respectively. Utility $U(\tilde{x}, z)$ (respectively cost $C(\tilde{y}, z)$) is upper (respectively lower) semicontinuous and bounded and normalized to zero in case of nonparticipation. In addition, the set of qualities $Z(\tilde{x}, \tilde{y})$ that maximize the joint surplus $U(\tilde{x}, z) - C(\tilde{y}, z)$ for each pair of types $(\tilde{x}, \tilde{y})$ is assumed to have a measurable selection. Then, Ekeland (2010) and Chiappori, McCann, and Nesheim (2010) show that an equilibrium exists in this market, in the form of a price function p on Z, a joint distribution $P_{\tilde{x}z}$ on $\tilde{X} \times Z$ and $P_{\tilde{y}z}$ on $\tilde{Y} \times Z$ such that their marginal on Z coincide, so that market clears for each traded quality $z \in Z$. Uniqueness is not guaranteed, in particular prices are not uniquely defined for non traded qualities in equilibrium. Purity is not guaranteed either: an equilibrium specifies a conditional distribution $P_{z|\tilde{x}}$ (respectively $P_{z|\tilde{y}}$) of qualities consumed by type $\tilde{x}$ consumers (respectively produced by type $\tilde{y}$ producers). The quality traded by a given producer-consumer pair $(\tilde{x}, \tilde{y})$ is not uniquely determined at equilibrium.

Ekeland (2010) and Chiappori, McCann, and Nesheim (2010) further show that a pure equilibrium exists and is unique, under the additional assumption that type distributions $P_{\tilde{x}}$ and $P_{\tilde{y}}$ are absolutely continuous and gradients of utility and cost, $\nabla_{\tilde{x}} U(\tilde{x}, z)$ and $\nabla_{\tilde{y}} C(\tilde{y}, z)$ exist and are injective as functions of quality z. The latter condition, also known as the *Twist Condition* in the optimal transport literature, ensures that all consumers of a given type $\tilde{x}$ (respectively all producers of a given type $\tilde{y}$) consume (respectively produce) the same quality z at equilibrium.

The identification problem consists in the recovery of structural features of preferences and technology from observation of traded qualities and their prices in a single market. The solution concept we impose in our identification analysis is the following feature of hedonic equilibrium, i.e., maximization of surplus generated by a trade.

Assumption EC. [Equilibrium concept] The joint distribution γ of $(\tilde{X}, Z, \tilde{Y})$ and the price function p form an hedonic equilibrium, i.e., they satisfy the following. The joint distribution γ has marginals $P_{\tilde{x}}$ and $P_{\tilde{y}}$ and for γ-almost all $(\tilde{x}, z, \tilde{y})$,

$$\begin{aligned} U(\tilde{x}, z) - p(z) &= \max_{z' \in Z} \left(U(\tilde{x}, z') - p(z') \right), \\ p(z) - C(\tilde{y}, z) &= \max_{z' \in Z} \left(p(z') - C(\tilde{y}, z) \right). \end{aligned}$$

In addition, observed qualities $z \in Z(\tilde{x}, \tilde{y})$ maximizing joint surplus $U(\tilde{x}, z) - C(\tilde{y}, z)$ for each $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}$, lie in the interior of the set of feasible qualities Z and $Z(\tilde{x}, \tilde{y})$ is assumed to have a measurable selection. The joint surplus $U(\tilde{x}, z) - C(\tilde{y}, z)$ is finite everywhere.

Remark 1. *An important implication of Assumption EC is that traded quality z maximizes the joint surplus $U(\tilde{x}, z) - C(\tilde{y}, z)$. This observation is the basis for the equivalent characterization of hedonic models as transferable utility matching models in Chiappori, McCann, and Nesheim (2010).*

Remark 2. *We assume full participation in the market. The possibility of non participation can modelled by adding isolated points to the sets of types and renormalizing distributions accordingly (see Section 1.1 of Chiappori, McCann, and Nesheim (2010) for details).*

Remark 3. *Under Assumption EC, we will denote throughout the projections of γ on $\tilde{X} \times Z$ and $\tilde{Y} \times Z$ by $P_{\tilde{x}z}$ and $P_{\tilde{y}z}$, respectively. The projections on $\tilde{X}$, $\tilde{Y}$ and Z will be denoted $P_{\tilde{x}}$, $P_{\tilde{y}}$ and P_z, respectively.*

Given observability of prices and the fact that producer type $\tilde{y}$ (respectively consumer type $\tilde{x}$) does not enter into the utility function $U(\tilde{x}, z)$ (respectively cost function $C(\tilde{y}, z)$) directly, we may consider the consumer and producer problems separately and symmetrically. We focus on the consumer problem and on identification of utility function $U(\tilde{x}, z)$. Under assumptions ensuring purity and uniqueness of equilibrium, the model predicts a deterministic choice of quality z for a given consumer type $\tilde{x}$. We do not impose such assumptions, but we need to account for heterogeneity in consumption patterns even in case of unique and pure equilibrium. Hence, we assume, as is customary, that consumer types $\tilde{x}$ are only partially observable to the analyst. We write $\tilde{x} = (x, \varepsilon)$, where $x \in X \subseteq \mathbb{R}^{d_x}$ is the observable part of the type vector, and $\varepsilon \in \mathbb{R}^{d_z}$ is the unobservable part, whose dimension is the same as the dimension of good quality z. We shall make a separability assumption that will allow us to specify constraints on the interaction between consumer unobservable type ε and good quality z in order to identify interactions between observable type x and good quality z.

Assumption H. [Unobservable heterogeneity] Consumer type $\tilde{x}$ is composed of observable type x with distribution P_x on $X \subseteq \mathbb{R}^{d_x}$ and unobservable type ε with a priori specified conditional distribution $P_{\varepsilon|x}$ on $\mathbb{R}^{d_z}$. The utility of consumers can be decomposed as $U(\tilde{x}, z) = \bar{U}(x, z) + \zeta(x, \varepsilon, z)$, where the functional form of ζ is known, but that of $\bar{U}$ is not.

Remark 4. *Despite the notation used, $\bar{U}$ should not necessarily be interpreted as* mean utility*, since we allow for a general choice of ζ and $P_{\varepsilon|x}$. If this interpretation is desirable in a particular application, ζ and $P_{\varepsilon|x}$ can be chosen in such a way that $\mathbb{E}[\zeta(x, \varepsilon, z)|x] = 0$.*

Remark 5. *Specification of $P_{\varepsilon|x}$ is a necessary normalization, which also arises in quantile identification strategies.*

We shall work throughout under the following regularity condition on the primitives of the hedonic equilibrium model.

Assumption R1. [Regularity of preferences and technology] The functions $U(\tilde{x}, z)$, $C(\tilde{y}, z)$ and $\zeta(x, \varepsilon, z)$ are twice continuously differentiable with respect to all their arguments.

The object of inference is the deterministic component of utility $\bar{U}(x, z)$.

Definition 1. *For each $(x, z) \in X \times Z$, we shall denote $V(x, z) := p(z) - \bar{U}(x, z)$ and call it the consumer's* potential.

We focus on identification of the function V, since under observability of price, it is equivalent to identification of the function $\bar{U}(x, z)$, which is our objective.

We shall work in stages, recalling first existing identification results in case of scalar z and clarifying which features we intend to extend and how. The guiding principle will be the characterization of shape restrictions on the function V that extend single crossing and monotonicity restrictions in the scalar case and remain just identifying in the multi-attribute case.

3. Single market identification with scalar attribute

In this section, we recall and reformulate results of Heckman, Matzkin, and Nesheim (2010) on identification of single attribute hedonic models. Suppose, for the purpose of this section, that $d_z = 1$, so that unobserved heterogeneity is scalar, as is the quality dimension. Suppose further (for ease of exposition) that ζ is twice continuously differentiable in z and ε (as in Assumption R1) and that V is twice continuously differentiable in z. Consumers take price schedule $p(z)$ as given and choose quality z to maximize $\zeta(x, \varepsilon, z) - V(x, z)$. We impose a single crossing condition on ζ.

Assumption S1. [Spence-Mirlees] Dimension d_z of Z is 1 and $\zeta_{\varepsilon z}(x, \varepsilon, z) > 0$ for all x, ε, z.

The first order condition of the consumer problem yields

$$\zeta_z(x,\varepsilon,z) = V_z(x,z), \tag{1}$$

which, under Assumption S1, implicitly defines an inverse demand function $z \mapsto \varepsilon(x,z)$, which specifies which unobserved type consumes quality z. Combining the second order condition $\zeta_{zz}(x,\varepsilon,z) < V_{zz}(x,z)$ and further differentiation of (1), i.e., $\zeta_{zz}(x,\varepsilon,z) + \zeta_{\varepsilon z}(x,\varepsilon,z)\varepsilon_z(x,z) = V_{zz}(x,z)$, yields

$$\varepsilon_z(x,z) = \frac{V_{zz}(x,z) - \zeta_{zz}(x,\varepsilon,z)}{\zeta_{\varepsilon z}(x,\varepsilon,z)} > 0.$$

Hence the inverse demand is increasing and is therefore identified as the unique increasing function that maps the distribution $P_{z|x}$ to the distribution $P_{\varepsilon|x}$, namely the quantile transform. Denoting F the cumulative distribution function corresponding to the distribution P, we therefore have identification of inverse demand according to the strategy put forward in Matzkin (2003) as:

$$\varepsilon(x,z) = F^{-1}_{\varepsilon|x}\left(F_{z|x}(z|x)\right).$$

The single crossing condition of Assumption S1 on the consumer surplus function $\zeta(x,\varepsilon,z)$ yields positive assortative matching, as in the Becker (1973) classical model. Consumers with higher taste for quality ε will choose higher qualities in equilibrium and positive assortative matching drives identification of demand for quality. The important feature of Assumption S1 is injectivity of $\zeta_z(x,\varepsilon,z)$ relative to ε and a similar argument would have carried through under $\zeta_{z\varepsilon}(x,\varepsilon,z) < 0$, yielding negative assortative matching instead.

Once inverse demand is identified, the consumer potential $V(x,z)$, hence the utility function $\bar{U}(x,z)$, can be recovered up to a constant by integration of the first order condition (1):

$$\bar{U}(x,z) = p(z) - V(x,z) = p(z) - \int_0^z \zeta_z(x,\varepsilon(x,z'),z')dz'.$$

We summarize the previous discussion in the following identification statement, originally due to Heckman, Matzkin, and Nesheim (2010).

Proposition 1. *Under Assumptions EC, H, R1, S1, $\bar{U}(x,z)$ is nonparametrically identified, in the sense that $z \mapsto \bar{U}_z(x,z)$ is the only marginal utility function compatible with the pair (P_{xz},p), i.e., any other marginal utility function coincides with it, $P_{z|x}$ almost surely.*

Unlike the demand function, which is identified without knowledge of the surplus function ζ, as long as the latter satisfies single crossing (Assumption S1), identification of the preference function $\bar{U}(x,z)$ does require a priori knowledge of the function ζ. This includes existing results in this literature. Ekeland, Heckman, and Nesheim (2004) cover the case where $\zeta(x,\varepsilon,z) = z\varepsilon$ and do so without assuming that the distribution of unobserved heterogeneity $P_{\varepsilon|x}$ is known. In that case, $\varepsilon(x,z)$ increasing in z, maximizes $\mathbb{E}[z\varepsilon|x]$ among all joint distributions for (z,ε), subject to the marginal restrictions that $z \sim P_{z|x}$ and $\varepsilon \sim P_{\varepsilon|x}$. This follows from the classical Hardy, Littlewood, and Pólya (1952) inequalities. This optimization formulation is dimension free and is the key to our extension of the identification argument to the case of a vector of qualities z. The maximization of $\mathbb{E}[z\varepsilon|x]$ has a natural extension in the multivariate case $\zeta(x,\varepsilon,z) = z'\varepsilon$, where quality z and taste for quality ε are conformable vectors. We shall examine the case $\zeta(x,\varepsilon,z) = z'\varepsilon$ in the next section, before moving to the general extension with arbitrary surplus function $\zeta(x,\varepsilon,z)$, allowing marginal utility to be nonlinear in unobserved taste, hence allowing interactions between consumer and good characteristics in the utility.

