On the Coexistence of Money and Higher-Return Assets and its Social Role*

Tai-Wei Hu
Northwestern University

Guillaume Rocheteau
University of California, Irvine

First version: December 2010.
This version: March 2013

Abstract

This paper adopts mechanism design to investigate the coexistence of fiat money and higher-return assets. We consider an economy with pairwise meetings where fiat money and risk-free capital compete as means of payment, as in Lagos and Rocheteau (2008). The trading mechanism in pairwise meetings is chosen among all individually rational, renegotiation-proof mechanisms to maximize society's welfare. We show that in any stationary monetary equilibrium capital commands a higher rate of return than fiat money.

JEL Classification: D82, D83, E40, E50
Keywords: money, capital, pairwise trades, rate-of-return dominance

*We thank Neil Wallace, an associate editor, and two anonymous referees for their comments on an earlier draft. We also benefited from the comments of seminar participants at Queen's University, the University of California at Davis and Santa Barbara, the University of Paris 2, the Federal Reserve Bank of Cleveland, the Federal Reserve Bank of Philadelphia, the Getulio Vargas Foundation, the second Summer Workshop in Macro Finance at Science-Po Paris, the Money, Banking, and Finance Summer Workshop at the Federal Reserve Bank of Chicago, and the West Coast Search-and-Matching Worskhop at UC Santa Cruz. We thank Monica Crabtree-Reusser for editorial assistance.
E-Mail: grochete@uci.edu.

1 Introduction

To paraphrase Banerjee and Maskin (1996), the coexistence of money and higher-return assets has always been something of an embarrassment to economic theory. Despite being a stubborn observation of monetary economies, it is not accounted for by standard economic paradigms. The dynamic general equilibrium models used for policy analysis evade the coexistence issue by either imposing cash-in-advance constraints or by adding money into the utility function.[1] Modern monetary theory is more careful in isolating the frictions that make fiat money essential (e.g., Kocherlakota, 1998), but these frictions do not appear to be sufficient to explain why economic agents hold both fiat money and capital goods that yield a positive rate of return. For instance, Wallace (1980) and Lagos and Rocheteau (2008) propose models in which fiat money and capital do compete as media of exchange, but find out that the two assets can coexist only if they have the same rate of return.[2] In Hicks's (1935) words, *"the critical question arises when we look for an explanation of the preference for holding money rather than capital goods. For capital goods will ordinarily yield a positive rate of return, which money does not."*

The objective of this paper is to apply mechanism design to an environment with pairwise meetings and multiple assets, similar to Lagos and Rocheteau (2008), in order to account for the coexistence of fiat money and higher-return capital. In contrast to the literature in which trading mechanisms in pairwise meetings are chosen arbitrarily, mechanism design focuses on socially optimal mechanisms taking as given key frictions, e.g., lack of commitment, limited enforcement, and lack of record keeping. This approach will allow us to isolate the properties of good allocations in monetary economies—such as rate-of-return dominance—from the ones resulting from (socially) inefficient trading mechanisms.

The environment to which we apply mechanism design is a two-asset economy version of Lagos and Wright (2005) with fiat money and capital. Agents trade alternatively in pairwise meetings, where a double-coincidence-of-wants problem creates a need for liquid assets, and in centralized meetings, where they have the opportunity to readjust their asset portfolios. Because of the lin-

[1] Such shortcuts are problematic, at best, as they introduce various hidden inconsistencies. See Wallace (1998).

[2] There are monetary models that obtain the coexistence of fiat money and higher-return capital goods by ruling out the use of capital, or claims on capital, as means of payment. Examples of such models include Shi (1999), Aruoba and Wright (2003), Molico and Zhang (2006), and Aruoba, Waller, and Wright (2011).

ear preferences in centralized meetings the model is tractable and the mechanism design problem manageable. Moreover, the rounds of pairwise meetings make the choice of a trading mechanism non-trivial as the set of Pareto-efficient trades within a meeting is non-degenerate, and it depends on agents' asset holdings.[3] Reciprocally, agents choose their portfolio of assets based on anticipated terms of trade in pairwise meetings. The main insight of the paper is that under an optimal trading mechanism, any stationary monetary equilibrium is such that capital commands a higher rate of return than fiat money.