4. Single market identification with multiple attributes

4.1. **Marginal utility linear in unobserved taste.**

We now turn to the main objective of the paper, which is to derive identifying shape restrictions in the multi-attribute case of quality $z \in \mathbb{R}^{d_z}$ and unobserved taste $\varepsilon \in \mathbb{R}^{d_z}$, with $d_z > 1$. We start with the case where marginal utility is linear in unobserved taste.

Assumption S2. The surplus function is $\zeta(x,\varepsilon,z) = z'\varepsilon$.

Remark 6. *A natural interpretation of this specification is that to each quality dimension z_j is associated a specific unobserved taste intensity ε_j for that particular quality dimension.*

Under Assumption S2, the consumer maximization problem is that of finding

$$\sup_{z\in Z}\{\bar{U}(x,z) + z'\varepsilon - p(z)\} = \sup_{z\in Z}\{z'\varepsilon - V(x,z)\} := V^*(x,\varepsilon). \tag{2}$$

For each fixed $x \in X$, the function $\varepsilon \mapsto V^*(x,\varepsilon)$ defined in (2) is called the *convex conjugate* (also known as *Legendre-Fenchel transform*) of $z \mapsto V(x,z)$. Still for fixed x, the convex conjugate $z \mapsto V^{**}(x, \varepsilon) := \sup_{\varepsilon \in \mathbb{R}^{d_z}} \{z'\varepsilon - V^*(x, \varepsilon)\}$ is called the *double conjugate* of the potential function $z \mapsto V(x, z)$. According to convex duality theory (see for instance Rockafellar (1970), Theorem 12.2 page 104), the double conjugate of V is V itself if and only if $z \mapsto V(x, z)$ is convex and lower-semi-continuous, as a function of z. Under suitable regularity, the first order condition from the program $\sup_\varepsilon \{z'\varepsilon - V^*(x, \varepsilon)\}$ yields a demand for quality function $z = \nabla_\varepsilon V^*(x, \varepsilon)$. Similarly, the first order condition from the consumer maximization problem (2) yields an inverse demand for quality function $\varepsilon = \nabla_z V(x, z)$. If the consumer's potential V is convex, as a function of z, we will show that the demand and inverse demand functions are uniquely determined. Convexity of V with respect to its second variable z, therefore, turns out to be the shape restriction that delivers identification in this case where marginal utility is linear in unobserved taste.

Assumption C2. [Convexity restriction] The function $V(x, z)$ is convex in z for all x.

Convexity of V implies that the inverse demand function $\varepsilon = \nabla_z V(x, z)$ is the gradient of a convex function (as a function of z, for each x). In the univariate case developed in the previous section, this corresponds to inverse demand being a non decreasing function. Hence, in the case where both z and ε are scalar, higher levels of quality z are chosen by consumers with higher levels of ε, interpreted as unobserved taste for quality. We therefore recover the property of assortative matching, delivered in Section 3 by the single crossing shape restriction on ζ, namely Assumption S1.

As discussed in Section 3, positive assortative matching (monotonicity of demand) is difficult to extend to the multi-attribute case, but not the efficiency result that comes with positive assortative matching. Imagine a social planner maximizing total surplus over the distribution of heterogeneous consumers. The planner's problem is to maximize $\mathbb{E}[z'\varepsilon|x]$ over all possible allocations of qualities z to consumer types ε, i.e., over all pairs of random vectors (ε, z) under the constraint that the marginal distributions $P_{\varepsilon|x}$ and $P_{z|x}$ are fixed. One of the central results of optimal transport theory, Brenier's Theorem (Theorem 1 in Appendix 6.1), shows precisely that such a planner's problem admits a unique pure allocation as solution, which takes the form of the inverse demand function $\varepsilon = \nabla_z V(x, z)$

with $V(x,z)$ convex in z. We see thereby that convexity of $V(x,z)$ in z is the shape restriction that delivers identification as summarized in the following theorem.

Theorem 1 (Identification with linear marginal utility in taste)**.** *Under Assumptions EC, H, R1, S2 and C2, the following statements hold:*

(1) *$\bar{U}(x,z)$ is nonparametrically identified, in the sense that $z \mapsto \nabla_z \bar{U}(x,z)$ is the only marginal utility function compatible with the pair (P_{xz}, p), i.e., any other marginal utility function coincides with it, $P_{z|x}$ almost surely.*

(2) *For all $x \in X$, $\bar{U}(x,z) = p(z) - V(x,z)$ and $z \mapsto V(x,z)$ is the convex solution to the optimization problem $\min_V \left(\mathbb{E}_z[V(x,z)|x] + \mathbb{E}_\varepsilon[V^*(x,\varepsilon)|x]\right)$, where $\varepsilon \mapsto V^*(x,\varepsilon)$ is the convex conjugate of $z \mapsto V(x,z)$ (convex conjugation with respect to the second variable).*

The structure of this identification proof is as follows (see the proof of Theorem 4, of which this is a special case). We show that there exists a unique allocation of qualities to tastes $z \mapsto \varepsilon(x,z)$ that maximizes the consumer problem (2). A significant portion of the proof is dedicated to showing that the endogenous price function $z \mapsto p(z)$, hence $z \mapsto V(x,z)$, is differentiable. Hence, once the allocation (inverse demand function) $\varepsilon(x,z)$ is identified, $V(x,z)$ satisfies the first order condition

$$\nabla_z V(x,z) = \nabla_z \zeta(x, \varepsilon(x,z), z) = \varepsilon(x,z), \quad P_{z|x}\text{-almost surely, for every } x.$$

The latter determines $V(x,z)$, and therefore $\bar{U}(x,z)$, up to a constant.

Remark 7. *The identification is constructive, as $V(x,z)$ is shown to be the convex solution to the minimization of $\mathbb{E}[V(x,z)|x] + \mathbb{E}[V^*(x,\varepsilon)|x]$, where V^* is the convex conjugate of V with respect to the second variable.*

Identification under primitive restrictions only. Since the consumer potential $V(x,z)$ is equal to $p(z) - \bar{U}(x,z)$, it involves the endogenous price function. Hence, Theorem 1 involves a condition on the potential V, which is undesirable, since it constrains a non primitive quantity in the model, namely the endogenous price function. However, we show that, as monotonicity of inverse demand in the scalar case was implied by the single crossing condition, here, convexity of V in z is "essentially" always true under the maintained primitive conditions, in the sense of the following lemma.

Lemma 1 (Convexity)**.** *Under Assumptions EC and H, for all x, $V(x,z) = V^{**}(x,z)$, $P_{z|x}$ almost surely, where ** denotes double convex conjugation with respect to the second variable.*

As we have seen above, the function V is equal to its double conjugate V^{**} if and only if it is convex and lower semi-continuous. Hence Lemma 1 implies in particular, that on any open subset of the support of $P_{z|x}$, $V(x,z)$ is a convex function of z as required. It also implies, as will be shown in Theorem 2, that the convexity assumption becomes unnecessary if the distribution of qualities traded at equilibrium is absolutely continuous. We therefore give conditions on the primitives of the model, such that the endogenous distribution of qualities traded is absolutely continuous. This will allow us to dispense entirely from restrictions on endogenous qualities in the statement of our identification theorem.

Assumption R2. [Conditions for absolute continuity of $P_{z|x}$] Assumption R1 holds.

(1) The Hessian of total surplus $D^2_{zz}\left(U(x,\epsilon,z) - C(\tilde{y},z)\right)$ is bounded above; that is $D^2_{zz}\left(U(x,\epsilon,z) - C(\tilde{y},z)\right) \leq M_1$ for all $x,\epsilon,z,\tilde{y}$, for some fixed M_1.
(2) The distribution of unobserved tastes $P_{\varepsilon|x}$ is absolutely continuous with respect to Lebesgue measure for all x.

Lemma 2. *Under Assumptions EC, H, S2 and R2, the endogenous distribution $P_{z|x}$ of qualities traded at equilibrium is absolutely continuous with respect to Lebesgue measure.*

Remark 8. *Lemma 2 is a crucial step in our identification strategy, but beyond its role in identification, it is an important result in its own right within the theory of hedonic equilibrium models.*

Lemma 2 allows us to dispense with the convexity assumption on $z \mapsto V(x,z)$, which implicitely involved a constraint on the endogenous price function $z \mapsto p(z)$. We can now state our identification theorem based exclusively on restrictions on primitive quantities, namely, preferences, technology and the distributions of consumer and producer types.

Theorem 2. *Under Assumptions EC, H, R2 and S2, the following statements hold:*

(1) $\bar{U}(x,z)$ *is nonparametrically identified, in the sense that* $z \mapsto \nabla_z \bar{U}(x,z)$ *is the only marginal utility function compatible with the pair* (P_{xz}, p)*, i.e., any other marginal utility function coincides with it,* $P_{z|x}$ *almost surely.*

(2) *For all* $x \in X$, $\bar{U}(x,z) = p(z) - V(x,z)$ *and* $z \mapsto V(x,z)$ *is* $P_{z|x}$ *almost everywhere equal to the convex solution to the problem* $\min_V (\mathbb{E}_z[V(x,z)|x] + \mathbb{E}_\varepsilon[V^*(x,\varepsilon)|x])$*, where* $\varepsilon \mapsto V^*(x,\varepsilon)$ *is the convex conjugate of* $z \mapsto V(x,z)$ *(convex conjugation with respect to the second variable).*

Remark 9. *Unlike Theorem 1, Theorem 2 above delivers an identification result based only on restrictions on the primities of the model. These restrictions are mild regularity conditions on preferences and technology, namely Assumption R2(1), and regularity of the conditional distribution* $P_{\varepsilon|x}$ *of unobserved types, namely Assumption R2(2). Note, in particular, that no regularity is assumed for the distribution of observable consumer types* P_x *or the distribution of producer types* $P_{\tilde{y}}$*, so that discrete observable characteristics are also covered by the result.*

Nonlinear simultaneous equations. The reasoning behind the identification result of Theorem 1 has implications beyond hedonic equilibrium models, as it provides identification conditions for a general nonlinear nonseparable simultaneous equations econometric model of the form $z = h(x,\varepsilon)$, where the vector of endogenous variables z has the same dimension as the vector of unobserved heterogeneity ε. Theorem 3 shows that in such models, h is nonparametrically identified within the class of gradients of convex functions.

Theorem 3 (Nonlinear simultaneous equations)**.** *In the simultaneous equations model* $z = h(x,\varepsilon)$*, with* $z, \varepsilon \in \mathbb{R}^{d_z}$ *and* $x \in \mathbb{R}^{d_x}$*, the function* $\varepsilon \mapsto h(x,\varepsilon)$ *is identified under the following conditions.*

(1) $\varepsilon \mapsto h(x,\varepsilon)$ *is the gradient of a convex function in* ε *for all* $x \in X$*.*

(2) *For all* $x \in X$, $P_{\varepsilon|x}$ *is specified a priori and is absolutely continuous with respect to Lebesgue measure and* $P_{\varepsilon|x}$ *and* $P_{z|x}$ *have finite variance.*

In the univariate case, gradients of convex functions are the increasing functions, so that our identifying shape restriction directly generalizes monotonicity in Matzkin (2003).

4.2. **General case.** The identification result of Theorem 1 can be easily extended to allow for variation in the quality-unobserved taste interaction with observed type x as in $\zeta(x,\varepsilon,z) = \phi(z)'\psi(x,\varepsilon)$, where ϕ and ψ are known functions and ϕ is invertible. Going beyond this requires a new type of shape restriction that can be interpreted as a multivariate extension of the single crossing "Spence-Mirlees" condition of Assumption S1.

Recalling our notation $V(x,z) = p(z) - \bar{U}(x,z)$, the consumer's program is to choose quality vector z to maximize

$$\sup_{z\in Z}\{\bar{U}(x,z) + \zeta(x\varepsilon,z) - p(z)\} = \sup_{z\in Z}\{\zeta(x,\varepsilon,z) - V(x,z)\}. \tag{3}$$

In the one dimensional case, single crossing condition $\zeta_{\varepsilon z}(x,\varepsilon,z) > 0$ delivered identification of inverse demand. We noted that the sign of the single crossing condition was not important for the identification result. Instead, what is crucial is the following, weaker, condition.

Assumption S3. [Twist Condition] For all x and z, the gradient $\nabla_z\zeta(x,\varepsilon,z)$ of $\zeta(x,\varepsilon,z)$ in z is injective as a function of ε on the support of $P_{\varepsilon|x}$.

Remark 10. *As is well known from Gale and Nikaido (1965), it is sufficient that $D^2_{z\varepsilon}\zeta(x,\varepsilon,z)$ be positive quasi-definite everywhere for Assumption S3 to be satisfied. A weaker set of conditions is given in Theorem 2 of Mas-Colell (1979).*

Remark 11. *Assumption S3, unlike the single crossing condition, is well defined in the multivariate case, and we shall show, using recent developments in optimal transport theory, that it continues to deliver the desired identification in the multivariate case.*

Before stating the theorem, we provide more intuition by further developing the parallel between this general case and the cases covered so far. Notice that the Twist Condition of Assumption S3 is satisfied in the particular case of marginal utility linear in taste, as considered in Section 4.1 above.

Consider, as before, the hedonic market from the point of view of a social planner, who allocates qualities z to tastes ε in a way that maximizes total consumer surplus. The distribution of consumer tastes is $P_{\varepsilon|x}$ and the distribution of qualities traded at equilibrium is $P_{z|x}$. For fixed observable type x, the variable surplus of a match between unobserved taste ε and quality z is $\zeta(x,\varepsilon,z)$. Hence, the planner's problem is to find an allocation of qualities to tastes, in the form of a joint probability μ over the pair of random vectors (ε,z), so as to maximize $\mathbb{E}_\mu[\zeta(x,\varepsilon,z)|x]$ under the constraint that ε has marginal distribution $P_{\varepsilon|x}$ and that z has marginal distribution $P_{z|x}$. This planner's problem

$$\max_{\mu} \mathbb{E}_\mu[\zeta(x,\varepsilon,z)|x] \;\text{ subject to } \varepsilon \sim P_{\varepsilon|x},\; z \sim P_{z|x} \tag{4}$$

is equal to its dual

$$\min_{V,W} \mathbb{E}[W(x,\varepsilon)|x] + \mathbb{E}[V(x,z)|x] \;\text{ subject to } W(x,\varepsilon) + V(x,z) \geq \zeta(x,\varepsilon,z) \tag{5}$$

and both primal (4) and dual (5) are attained under the conditions of the Monge-Kantorovitch Theorem (Theorem 6 in Appendix 6.1). Notice that the constraint in (5) can be written as

$$W(x,\varepsilon) = \sup_{z\in Z}\{\zeta(x,\varepsilon,z) - V(x,z)\} := V^{\zeta}(x,\varepsilon), \tag{6}$$

so that $W(x,\varepsilon)$ is a candidate for the demand function mapping tastes ε into qualities z derived from the consumer's program (3). Equation (6) defines a generalized notion of convex conjugation, which can be inverted, similarly to convex conjugation, into:

$$V^{\zeta\zeta}(x,z) = \sup_{\varepsilon\in\mathbb{R}^{d_z}}\{\zeta(x,\varepsilon,z) - V^{\zeta}(x,\varepsilon)\}, \tag{7}$$

where $V^{\zeta\zeta}$ is called the *double conjugate* of V by analogy with V^{**}.

Definition 2 (ζ-convexity)**.** *A function V is called ζ-convex if and only if $V^{\zeta\zeta} = V$.*

The requirement of ζ-convexity, therefore, is a shape restriction that directly generalizes the convexity restriction of Assumption C2 in case of marginal utility linear in unobserved taste.

Assumption C3. [ζ-convexity] The potential V is ζ-convex as a function of z for all x.

Under Assumptions EC, H, R1, S3 and C3, we show that there exists a unique allocation of qualities to tastes $z \mapsto \varepsilon(x,z)$ that maximizes the consumer problem (6). Moreover, this allocation is such that markets clear, since $\varepsilon(x,z)$ is distributed according to $P_{\varepsilon|x}$

when z is distributed according to $P_{z|x}$. Heuristically, by the envelope theorem applied to $V(x,z) = \sup_\varepsilon \{\zeta(x,\varepsilon,z) - V^\zeta(x,\varepsilon)\}$, for a small variation δV^ζ of V^ζ, the variation in V is $\delta V(x,z) = -\delta V^\zeta(x, \varepsilon(x,z), z)$. Plugging the latter into the first order condition for (5) yields $\mathbb{E}[\delta V^\zeta(x,\varepsilon)|x] = \mathbb{E}[\delta V^\zeta(x,\varepsilon(x,z))|x]$. The latter holds for any small variation δV^ζ, so that the distribution of allocation $\varepsilon(x,z)$ is the same as the exogenously given distribution of unobserved tastes $P_{\varepsilon|x}$, so that the market clears.

Finally, once the allocation (inverse demand function) $\varepsilon(x,z)$ is identified, $V(x,z)$ satisfies the first order condition of the consumer's program

$$\nabla_z V(x,z) = \nabla p(z) - \nabla_z \bar{U}(x,z) = \nabla_z \zeta(x, \varepsilon(x,z), z), \quad P_{z|x}\text{-almost surely, for every } x.$$

The latter determines $V(x,z)$, and therefore $\bar{U}(x,z)$, up to a constant. We are now ready to state our main theorem, relating the Twist Condition and the ζ-convex shape restriction to nonparametric identification of preferences.

Theorem 4 (Identification of preferences)**.** *Under Assumptions EC, H, R1, S3 and C3, the following statements hold:*

(1) *$\bar{U}(x,z)$ is nonparametrically identified, in the sense that $z \mapsto \nabla_z \bar{U}(x,z)$ is the only marginal utility function compatible with the pair (P_{xz}, p), i.e., any other marginal utility function coincides with it, $P_{z|x}$ almost surely.*
(2) *For all $x \in X$, $\bar{U}(x,z) = p(z) - V(x,z)$ and $z \mapsto V(x,z)$ is the ζ-convex solution to the problem $\min_V \left(\mathbb{E}_z[V(x,z)|x] + \mathbb{E}_\varepsilon[V^\zeta(x,\varepsilon)|x]\right)$, with V^ζ defined in (6).*

Remark 12. *As before, the identification strategy is constructive and efficient computation of $\bar{U}(x,z) = p(z) - V(x,z)$ is based on the identification of V as the solution to the optimization problem of Theorem 4(2).*

4.2.1. *Identification under primitive restrictions only.* As discussed in the previous section, Assumption C3 involves restrictions on the price function $p(z)$, which is endogenously determined at equilibrium. We show, however, that the Twist Condition of Assumption S3 is the relevant shape restriction, corresponding to single crossing, and that ζ-convexity follows, in the sense of the following lemma.

Lemma 3. *Under Assumptions EC and H,* $V(x,z) = V^{\zeta\zeta}(x,z)$, $P_{z|x}$*-a.s., for all* x.

Again, this lemma allows us to dispense completely with conditions on non primitive quantities (here, the function V, which depends on the endogenous price function) under conditions such that the endogenous distribution of qualities traded at equilibrium is absolutely continuous.

Assumption R3. [Conditions for absolute continuity of $P_{z|x}$] Assumption R2 holds.

(1) The function $z \mapsto \nabla_\epsilon \zeta(x,\epsilon,z)$ is injective, as a function of $z \in Z$.
(2) For each $x \in X$, $\nabla_z \zeta(x,\varepsilon,z) \to \infty$ as $\|\varepsilon\| \to \infty$, uniformly in $z \in Z$.
(3) The matrix $D^2_{\epsilon z}\zeta(x,\varepsilon,z)$ has full rank for all x,ε,z. Its inverse $[D^2_{\epsilon z}\zeta(x,\varepsilon,z)]^{-1}$ has uniform upper bound M_0.

Lemma 4. *Under Assumptions EC, H, S3 and R3, the endogenous distribution* $P_{z|x}$ *of qualities traded at equilibrium is absolutely continuous with respect to Lebesgue measure.*

Under the assumptions of Lemma 4, we obtain an identification result that relies only on constraints on the primitive quantities of the model, namely preferences, technology and the distributions of consumer and producer types.

Theorem 5. *Under Assumptions EC, H, S3 and R3, the following statements hold:*

(1) $\bar{U}(x,z)$ *is nonparametrically identified, in the sense that* $z \mapsto \nabla_z \bar{U}(x,z)$ *is the only marginal utility function compatible with the pair* (P_{xz}, p)*, i.e., any other marginal utility function coincides with it,* $P_{z|x}$ *almost surely.*
(2) *For all* $x \in X$, $\bar{U}(x,z) = p(z) - V(x,z)$ *and* $z \mapsto V(x,z)$ *is* $P_{z|x}$ *almost everywhere equal to the* ζ*-convex solution to the problem* $\min_V \left(\mathbb{E}_z[V(x,z)|x] + \mathbb{E}_\varepsilon[V^\zeta(x,\varepsilon)|x]\right)$, *with* V^ζ *defined in (6).*

Again, the identification result of Theorem 4 has ramifications beyond the framework of hedonic equilibrium models. Indeed, it provides the just identifying shape restriction for general consumer problems with multivariate unobserved preference heterogeneity, where consumers choose within a set of goods, differentiated along more than one dimension.

Theorem 4 tells us that the shape of interactions between good qualities and unobserved tastes governs the shape restriction that just identifies the utility function.

5. Discussion

The identification results in the paper rely on observations from a single price schedule and the structural functions are just identified under normalization of the distribution of unobserved heterogeneity. Although $\bar{U}(x,z)$ is not identified without such a normalization, or additional restrictions, there are features of preferences that are identified. Consider our model under Assumption S2, with $U(x,\varepsilon,z) = \bar{U}(x,z) + z'\varepsilon$. From Theorem 8 in Appendix 6.1, the inverse demand $\varepsilon(x,z) = \nabla_z \left[p(z) - \bar{U}(x,z)\right]$ satisfies the following: for all bounded continuous functions ξ, $\int \xi(\varepsilon) f_\varepsilon(\varepsilon) d\varepsilon = \int \xi\left(\nabla_z p(z) - \nabla_z \bar{U}(x,z)\right) f_{z|x}(z|x)dz$. Hence, taking ξ equal to the identity above and assuming only that $P_{\varepsilon|x}$ has mean zero, instead of fixing the whole distribution, yields identification of the averaged partial effects $\mathbb{E}\left[\nabla_z \bar{U}(x,Z)|X=x\right] = \mathbb{E}\left[\nabla p(Z)|X=x\right]$ from the fact that $p(z)$ and $P_{z|x}$ are identified.