We first show, in accordance with Lagos and Rocheteau (2008), that fiat money is essential when the economy faces a shortage of assets because the first-best capital stock does not provide enough means of payment to compensate producers for their costs in pairwise meetings.[4] If the shortage of capital is not too large, then a constant stock of fiat money supplements the capital stock and it allows the implementation of a first-best allocation. In this case the rate of return of capital is equal to the rate of time preference, which is larger than the rate of return of money.

If the shortage of capital is large, then individuals lack incentives to hold enough real balances to supplement the capital stock and achieve the first-best level of output in pairwise meetings. In this case the gains from trade in pairwise meetings are too small relative to the cost of holding real balances, as measured by the rate of time preference. As a result there is a trade-off between the role of capital as a liquid asset to finance consumption and its role as a costly input factor to generate future output. When trading off the inefficiently low output in pairwise meetings and the inefficiently low net output in centralized meetings the optimal mechanism can lead to over-accumulation of capital relative to the first best. Crucially, it never drives the rate-of-return of capital down to the rate of return of fiat money. If rates of return were equalized, then the mechanism designer could induce agents in centralized meetings to substitute some capital for additional real balances without violating individual rationality constraints, thereby generating a

[3]A similar analysis could be conducted in the context of the large-household model of Shi (1997). The tractability of the model comes at a cost: It shuts down the distributional effects of monetary policy. These distributional effects, however, do not play a role in the argument developed in this paper, and while models with a nondegenerate distribution of asset holdings can be solved numerically (e.g., Molico and Zhang, 2006), designing the optimal trading mechanism for this class of models is currently out of reach.

[4]The condition that the available supply of capital that can be used as medium of exchange is relatively scarce is empirically relevant if one takes a broad view of media of exchange as means of payment or collateral. See, e.g., Caballero (2006) and Geanakoplos and Zame (2010).

welfare improvement. A corollary of this result is that trading mechanisms that predict rate-of-return equality are suboptimal mechanisms.

Under the most commonly used trading protocols—axiomatic or strategic bargaining solutions in random matching models or Walrasian pricing in overlapping generation models—it is not individually rational to hold real balances if capital yields a positive rate of return. Indeed, these standard trading mechanisms treat real balances and capital as perfect substitutes for payment purposes, making it individually optimal to accumulate the assets with the highest rate of return. In contrast, in economies with pairwise meetings, the optimal mechanism specifies terms of trade that are contingent on the composition of agents' portfolios. As a result the buyer's surplus from a trade depends on whether he is paying with money, capital, or a combination of the two assets. We show that an optimal trading mechanism punishes agents who do not hold enough assets, or who accumulate too much capital, the highest-return asset, by selecting their least-preferred trade in the (pairwise) core.

Relative to a pure currency economy, the use of capital goods as means of payment can be welfare enhancing because the substitution of low-return assets (fiat money) with high-return ones (capital) relaxes individuals' participation constraints and hence allows them to hold a larger quantity of assets in pairwise meetings. An alternative way to relax agents' participation constraints is by engineering a positive rate of return for fiat money. To analyze this possibility we consider the case in which the money supply grows, or shrinks, at a constant rate. Under a socially optimal trading mechanism, the Friedman rule is not necessary to maximize society's welfare. There is a threshold for the inflation rate, below which the first-best allocation is implementable, and capital is unaffected by changes in the money growth rate, i.e., there is no Tobin effect. However, if inflation is sufficiently large, an increase in inflation reduces real balances and welfare, and it raises the aggregate capital stock.