Going beyond the latter result requires additional assumptions, which can take either of the following forms. (1) A separability assumption of the form $\bar{U}(x,z) := z'\alpha(x) + \beta(z)$ can be imposed, in which case, the strategy outlined in Ekeland, Heckman, and Nesheim (2004) can be extended to the case of multiple attributes and yield identification without normalization of the distribution of unobserved tastes for quality ε. (2) Data from multiple markets can be brought to bear on the identification problem, or more generally a variable that shifts underlying distributions of producers and consumers, without affecting preferences and technology. To fix ideas, suppose two seperate markets m_1 and m_2 (separate in the sense that producers, consumers or goods cannot move between markets) with underlying producer and consumer distributions $(P_x^{m_1}, P_{\tilde{y}}^{m_1})$ and $(P_x^{m_2}, P_{\tilde{y}}^{m_2})$, are at equilibrium, with respective price schedules $p^{m_1}(z)$ and $p^{m_2}(z)$. The additional identification assumption is that utility $\bar{U}(x,z)$ and unobserved taste distribution $P_{\varepsilon|x}$ are common across the two markets. In each market, we recover a nonparametrically identified utility function $\bar{U}^m(x,z;P_{\varepsilon|x})$, where the dependence in the unknown distribution of tastes $P_{\varepsilon|x}$ is emphasized. Therefore, the additional identifying restriction associated with multiple markets data takes the form $\bar{U}^{m_1}(x,z;P_{\varepsilon|x}) = \bar{U}^{m_2}(x,z;P_{\varepsilon|x})$ which defines an identified set for the pair $(\bar{U}, P_{\varepsilon|x})$. Further research is needed to characterize this identified set and derive conditions for point identification in this setting.

6. Appendix

Throughout Appendix 6.1, when there is no ambiguity, we drop the conditioning variable x from the notation and consider the theory of optimal transportation of distribution P_z of quality vector $z \in \mathbb{R}^d$ to distribution P_ε of vector of unobserved tastes $\varepsilon \in \mathbb{R}^d$. In Appendix 6.2, where we prove the results in the main text, the conditioning variable will be reintroduced.

6.1. **Optimal transportation results.**

Monge-Kantorovich problem. We first consider the Kantorovich problem, which is the probabilistic allocation of qualities to tastes so as to maximize total surplus, where the surplus of a pair (ε, z) is given by the function $\zeta(\varepsilon, z)$, and the marginal distributions of qualities P_z and tastes P_ε are fixed constraints. We therefore define the set of allocations that satisfy the constraints as follows.

Definition 3 (Probabilities with given marginals). *We denote $\mathcal{M}(P_\varepsilon, P_z)$ the set of probability distributions on $\mathbb{R}^d \times \mathbb{R}^d$ with marginal distributions P_ε and P_z.*

With this definition, we can formally state the Kantorovich problem as follows.

$$\text{(PK)} = \sup_{\pi \in \mathcal{M}(P_\varepsilon, P_z)} \int \zeta(\varepsilon, z) d\pi(\varepsilon, z).$$

If we consider the special case of surpluses that are separable in ε and z and dominate $\zeta(\varepsilon, z)$, i.e., of the form $W(\varepsilon) + V(z) \geq \zeta(\varepsilon, z)$, the integral yields $\int W(\varepsilon) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z)$. We denote Φ_ζ the set of such functions.

Definition 4 (Admissible set). *A pair of function (W, V) on $\mathbb{R}^d$ belongs to the admissible set Φ_ζ if and only if $W \in L^1(P_\varepsilon)$, $V \in L^1(P_z)$ and $W(\varepsilon) + V(z) \geq \zeta(\varepsilon, z)$ for P_ε-almost all ε and P_z-almost all z.*

The integral over separable surpluses

$$\text{(DK)} = \inf_{(W,V) \in \Phi_\zeta} \int W(\varepsilon) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z)$$

will in general yield a weakly larger total surplus than (PK), but it turns out that under very weak conditions, the two coincide.

Theorem 6 (Kantorovich duality). *If ζ is upper semi-continuous, then (PK)=(MK) and there exists an allocation $\pi \in \mathcal{M}(P_\varepsilon, P_z)$ that achieves the maximum in (PK).*

A proof of the Kantorovich duality theorem can be found in Chapter 1 of Villani (2003). We give here the intuition of the result based on switching infimum and supremum operations. First, if ζ is a continuous function, the mapping $\mu \mapsto \int \zeta d\mu$ is weakly continuous. Since $\mathcal{M}(P_\varepsilon, P_z)$ is weakly compact, the maximum in (PK) is achieved for some π by the Weierstrass Theorem. Hence, an optimal allocation exists. However, continuity of ζ is not necessary.

To see the duality result, denote $\chi_A(x) = 0$ if $x \in A$ and $-\infty$ otherwise. Then, we verify that

$$\chi_{\mathcal{M}(P_\varepsilon, P_z)}(\pi) = \inf_{(W,V)} \left\{ \int W(\varepsilon) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z) - \int (W(\varepsilon) + V(z))\, d\pi(\varepsilon, z) \right\},$$

where the infimum is over all integrable functions, say. Now we can rewrite (PK) as follows:

$$\text{(PK)} = \sup_\pi \left\{ \int \zeta(\varepsilon, z) d\pi(\varepsilon, z) + \chi_{\mathcal{M}(P_\varepsilon, P_z)}(\pi) \right\},$$

where the supremum is taken over positive measures π. Assuming the infimum and supremum operations can be switched yields:

$$\begin{aligned}
\text{(PK)} \quad &= \quad \inf_{(W,V)} \sup_\pi \left\{ \int \zeta(\varepsilon, z) d\pi(\varepsilon, z) \right. \\
&\qquad \left. + \int W(z) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z) - \int (W(\varepsilon) + V(z))\, d\pi(\varepsilon, z) \right\} \\
&= \quad \inf_{(W,V)} \left\{ \int W(z) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z) \right. \\
&\qquad \left. - \inf_\pi \int (W(\varepsilon) + V(z) - \zeta(\varepsilon, z))\, d\pi(\varepsilon, z) \right\}.
\end{aligned}$$

Consider the second infimum in the last display. If the function $W(\varepsilon) + V(z) - \zeta(\varepsilon, z)$ takes a negative value, then, choosing for π the Dirac mass at that point will yield an infimum of $-\infty$. Therefore, we have:

$$\inf_\pi \int (W(\varepsilon) + V(z) - \zeta(\varepsilon, z))\, d\pi(\varepsilon, z) = \chi_{\Phi_\zeta}(W, V),$$

so that

$$\text{(PK)} = \inf_{(W,V)} \left\{ \int W(z) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z) - \chi_{\Phi_\zeta}(W, V) \right\} = \text{(DK)}$$

as required.

We now see that the dual is also achieved.

Theorem 7 (Kantorovich duality (continued))**.** *If (PK)$< \infty$ and there exist integrable functions ζ^ε and ζ^z such that $\zeta(\varepsilon, z) \geq \zeta^\varepsilon(\varepsilon) + \zeta^z(z)$, then there exists a ζ-convex function (see Definition 2) V such that*

$$\int V^\zeta(\varepsilon) dP_\varepsilon(\varepsilon) + \int V(z) dP_z(z)$$

achieves (DK). In addition, if π is an optimal allocation, i.e., achieves (PK), and (V^ζ, V) is an optimal ζ-conjugate pair, i.e., achieves (DK), then

$$V^\zeta(\varepsilon) + V(z) \geq \zeta(\varepsilon, z) \text{ with equality } \pi\text{-almost surely.} \tag{8}$$

Idea of the proof. The proof can be found in Chapter 5 of Villani (2009). The last statement of Theorem 7 is easy to see. If π achieves (PK) and (V^ζ, V) achieves (DK), then, as (PK)=(DK) by Theorem 6, we have $\int [V^\zeta(\varepsilon) + V(z) - \zeta(\varepsilon, z)] d\pi(\varepsilon, z) = 0$. The integrand is non negative, since $(V^\zeta, V) \in \Phi_\zeta$. Hence, $V^\zeta + V = \zeta$, π-almost surely, as desired. The proof of existence of an optimal pair of ζ-convex functions achieving (DK) revolves around the notion of cyclical monotonicity.

In view of the above, if (ϕ, ψ) achieve (DK) and a sequence of pairs $(\varepsilon_i, z_i)_{i=1,\ldots,m}$ belong to the support of the optimal allocation π, then $\phi(\varepsilon_i) + \psi(z_i) = \zeta(\varepsilon_i, z_i)$ for each $i = 1, \ldots, m$. On the other hand, since $(\phi, \psi) \in \Phi_\zeta$, we have $\phi(\varepsilon_i) + \psi(z_{i+1}) \geq \zeta(\varepsilon_i, z_{i+1})$ for each $i = 1, \ldots, m-1$, and $\phi(\varepsilon_m) + \psi(z) \geq \zeta(\varepsilon_m, z)$ for an arbitrary z. Substracting and adding up yields

$$\psi(z) \geq \psi(z_1) + [\zeta(\varepsilon_m, z) - \zeta(\varepsilon_m, z_m)] + \ldots + [\zeta(\varepsilon_1, z_2) - \zeta(\varepsilon_1, z_1)].$$

Since the functions in the pair (ϕ, ψ) are only determined up to a constant, normalize $\psi(z_1) = 0$ and define V as the infimum of all functions ψ satisfying $\psi(z) \geq [\zeta(\varepsilon_m, z) - \zeta(\varepsilon_m, z_m)] + \ldots + [\zeta(\varepsilon_1, z_2) - \zeta(\varepsilon_1, z_1)]$ over all choices of $(\varepsilon_i, z_i)_{i=1,\ldots,m}$ in the support of π and all $m \geq 0$. It turns out that $V^\zeta(\varepsilon) + V(z) = \zeta(\varepsilon, z)$, π-almost surely, so that integration over π yields the fact that (V^ζ, V) achieves (DK) as desired. □

The quadratic case and Brenier's Theorem. In the special case of Assumption S2, where $\zeta(\varepsilon, z) = z'\varepsilon$, the planner's program (PK) writes

$$\sup_{\pi \in \mathcal{M}(P_\varepsilon, P_z)} \int z'\varepsilon d\pi(\varepsilon, z)$$

and the set Φ_ζ becomes

$$\Phi = \{(W, V) : \; W(\varepsilon) + V(z) \geq z'\varepsilon\}.$$

The pair $(W, W^*) \in \Phi$ defined by

$$\begin{aligned} W^*(z) &= \sup_\varepsilon \{z'\varepsilon - W(\varepsilon)\}, \\ W(\varepsilon) &= \sup_z \{z'\varepsilon - W^*(z)\} \end{aligned}$$

achieves the minimum in the dual problem (DK). Notice that W and W^* are standard Fenchel-Legendre convex conjugates of each other and that $W = W^{**}$ and is hence convex.

In this case, $\nabla W^*(z) = \nabla_z \zeta(\varepsilon, z)$ simplifies to $\nabla W^*(z) = \varepsilon$, which guarantees uniqueness and purity of the optimal assignment $z = \nabla W(\varepsilon)$, where V is convex. As a corollary, ∇W is a P_ε-almost surely uniquely determined gradient of a convex function.

Theorem 8 (Brenier)**.** *Suppose P_ε is absolutely continuous with respect to Lebesgue measure and that P_ε and P_z have finite second order moments. Then, there exists a P_ε-almost surely unique map of the form ∇W, where W is convex, such that $\int \varepsilon' \nabla W(\varepsilon) dP_\varepsilon(\varepsilon)$ achieves the maximum in (PK) with $\zeta(\varepsilon, z) = z'\varepsilon$. The* *function W is the P_ε-almost everywhere uniquely determined convex map such that $P_\varepsilon(\nabla W^{-1}(B)) = P_z(B)$ for all Borel subsets B of the support of P_z. Moreover, (W, W^*) achieves the dual program (DK).*

A proof of Theorem 8 can be found in Section 2.1.5 of Villani (2003). Note that W is not only the unique convex map that solves the optimization problem. However, ∇W is the unique gradient of a convex map that pushes forward probability distribution P_ε to P_z. Hence, identification can be achieved for a nonlinear simultaneous equations model $z = f(x, \varepsilon)$ without an underlying assumption about how choices z were generated from tastes (x, ε). This is the content of Theorem 3, which is therefore seen to be a straightforward application of Theorem 8.

6.2. **Proof of results in the main text.**

Proof of Lemma 1. This is a direct corollary of Lemma 3. □

Proof of Lemma 2. This is a corollary of Lemma 4, since Assumptions S2,R2 imply R3. □

Proof of Theorems 1 and 2. These are corollaries of Theorems 4 and 5, since Assumptions S2,C2 imply S3,C3 and Assumptions S2,R2 imply R3. □

Proof of Theorem 3. Since $z = h(x, \varepsilon)$, for any Borel set B,

$$\mathbb{P}(Z \in B | X = x) \quad = \quad \mathbb{P}(h(x, \varepsilon) \in B | X = x). \tag{9}$$

By Theorem 8, a gradient of a convex function satisfying (9) is $P_{\varepsilon|x}$-almost everywhere uniquely determined, hence the result. □

Proof of Lemma 3. By definition of V^ζ, we have

$$V(x, z) \geq \zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon). \tag{10}$$

As, by definition of ζ-conjugation, $V^{\zeta\zeta}(x, z) = \sup_\varepsilon \left[\zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon)\right]$, we have

$$V(x, z) \geq V^{\zeta\zeta}(x, z), \tag{11}$$

by taking supremum over ε in (10).

Let γ be an hedonic equilibrium probability distribution on $\tilde{X} \times Z \times \tilde{Y}$. By Assumption EC,

$$\zeta(x, \varepsilon, z) - V(x, z) = U(x, \varepsilon, z) - p(z) = \max_{z \in Z} (U(x, \varepsilon, z) - p(z)) = \max_{z \in Z} (\zeta(x, \varepsilon, z) - V(x, z)) = V^\zeta(x, \varepsilon)$$

is true γ-almost everywhere. Hence, there is equality in (10) γ-almost everywhere. Hence, for $P_{z|x}$ almost every z, and ε such that (z, ε) is in the support of γ, we have $V(x, z) = \zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon)$. But the right hand side is bounded above by $V^{\zeta\zeta}(x, z)$ by definition, so we get $V(x, z) \leq V^{\zeta\zeta}(x, z)$. Combined with (11), this tells us $V(x, z) = V^{\zeta\zeta}(x, z)$, $P_{z|x}$ almost everywhere. □

Proof of Lemma 4. For an upper semi-continuous map $S : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$, possibly indexed by x, and probability distributions μ on $\mathbb{R}^{d_1}$ and ν on $\mathbb{R}^{d_2}$, possibly conditional on x, let $T_S(\mu,\nu)$ be the solution of the Kantorovich problem, i.e.,

$$T_S(\mu,\nu) = \sup_{\gamma\in\mathcal{M}(\mu,\nu)} \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} S(\varepsilon,z)d\gamma(\varepsilon,z).$$

We state and prove three intermediate results in Steps 1, 2 and 3 before completing the proof of Lemma 4.

Step 1: We first show that P_z achieves

$$\max_{\nu\in\Delta(Z)} [T_U(P_{\tilde{x}},\nu) + T_{-C}(P_{\tilde{y}},\nu)]. \tag{12}$$

This step follows the method of proof of Proposition 3 in Carlier and Ekeland (2010). Take any probability distribution $\nu \in \Delta(Z)$, $\mu_1 \in \mathcal{M}(P_{\tilde{x}},\nu)$ and $\mu_2 \in \mathcal{M}(P_{\tilde{y}},\nu)$. By the Disintegration Theorem (see for instance Pollard (2002), Theorem 9, page 117), there are families of conditional probabilities μ_1^z and μ_2^z, $z \in Z$, such that $\mu_1 = \mu_1^z \otimes \nu$ and $\mu_2 = \mu_2^z \otimes \nu$. Define the probability $\gamma \in \Delta(\tilde{X}\times\tilde{Y}\times Z)$ by

$$\int_{\tilde{X}\times\tilde{Y}\times Z} F(\tilde{x},\tilde{y},z)d\gamma(\tilde{x},\tilde{y},z) = \int_{\tilde{X}\times\tilde{Y}\times Z} F(\tilde{x},\tilde{y},z)d\mu_1^z(\tilde{x})d\mu_2^z(\tilde{y})d\nu(z),$$

for each $F \in C^0(\tilde{X},\tilde{Y},Z)$. By construction, the projection μ of γ on $\tilde{X}\times\tilde{Y}$ belongs to $\mathcal{M}(P_{\tilde{x}},P_{\tilde{y}})$. We therefore have:

$$\begin{aligned}
\int_{\tilde{X}\times Z} U(\tilde{x},z)d\mu_1(\tilde{x},z) - \int_{\tilde{Y}\times Z} C(\tilde{y},z)d\mu_2(\tilde{y},z) &= \int_{\tilde{X}\times\tilde{Y}\times Z} [U(\tilde{x},z) - C(\tilde{y},z)]\, d\gamma(\tilde{x},\tilde{y},z) \\
&\le \int_{\tilde{X}\times\tilde{Y}\times Z} \sup_{z\in Z} [U(\tilde{x},z) - C(\tilde{y},z)]\, d\gamma(\tilde{x},\tilde{y},z) \\
&= \int_{\tilde{X}\times\tilde{Y}} \sup_{z\in Z} [U(\tilde{x},z) - C(\tilde{y},z)]\, d\mu(\tilde{x},\tilde{y}).
\end{aligned}$$

Since ν, μ_1 and μ_2 are arbitrary, the latter sequence of displays shows that the value of program (12) is smaller than or equal to the value of program (13) below.

$$\sup_{\mu\in\mathcal{M}(P_{\tilde{x}},P_{\tilde{y}})} \int_{\tilde{X}\times\tilde{Y}} \sup_{z\in Z} [U(\tilde{x},z) - C(\tilde{y},z)]\, d\mu(\tilde{x},\tilde{y}). \tag{13}$$

By Assumption EC, Program (13) is attained by the projection $\bar{\mu}$ on $\tilde{X}\times\tilde{Y}$ of the equilibrium probability distribution γ. Let μ_1 and μ_2 be the projections of the equilibrium probability distribution γ onto $\tilde{X}\times Z$ and $\tilde{Y}\times Z$. Now, since the value of (12) is smaller than or equal to the value of (13), and the latter is attained by $\bar{\mu}$, we have (12) smaller than or equal to

$$\begin{aligned}
&\int_{\tilde{X}\times\tilde{Y}} \sup_{z\in Z} [U(\tilde{x},z) - C(\tilde{y},z)]\, d\bar{\mu}(\tilde{x},\tilde{y}) \\
=\; &\int_{\tilde{X}\times Z} U(\tilde{x},z)d\mu_1(\tilde{x},z) + \int_{\tilde{Y}\times Z} [-C(\tilde{y},z)]\, d\mu_2(\tilde{y},z) \\
\le\; &\sup_{\mu_1\in\mathcal{M}(P_{\tilde{x}},P_z)} \int_{\tilde{X}\times Z} U(\tilde{x},z)d\mu_1(\tilde{x},z) + \sup_{\mu_2\in\mathcal{M}(P_{\tilde{y}},P_z)} \int_{\tilde{Y}\times Z} [-C(\tilde{y},z)]\, d\mu_2(\tilde{y},z)
\end{aligned}$$

which is smaller than or equal to (12), so that equality holds throughout, and P_z solves (12).

Step 2: We then show by contradiction that for P_x almost every x, $P_{z|x}$ achieves

$$\max_{\nu}[T_U(P_{\varepsilon|x},\nu)+T_{-C}(P_{\tilde{y}|x},\nu)]. \tag{14}$$

Assume the conclusion is false; that is, that there is some set $E\subset X$, with $P_x(E)>0$ such that, for each x in E there is some probability distribution $P'_{z|x}$ on Z such that

$$T_U(P_{\varepsilon|x},P'_{z|x})+T_{-C}(P_{\tilde{y}|x},P'_{z|x})>T_U(P_{\varepsilon|x},P_{z|x})+T_{-C}(P_{\tilde{y}|x},P_{z|x}).$$

For $x\notin E$, we set $P'_{z|x}=P_{z|x}$. This means that for every x, $P'_{z|x}$ is defined and we have

$$T_U(P_{\varepsilon|x},P'_{z|x})+T_{-C}(P_{\tilde{y}|x},P'_{z|x})\geq T_U(P_{\varepsilon|x},P_{z|x})+T_{-C}(P_{\tilde{y}|x},P_{z|x});$$

moreover, the inequality is strict on a set E of positive P_x measure, so that

$$\int_X[T_U(P_{\varepsilon|x},P'_{z|x})+T_{-C}(P_{\tilde{y}|x},P'_{z|x})]dP_x>\int_X[T_U(P_{\varepsilon|x},P_{z|x})+T_{-C}(P_{\tilde{y}|x},P_{z|x})]dP_x. \tag{15}$$

We define a new probability distribution P'_z on Z by $P'_z(A)=\int_X P'_{z|x}(A)dP_x(x)$, for each Borel set $A\subseteq Z$. We claim that

$$T_U(P_{\tilde{x}},P'_z)+T_{-C}(P_{\tilde{y}},P'_z)>T_U(P_{\tilde{x}},P_z)+T_{-C}(P_{\tilde{y}},P_z),$$

which contradicts the maximality of P_z in (12) established in Step 1. Let $P'_{\varepsilon z|x}$ achieve the optimal transportation between $P_{\varepsilon|x}$ and $P'_{z|x}$; that is,

$$T_U(P_{\varepsilon|x},P'_{z|x})=\int_{\mathbb{R}^{d_\varepsilon}\times Z}U(x,\varepsilon,z)dP_{\varepsilon z|x}.$$

Existence of such a $P_{\varepsilon z|x}$ is given by Theorem 6. By construction, the marginals of $P'_{\varepsilon z|x}$ are $P_{\varepsilon|x}$ and $P'_{z|x}$. We define $P'_{\tilde{x}z}$ on $\tilde{X}\times Z$ for any Borel subset $B\subseteq\tilde{X}\times Z$ by $P'_{\tilde{x}z}(B):=\int_X P'_{\varepsilon z|x}(B_x)dP_x$, where we define $B_x:=\{(\varepsilon,z)\in\mathbb{R}^{d_z}\times Z:\ (x,\varepsilon,z)\in B\}$. Then, for any Borel subset $A\subseteq Z$,

$$P'_{\tilde{x}z}(\mathbb{R}^{d_z}\times X\times A)=\int_X P'_{\varepsilon z|x}(\mathbb{R}^{d_z}\times A)dP_x=\int_X P'_{z|x}(A)dP_x=P'_z(A).$$

Similarly, for any Borel subset $B\subseteq\mathbb{R}^{d_z}\times X$,

$$P'_{\tilde{x}z}(B\times Z)=\int_X P'_{\varepsilon z|x}(B_x\times Z)dP_x=\int_X P_{\varepsilon|x}(B_x)dP_x=P_{\tilde{x}}(B).$$

The last two calculations show that $P_{\tilde{x}}$ and P'_z are the $\tilde{x}$ and z marginals of $P'_{\tilde{x}z}$, respectively, and so it follows by definition that

$$T_U(P_{\tilde{x}},P'_z)\geq\int_{X\times\mathbb{R}^{d_z}\times Z}U(\tilde{x},z)dP'_{\tilde{x}z}. \tag{16}$$

Similarly, we let $P'_{\tilde{y}z|x}$ achieve the optimal transportation between $P_{\tilde{y}|x}$ and $P'_{z|x}$ and define $P'_{\tilde{y}z}$ on $\tilde{Y}\times Z$ by $P'_{\tilde{y}z}=\int_X P'_{\tilde{y}z|x}dP_x$; a similar argument to above shows that the $\tilde{y}$ and z marginals of $P'_{\tilde{y}z}$ are $P_{\tilde{y}}$ and P'_z, respectively. Therefore,

$$T_{-C}(P_{\tilde{y}},P'_z)\geq\int_{\tilde{Y}\times Z}[-C(\tilde{y},z)]\,dP'_{\tilde{y}z}. \tag{17}$$

Now, equations (16) and (17) yield

$$\begin{aligned}
T_U(P_{\tilde{x}},P'_z)+T_{-C}(P_{\tilde{y}},P'_z) &\geq \int_{\tilde{X}\times Z} U(\tilde{x},z)dP'_{\tilde{x}z}+\int_{\tilde{Y}\times Z}[-C(\tilde{y},z)]\,dP'_{\tilde{y}z} \\
&= \int_X\left[\int_{\mathbb{R}^{d_z}\times Z} U(x,\varepsilon,z)dP'_{\varepsilon z|x}\right]dP_x+\int_X\left[\int_{\tilde{Y}\times Z}[-C(\tilde{y},z)]\,dP'_{\tilde{y}z|x}\right]dP_x \\
&= \int_X T_U(P_{\varepsilon|x},P'_{z|x})dP_x+\int_X[T_{-C}(P_{\tilde{y}|x},P'_{z|x})dP_x \\
&> \int_X T_U(P_{\varepsilon|x},P_{z|x})dP_x+\int_X[T_{-C}(P_{\tilde{y}|x},P_{z|x})dP_x \\
&\geq \int_X\left[\int_{\mathbb{R}^{d_z}\times Z} U(x,\varepsilon,z)dP_{\varepsilon z|x}\right]dP_x+\int_X\left[\int_{\tilde{Y}\times Z}[-C(\tilde{y},z)]\,dP_{\tilde{y}z|x}\right]dP_x \\
&= \int_{\tilde{X}\times Z} U(\tilde{x},z)dP_{\tilde{x}z}+\int_{\tilde{Y}\times Z}[-C(\tilde{y},z)]\,dP_{\tilde{y}z} \\
&= T_U(P_{\tilde{x}},P_z)+T_{-C}(P_{\tilde{y}},P_z),
\end{aligned}$$

where we have used (15) in the fourth line above. This establishes the contradiction and completes the proof.

Step 3: We finally now show that for P_x almost all x, $P_{z|x}$ is the unique solution of Program (14). This step follows the method of proof of Proposition 4 in Carlier and Ekeland (2010). We know from Step 2 that $P_{z|x}$ is a solution to (14). Suppose $\nu\in\Delta(Z)$ also solves (14). Let $\mu_{\varepsilon z|x}$ and $\mu_{\tilde{y}z|x}$ achieve $T_U(P_{\varepsilon|x},\nu)$ and $T_{-C}(P_{\tilde{y}|x},\nu)$. The latter exist by Theorem 6. We therefore have

$$\int_{\mathbb{R}^{d_z}\times Z} U(x,\varepsilon,z)d\mu_{\varepsilon z|x}(\varepsilon,z|x)+\int_{\tilde{Y}\times Z}[-C(\tilde{y},z)]d\mu_{\tilde{y}z|x}(\tilde{y},z|x)=T_U(P_{\varepsilon|x},\nu)+T_{-C}(P_{\tilde{y}|x},\nu)=(14).$$

Let φ_x^U denote the U-conjugate of a function φ on Z, as defined by $\varphi_x^U(\varepsilon):=\sup_{z\in Z}\{U(x,\varepsilon,z)-\varphi(z)\}$. Similarly, let $(-\varphi)^{-C}$ be the $(-C)$-conjugate of $(-\varphi)$, defined as $(-\varphi)^{-C}(\tilde{y}):=\sup_{z\in Z}\{-C(\tilde{y},z)+\varphi(z)\}$. By definition, for any function φ on Z, and any $(\varepsilon,\tilde{y},z)$ on $\mathbb{R}^{d_z}\times\tilde{Y}\times Z$, we have

$$\varphi_x^U(\varepsilon)+\varphi(z)\geq U(x,\varepsilon,z),\ \text{and}\ [-\varphi]^{-C}(\tilde{y})-\varphi(z)\geq -C(\tilde{y},z). \tag{18}$$

By Assumption EC, the price function p of the hedonic equilibrium satisfies

$$p_x^U(\varepsilon)+p(z)=U(x,\varepsilon,z),P_{\varepsilon z|x}\text{-a.s. and } [-p]^{-C}(\tilde{y})-p(z)=-C(\tilde{y},z),P_{\tilde{y}z|x}\text{-almost surely.} \tag{19}$$

Therefore, p achieves the minimum in the program

$$\begin{aligned}
&\inf_\varphi\left\{\int_{\mathbb{R}^{d_z}\times Z}[\varphi_x^U(\varepsilon)+\varphi(z)]dP_{\varepsilon z|x}(\varepsilon,z|x)+\int_{\tilde{Y}\times Z}[(-\varphi)^{-C}(\tilde{y})-\varphi(z)]dP_{\tilde{y}z|x}(\tilde{y},z|x)\right\} \\
=\ &\inf_\varphi\left\{\int_{\mathbb{R}^{d_z}}\varphi_x^U(\varepsilon)dP_{\varepsilon|x}(\varepsilon|x)+\int_Z\varphi(z)dP_{z|x}(z|x)+\int_{\tilde{Y}}(-\varphi)^{-C}(\tilde{y})dP_{\tilde{y}|x}(\tilde{y}|x)-\int_Z\varphi(z)dP_{z|x}(z|x)\right\} \\
=\ &\inf_\varphi\left\{\int_{\mathbb{R}^{d_z}}\varphi_x^U(\varepsilon)dP_{\varepsilon|x}(\varepsilon|x)+\int_{\tilde{Y}}(-\varphi)^{-C}(\tilde{y})dP_{\tilde{y}|x}(\tilde{y}|x)\right\}.
\end{aligned} \tag{20}$$

Since p achieves (20), we have

$$\begin{aligned}(20) \quad &= \int_{\mathbb{R}^{d_z}} p_x^U(\varepsilon) dP_{\varepsilon|x}(\varepsilon|x) + \int_{\tilde{Y}} (-p)^{-C}(\tilde{y}) dP_{\tilde{y}|x}(\tilde{y}|x) \\ &= \int_{\mathbb{R}^{d_z}\times Z} [p_x^U(\varepsilon) + p(z)] d\mu_{\varepsilon z|x}(\varepsilon, z|x) + \int_{\tilde{Y}\times Z} [(-p)^{-C}(\tilde{y}) - p(z)] d\mu_{\tilde{y}z|x}(\tilde{y}, z|x),\end{aligned}$$

since $\mu_{\varepsilon z|x} \in \mathcal{M}(P_{\varepsilon|x}, \nu)$ and $\mu_{\tilde{y}z|x} \in \mathcal{M}(P_{\tilde{y}|x}, \nu)$ by construction. By the strong duality result in Theorem 3 of Carlier and Ekeland (2010), we have (14)=(20). Hence

$$\begin{aligned}&\int_{\mathbb{R}^{d_z}\times Z} U(x, \varepsilon, z) d\mu_{\varepsilon z|x}(\varepsilon, z|x) + \int_{\tilde{Y}\times Z} [-C(\tilde{y}, z)] d\mu_{\tilde{y}z|x}(\tilde{y}, z|x) \\ &\qquad = \int_{\mathbb{R}^{d_z}\times Z} [p_x^U(\varepsilon) + p(z)] d\mu_{\varepsilon z|x}(\varepsilon, z|x) + \int_{\tilde{Y}\times Z} [(-p)^{-C}(\tilde{y}) - p(z)] d\mu_{\tilde{y}z|x}(\tilde{y}, z|x),\end{aligned}$$

which, given the inequalities (18), implies

$$p_x^U(\varepsilon) + p(z) = U(x, \varepsilon, z), \mu_{\varepsilon z|x} \text{ almost surely, and } [-p]^{-C}(\tilde{y}) - p(z) = -C(\tilde{y}, z), \mu_{\tilde{y}z|x} \text{ almost surely.}$$

We therefore have $p_x^U(\varepsilon) + p(z) = U(x, \varepsilon, z)$ both $P_{\varepsilon z|x}$ and $\mu_{\varepsilon z|x}$ almost surely. The U-conjugate p_x^U of p is locally Lipschitz by Lemma C.1 of Gangbo and McCann (1996), hence differentiable $P_{\varepsilon|x}$-almost everywhere by Rademacher's Theorem (see for instance Villani (2009), Theorem 10.8(ii)). We can therefore apply the Envelope Theorem to obtain the equation below.

$$\nabla p_x^U(\varepsilon) = \nabla_\varepsilon U(x, \varepsilon, z) = \nabla_\varepsilon \zeta(x, \varepsilon, z), \text{ both } P_{\varepsilon z|x} \text{ and } \mu_{\varepsilon z|x} \text{ almost everywhere.} \tag{21}$$

By Assumption R3(1), $\nabla_\varepsilon \zeta(x, \varepsilon, z)$ is injective as a function of z. Therefore, for each ε, there is a unique z that satisfies (21). This defines a map $T : \mathbb{R}^{d_z} \to Z$, such that $z = T(\varepsilon)$ both $P_{\varepsilon z|x}$ and $\mu_{\varepsilon z|x}$ almost everywhere. Since the projections of $P_{\varepsilon z|x}$ and $\mu_{\varepsilon z|x}$ with respect to ε are the same, namely $P_{\varepsilon|x}$, we therefore have

$$P_{\varepsilon z|x} = (id, T)\# P_{\varepsilon|x} = \mu_{\varepsilon z|x}.$$

The two probability distributions $P_{\varepsilon z|x}$ and $\mu_{\varepsilon z|x}$ being equal, they must also share the same projection with respect to z and $\nu = P_{z|x}$ as a result.

Armed with the results in Steps 1 to 3, we are ready to prove Lemma 4. Fix $x \in X$ such that $P_{z|x}$ is the unique solution to Program (14). Let $P_{\tilde{y}}^N$ be a sequence of discrete probability distributions with N points of support on $\tilde{Y} \subseteq \mathbb{R}^{d_{\tilde{y}}}$ converging weakly to $P_{\tilde{y}|x}$. The set of probability distributions on the compact set Z is compact relative to the topology of weak convergence. By Assumption R3, U and C are continuous, hence, by Theorem 5.20 of Villani (2009), the functional $\nu \mapsto T_U(P_{\varepsilon|x}, \nu) + T_{-C}(P_{\tilde{y}}^N, \nu)$ is continuous with respect to the topology of weak convergence. Program $\max_{\nu \in \Delta(Z)} [T_U(P_{\varepsilon|x}, \nu) + T_{-C}(P_{\tilde{y}}^N, \nu)]$, therefore, has a solution we denote P_z^N.

We first show that P_z^N converges weakly to $P_{z|x}$. For any probability measure ν on Z, we have

$$T_U(P_{\varepsilon|x}, \nu) + T_{-C}(P_{\tilde{y}}^N, \nu) \leq T_U(P_{\varepsilon|x}, P_z^N) + T_{-C}(P_{\tilde{y}}^N, P_z^N). \tag{22}$$

Since Z is compact, $\Delta(Z)$ is compact with respect to weak convergence. Hence, we can extract from (P_z^N) a convergent subsequence, which we also denote (P_z^N), as is customary, and we call the limit $\bar{P}$. By the stability of optimal transport (Villani (2009), Theorem 5.20), we have $T_{-C}(P_{\tilde{y}}^N, \nu) \to T_{-C}(P_{\tilde{y}|x}, \nu)$, $T_U(P_{\varepsilon|x}, P_z^N) \to T_U(P_{\varepsilon|x}, \bar{P})$ and $T_{-C}(P_{\tilde{y}}^N, \bar{P}) \to T_{-C}(P_{\tilde{y}|x}, \bar{P})$, and so passing to the limit in inequality (22) yields

$$T_U(P_{\varepsilon|x}, \nu) + T_{-C}(P_{\tilde{y}|x}, \nu) \leq T_U(P_{\varepsilon|x}, \bar{P}) + T_{-C}(P_{\tilde{y}|x}, \bar{P}).$$

As this holds for any probability distribution ν on $\mathbb{R}^{d_z}$, it implies that $\bar{P}$ is optimal in Program (14). By uniqueness proved in Step 3, we then have $\bar{P} = P_{z|x}$, as desired.

We are now ready to complete the proof of Lemma 4. Combining Steps 1 to 3 above, we know that P_z^N is the marginal with respect to z of a hedonic equilibrium distribution γ^N on $\mathbb{R}^{d_z} \times Z \times \{\tilde{y}_1, \ldots, \tilde{y}_N\}$, with consumer and producer distributions $P_{\varepsilon|x}$ and $P_{\tilde{y}}^N$, respectively. By Step 1 in the proof of Theorem 4(1) applied to this hedonic equilibrium with producer type distribution $P_{\tilde{y}}^N$, for each N, there is an optimal map F_N pushing P_z^N forward to $P_{\varepsilon|x}$ i.e., $P_{\varepsilon|x} = F_N \# P_z^N$. For $i = 1, 2, ...N$, we define the subsets $S_i^N \subseteq Z$ by

$$S_i^N = \{z \in Z : \; z \in \arg\max_w (U(x, F_N(z), w) - C(\tilde{y}_i, w))\};$$

note that S_i^N is the set of quality vectors that are produced by producer type $\tilde{y}_i$. Since $P_{\tilde{y}}^N$ has finite support $\{\tilde{y}_1, \ldots, \tilde{y}_N\}$, P_z^N almost all z belong to some S_i, $i = 1, \ldots, N$. We then set

$$E_i^N = \{z \in S_i^N \text{ and } z \notin S_j^N \text{ for all } j < i\},$$

for each $i = 1, \ldots, N$, with the convention $S_0^N = \varnothing$. The E_i^N are disjoint, and $E^N = \cup_{i=1}^N E_i^N$ has full P_z^N measure, $P_z^N(E^N) = 1$. On each E_i^N, F_N coincides with a map G_i, which satisfies

$$\nabla_z U(x, G_i(z), z) - \nabla_z C(\tilde{y}_i, z) = 0$$

or

$$\nabla_z \bar{U}(x, z) + \nabla_z \zeta(x, G_i(z), z) - \nabla_z C(\tilde{y}_i, z) = 0.$$

By Assumption R3(3) the Implicit Function Theorem, G_i is differentiable and we have

$$[D^2_{zz}\bar{U}(x, z) + D^2_{zz}\zeta(x, G_i(z), z) - D^2_{zz}C(\tilde{y}_i, z)] + D^2_{z\varepsilon}\zeta(x, G_i(z), z) DG_i(z) = 0$$

so that

$$DG_i(z) \quad = \quad [D^2_{z\varepsilon}\zeta(x, G_i(z), z)]^{-1}[D^2_{zz}\bar{U}(x, z) + D^2_{zz}\zeta(x, G_i(z), z) - D^2_{zz}C(\tilde{y}_i, z)].$$

Therefore, $|DG_i(z)| \leq M_0 M_1 := C$ by Assumptions R2(1) and R3(3). Now, this implies that G_i is Lipschitz with constant C, and therefore, F_N restricted to E_i^N is also Lipschitz with constant C.

Now, for any Borel $A \subset \mathbb{R}^{d_z}$, we can write $A \cap E^N = \cup_{i=1}^N (A \cap E_i^N)$. Therefore,

$$
\begin{aligned}
P_z^N(A) = P_z^N(A \cap E^N) &\leq P_z^N(F_N^{-1}(F_N(A \cap E^N))) \\
&= P_{\varepsilon|x}(F_N(A \cap E^N)) \\
&= P_{\varepsilon|x}(F_N(\cup_{i=1}^N (A \cap E_i^N)) \\
&= P_{\varepsilon|x}(\cup_{i=1}^N F_N(A \cap E_i^N)). \qquad (23)
\end{aligned}
$$

Denote by $|A|$ the Lebesgue measure of a set A in the rest of this proof. We now show the absolute continuity of of $P_{z|x}$ by contradiction. Assume not; then there is a set A with $|A| = 0$ but $\delta = P_{z|x}(A) > 0$. We can choose open neighbourhoods, $A \subseteq A_k$, with $|A_k| \leq \frac{1}{k}$; by weak convergence of P_z^N to $P_{z|x}$, we have

$$\liminf_{N \to \infty} P_z^N(A_k) \geq P_{z|x}(A_k) \geq \delta$$

and so for sufficiently large N, we have $P_z^N(A_k) \geq \delta/2$. On the other hand, by the Lipschitz property of F_N on E_i^N, we have

$$|F_N(A_k \cap E_i^N)| \leq C^{d_z} |A_k \cap E_i^N|,$$

so that

$$| \cup_{i=1}^N F_N(A_k \cap E_i^N)| \leq C^{d_z} \sum_{i=1}^N |A_k \cap E_i^N| = C^{d_z} |A_k \cap E^N| \leq C^{d_z} |A_k| \leq \frac{C^{d_z}}{k}.$$

Now, $P_{\varepsilon|x}$ is absolutely continuous, so that $P_{\varepsilon|x}(\cup_{i=1}^N F_N(A_k \cap E_i^N)) \to 0$ as $k \to \infty$ (because $| \cup_{i=1}^N F_N(A_k \cap E_i^N)| \to 0$ as $k \to \infty$). On the other hand, by (23), $P_{\varepsilon|x}(\cup_{i=1}^N F_N(A_k \cap E_i^N)) \geq P_z^N(A_k) \geq \frac{\delta}{2}$ for all k, which is a contradiction, completing the proof. □

Proof of Theorems 4 and 5. Step 1: identification of inverse demand. For a fixed observable type x, assume that the types $\tilde{x}_0 := (x, \varepsilon_0)$ and $\tilde{x}_1 := (x, \varepsilon_1)$ both choose the same good, $\bar{z} \in Z$, from producers $\tilde{y}_0$ and $\tilde{y}_1$, respectively.

We want to prove that this implies the unobservable types are also the same; that is, that $\varepsilon_0 = \varepsilon_1$. This property is equivalent to having a map from the good qualities Z to the unobservable types, for each fixed observable type.

Note that $\bar{z}$ must maximize the joint surplus for both ε_0 and ε_1. That is, setting

$$S(x, \varepsilon, \tilde{y}) = \sup_{z \in Z} [\bar{U}(x, z) + \zeta(x, \varepsilon, z) - C(\tilde{y}, z)]$$

we have,

$$S(x, \varepsilon_0, \tilde{y}_0) = \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_0, \bar{z}) \qquad (24)$$

and

$$S(x, \varepsilon_1, \tilde{y}_1) = \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_1, \bar{z}). \qquad (25)$$

By Assumption EC, we can apply Lemma 1 of Chiappori, McCann, and Nesheim (2010), so that the pair of indirect utilities (V, W), where

$$\begin{aligned} V(\tilde{x}) &= \sup_{z \in Z} \left(U(\tilde{x}, z) - p(z) \right) \\ W(\tilde{y}) &= \sup_{z \in Z} \left(p(z) - C(\tilde{y}, z) \right), \end{aligned}$$

achieve the dual (DK) of the optimal transportation problem

$$\sup_{\pi \in \mathcal{M}\left(P_{\tilde{x}}, P_{\tilde{y}}\right)} \int S(\tilde{x}, \tilde{y}) d\pi(\tilde{x}, \tilde{y}),$$

with solution π. This implies, from Theorem 7, that for π-almost all pairs $(\tilde{x}_0, \tilde{y}_0)$ and $(\tilde{x}_1, \tilde{y}_1)$,

$$\begin{aligned} V(\tilde{x}_0) + W(\tilde{y}_0) &= S(\tilde{x}_0, \tilde{y}_0), \\ V(\tilde{x}_1) + W(\tilde{y}_1) &= S(\tilde{x}_1, \tilde{y}_1), \\ V(\tilde{x}_0) + W(\tilde{y}_1) &\geq S(\tilde{x}_0, \tilde{y}_1), \\ V(\tilde{x}_1) + W(\tilde{y}_0) &\geq S(\tilde{x}_1, \tilde{y}_0). \end{aligned}$$

We therefore deduce the condition (called the 2-monotonicity condition):

$$S(x, \varepsilon_0, \tilde{y}_0) + S(x, \varepsilon_1, \tilde{y}_1) \geq S(x, \varepsilon_1, \tilde{y}_0) + S(x, \varepsilon_0, \tilde{y}_1),$$

recalling that $\tilde{x}_0 = (x, \varepsilon_0)$ and $\tilde{x}_1 = (x, \varepsilon_1)$. Now, by definition of S as the maximized surplus, we have

$$S(x, \varepsilon_1, \tilde{y}_0) \geq \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_0, \bar{z}) \tag{26}$$

and

$$S(x, \varepsilon_0, \tilde{y}_1) \geq \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_1, \bar{z}). \tag{27}$$

Inserting this, as well as (24) and (25) into the 2-monotonicity inequality yields

$$\begin{aligned} \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_0, \bar{z}) \quad &+ \quad \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_1, \bar{z}) \\ &\geq \quad S(x, \varepsilon_1, \tilde{y}_0) + S(x, \varepsilon_0, \tilde{y}_1) \\ &\geq \quad \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_0, \bar{z}) + \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_1, \bar{z}). \end{aligned}$$

But the left and right hand sides of the preceding string of inequalities are identical, so we must have equality throughout. In particular, we must have equality in (26) and (27). Equality in (26), for example, means that $\bar{z}$ maximizes $z \mapsto \bar{U}(x, z) + \zeta(x, \varepsilon_1, z) - C(\tilde{y}_0, z)$, and so, as $\bar{z}$ is in the interior of Z by Assumption EC, we have

$$\nabla_z \zeta(x, \varepsilon_1, \bar{z}) = \nabla_z C(\tilde{y}_0, \bar{z}) - \nabla_z \bar{U}(x, \bar{z}). \tag{28}$$

Since $\bar{z}$ also maximizes $z \mapsto \bar{U}(x, z) + \zeta(x, \varepsilon_0, z) - C(\tilde{y}_0, z)$, we also have

$$\nabla_z \zeta(x, \varepsilon_0, \bar{z}) = \nabla_z C(\tilde{y}_0, \bar{z}) - \nabla_z \bar{U}(x, \bar{z}). \tag{29}$$

Equations (28) and (29) then imply

$$\nabla_z \zeta(x, \varepsilon_1, \bar{z}) = \nabla_z \zeta(x, \varepsilon_0, \bar{z})$$

and Assumption S3 implies $\varepsilon_1 = \varepsilon_0$.

Step 2: Differentiability of V in z. In this step, we work under Assumptions EC, H, R1, S3 and C3. The proof under Assumptions EC, H, S3 and R3 is presented in Step 2 alternative below. The method of proof of Step 2 is to prove that the subdifferential at each z_0 is a singleton, which is equivalent to differentiability at z_0.

Definition 5 (Subdifferential)**.** *The* subdifferential $\partial\psi(x_0)$ *of a function* $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *at* $x_0 \in \mathbb{R}^d$ *is the set of vectors* $p \in \mathbb{R}^d$*, called* subgradients*, such that* $\psi(x) - \psi(x_0) \geq p'(x - x_0) + o(\|x - x_0\|)$.

Remark 13. *The subdifferential is always closed and convex. If f is continuous, its subdifferential is also bounded.*

From Assumption C3, $V(x, z)$ is ζ-convex, and hence locally semiconvex, by Proposition C.2 in Gangbo and McCann (1996). We recall the definition of local semiconvexity from the latter paper.

Definition 6 (Local semiconvexity)**.** *A function* $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *is called* locally semiconvex *at* $x_0 \in \mathbb{R}^d$ *if there is a scalar* $\lambda > 0$ *such that* $\psi(x) + \lambda\|x\|^2$ *is convex on some open ball centered at* x_0.

Since the term $\lambda\|x\|^2$ in the definition of local semiconvexity simply shifts the subdifferential by $2\lambda x$, we can extend Theorem 25.6 in Rockafellar (1970) to locally semiconvex functions and obtain the following lemma.

Lemma 5. *Let* $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *be a locally semiconvex function, and suppose that* $q \in \mathbb{R}^d$ *is an extremal point in the subdifferential* $\partial\psi(x_0)$ *of* ψ *at* x_0*. Then there exists a sequence* x_n *converging to* x_0*, such that* ψ *is differentiable at each* x_n *and the gradient* $\nabla\psi(x_n)$ *converges to* q*.*

Now, Step 1 shows that for each fixed z, the set

$$\{\varepsilon \in \mathbb{R}^{d_z} : V(x, z) + V^\zeta(x, \varepsilon) = \zeta(x, \varepsilon, z)\} := \{f(z)\} \tag{30}$$

is a singleton. We claim that this means V is differentiable with respect to z everywhere. Fix a point z_0. We will prove that the subdifferential $\partial_z V(x, z_0)$ contains only one extremal point (for a definition, see Rockafellar (1970), Section 18). This will yield the desired result. Indeed, the subdifferential of V is closed and convex by Remark 13. By Assumption C3, V is ζ-convex, hence continuous, by the combination of Propositions C.2 and C.6(i) in Gangbo and McCann (1996). Hence, still by Remark 13, the subdifferential is also bounded. Hence, it is equal to the convex hull of its extreme points (see Rockafellar (1970), Theorem 18.5). The subdifferential of V at z_0 must therefore be a singleton, and V must be differentiable at z_0 (Theorem 25.1

in Rockafellar (1970) can be easily extended to locally semiconvex functions). Let q be any extremal point in $\partial_z V(x,z_0)$. Let z_n be a sequence satisfying the conclusion in Lemma 5. Now, as V is differentiable at each point z_n, we have the envelope condition

$$\nabla V_z(x,z_n) = \nabla_z \zeta(x,\varepsilon_n,z_n) \tag{31}$$

where $\varepsilon_n = f(z_n)$ is the unique point giving equality in (30).

As the sequence $\nabla_z \zeta(x,\varepsilon_n,z_n)$ converges, the growth condition in Assumption R3 implies that the ε_n remain in a bounded set. We can therefore pass to a convergent subsequence $\varepsilon_n \to \varepsilon_0$. By continuity of $\nabla_z \zeta$, we can pass to the limit in (31) and, recalling that the left hand side tends to q, we obtain $q = \nabla_z \zeta(x,\varepsilon_0,z_0)$. Now, by definition of ε_n, we have the equality $V(x,z_n) + V^\zeta(x,\varepsilon_n) = \zeta(x,\varepsilon_n,z_n)$. By Assumption C3, V and V^ζ are ζ-convex, hence continuous, by the combination of Propositions C.2 and C.6(i) in Gangbo and McCann (1996). Hence, we can pass to the limit to obtain $V(x,z_0) + V^\zeta(x,\varepsilon_0) = \zeta(x,\varepsilon_0,z_0)$. But this means $\varepsilon_0 = f(z_0)$, and so $q = \nabla_z \zeta(x,\varepsilon_0,z_0) = \nabla_z \zeta(x,f(z_0),z_0)$ is uniquely determined by z_0. This means that the subdifferential can only have one extremal point, completing the proof of differentiability of V.

Step 2 alternative: We now work under Assumptions EC, H, S3 and R3. By Lemma 3, we know that for each x, $V(x,z) = V^{\zeta\zeta}(x,z)$, $P_{z|x}$ almost everywhere, and $P_{z|x}$ is absolutely continuous with respect to Lebesgue measure, by Lemma 4. We will prove that V is therefore approximately differentiable $P_{z|x}$ almost everywhere, with $\nabla_{ap,z} V(x,z) = \nabla_z V^{\zeta\zeta}(x,z)$, where $\nabla_{ap,z} V(x,z)$ denotes the approximate gradient of $V(x,z)$ with respect to z.

Definition 7 (Approximate differentiability)**.** *If we have*

$$\lim_{r\downarrow 0} \frac{|B_r(x)\cap E|}{|B_r(x)|} = 1,$$

where $|B|$ is Lebesgue measure of B, $B_r(x)$ is an open ball of radius r centered at x, and $E \subseteq \mathbb{R}^d$ is a measurable set, x is called density point *of E. Let x_0 be a density point of a measurable set $E \subseteq \mathbb{R}^d$ and let $f: E \to \mathbb{R}$ be a measurable map. If there is a linear map $A: \mathbb{R}^d \to \mathbb{R}$ such that, for each $\eta > 0$, x_0 is a density point of*

$$\left\{ x \in E:\ -\eta \le \frac{f(x) - f(x_0) - A(x-x_0)}{\|x-x_0\|} \le \eta \right\},$$

then f is said to be approximately differentiable at x_0*, and A is called the* approximate gradient *of f at x_0.*

Remark 14. *The approximate differential is uniquely defined, as shown below Definition 10.2 page 218 of Villani (2009).*

By Lemma 3, we have $V = V^{\zeta\zeta}$, $P_{z|x}$ almost everywhere. Moreover, as a ζ conjugate, $V^{\zeta\zeta}$ is locally Lipschitz by Lemma C.1 of Gangbo and McCann (1996), hence differentiable $P_{z|x}$-almost everywhere by Rademacher's Theorem (see for instance Villani (2009), Theorem 10.8(ii)). Hence, there exists a set S of full $P_{z|x}$ measure such that, for each $z_0 \in S$,

(1) $V^{\zeta\zeta}(x,z)$ is differentiable with respect to z at z_0.
(2) $V(x,z_0) = V^{\zeta\zeta}(x,z_0)$.

By Lebesgue's density theorem (still denoting Lebesgue measure of B by $|B|$),

$$\lim_{r\downarrow 0} \frac{|S\cap B_r(z)|}{|B_r(z)|} = 1$$

for Lebesgue almost every $z \in Z$, hence also for $P_{z|x}$ almost every z, by the absolute continuity of $P_{z|x}$. Since S has $P_{z|x}$ measure 1, the set $\bar{S}$ of density points of S,

$$\bar{S} = \left\{ z\in S : \lim_{r\downarrow 0} \frac{|S\cap B_r(z)|}{|B_r(z)|} = 1 \right\},$$

therefore has $P_{z|x}$ measure 1. Take any density point z_0 of S, i.e., $z_0 \in \bar{S}$. Fix $\eta > 0$. Since $z_0 \in \bar{S} \subseteq S$, $V^{\zeta\zeta}$ is differentiable at z_0. Hence there is $r > 0$ such that for all $z \in B_r(z_0)$,

$$-\eta \le \frac{V^{\zeta\zeta}(z) - V^{\zeta\zeta}(z_0) - \nabla_z V^{\zeta\zeta}(z_0)\cdot(z-z_0)}{\|z-z_0\|} \le \eta.$$

Since $V^{\zeta\zeta} = V$ on S, for all $z \in B_r(z_0)\cap S$, we have

$$-\eta \le \frac{V(z) - V(z_0) - \nabla_z V^{\zeta\zeta}(z_0)\cdot(z-z_0)}{\|z-z_0\|} \le \eta.$$

Therefore $B_r(z_0)\cap S = B_r(z_0)\cap \tilde{S}$, where

$$\tilde{S} := \left\{ z\in S : -\eta \le \frac{V(z) - V(z_0) - \nabla_z V^{\zeta\zeta}(z_0)\cdot(z-z_0)}{\|z-z_0\|} \le \eta \right\}.$$

As z_0 is a density point of S,

$$\lim_{r\downarrow 0} \frac{|\tilde{S}\cap B_r(z_0)|}{|B_r(z_0)|} = \lim_{r\downarrow 0} \frac{|S\cap B_r(z_0)|}{|B_r(z_0)|} = 1,$$

so that z_0 is also a density point of $\tilde{S}$, which means, by definition, that V is approximately differentiable at z_0, and that its approximate gradient is $\nabla_{ap,z}V(x,z_0) = \nabla_z V^{\zeta\zeta}(x,z_0)$. The latter is true for any $z_0 \in \bar{S}$ and $\bar{S}$ has $P_{z|x}$ measure 1. Hence, V is approximately differentiable $P_{z|x}$ almost everywhere.

Step 3: Since by Step 2 or Step 2 alternative, $V(x,z)$ is approximately differentiable $P_{z|x}$ almost surely, and since $\bar{U}(x,z)$ is differentiable by assumption, $p(z) = V(x,z) - \bar{U}(x,z)$ is also approximately differentiable $P_{z|x}$ almost surely. Since, by Step 1, the inverse demand function $\varepsilon(x,z)$ is uniquely determined, the first order condition $\nabla_z\zeta(x,\varepsilon(x,z),z) = \nabla_{ap}p(z) - \nabla_z\bar{U}(x,z)$ identifies $\nabla_z\bar{U}(x,z)$, $P_{z|x}$ almost everywhere, as required. □

Proof of Theorem 4(2). In Part (1), we have shown uniqueness (up to location) of the pair (V, V^ζ) such that $V(x,z) + V^\zeta(x,\varepsilon) = \zeta(x,\varepsilon,z)$, π-almost surely. By Theorem 7, this implies that (V, V^ζ) is the unique (up to location) pair of ζ-conjugates that solves the dual Kantorovich problem as required.

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