The rest of the paper is organized as follows. We first review the literature on the coexistence of money and higher return assets in Section 1.1. Section 2 describes the environment. Section 3 determines the set of stationary, incentive-feasible allocations. The optimal, incentive-feasible allocation and the main result in terms of rate-of-return dominance appear in Section 4. The relationship between inflation and capital accumulation is studied in Section 5. Finally, a comparison between the optimal mechanism derived in this paper and other trading mechanisms used in the literature is provided in Section 6.

1.1 Literature

There are several approaches to explain rate-of-return differences across assets with similar risk characteristics. In the following we review some of them succinctly.

Legal restrictions A first approach to rationalize the coexistence of interest-bearing bonds and fiat money is to assume that bonds exist in large denominations and the government places restrictions on intermediation activities to transform these bonds into smaller-denomination ones (e.g., Wallace, 1983). In the context of random matching models with indivisible assets, Aiyagari, Wallace, and Wright (1996) and Li and Wright (1998) introduce government agents who can accept or reject assets to affect their liquidity and prices. In contrast to this approach we will place no restrictions on the use of assets as means of payment.

Physical properties of assets In his dictum for monetary theory Wallace (1998) called for theories that specify assets by their physical properties in order to endogenize their role in exchange. Renero (1999) shows in the context of the Kiyotaki-Wright (1989) model the existence of mixed-strategy equilibria where commodities with higher storage costs have higher acceptability. In Wallace (2000) a liquidity structure of asset returns arises from asset indivisibilities and portfolio restrictions. Freeman (1985), Rocheteau (2011b), Lester, Postlewaite, and Wright (2012), Hu (2013), and Li, Rocheteau, and Weill (2012), among others, explain the coexistence of money with higher-return assets based on various notions of imperfect recognizability of assets and private information problems.[5] Aruoba, Waller, and Wright (2011) justify the illiquidity of capital by its lack of portability. In contrast, in this paper rate-of-return dominance emerges even though capital goods have the same physical properties as fiat money, i.e., they are perfectly divisible, recognizable, and portable.

[5] The literature on monetary models with pairwise meetings and multiple assets is reviewed in Nosal and Rocheteau (2011). See also the survey by Williamson and Wright (2010).

Self-fulfilling beliefs about assets' liquidity In Kiyotaki and Wright (1989) and Aiyagari, Wallace, and Wright (1996), rate-of-return dominance can be explained in the absence of legal restrictions by self-fulfilling beliefs. For instance, in model B of Kiyotaki and Wright (1989) there can be multiple equilibria, including a so-called speculative one where the commodities with the highest storage costs serve as media of exchange. In Aiyagari, Wallace, and Wright (1996) there is a steady state with matured bonds circulating at par and, for some parameters, another steady state where matured bonds circulate at a discount. The second steady state generates rate-of-return dominance—in the sense of newly-issued bonds being sold at a discount—even if the government does not adopt a discriminatory trading policy against bonds. Vayanos and Weill (2008) show that bonds with identical cash flows can be traded at different prices when there are increasing returns to scale in the matching technology in asset markets that generate multiple equilibria. Finally, Lagos (2013) describes an economy where fiat monies are heterogeneous in an extraneous attribute and shows the existence of equilibria in which money coexists with interest-bearing bonds. In contrast, our explanation will not rely on self-fulfilling beliefs since we focus on constrained-efficient allocations that are unique (for all relevant variables).

Pricing of assets in pairwise meetings A recent approach considers trading mechanisms in economies with pairwise meetings that treat assets asymmetrically. Zhu and Wallace (2007) construct a pairwise Pareto optimal mechanism with the property that agents do not receive any surplus from holding nominal bonds even though bonds serve as means of payment. As a consequence bonds generate no liquidity value to their holders and pay a positive interest rate. Nosal and Rocheteau (2013) generalize this mechanism to allow the liquidity premium of an asset to be controlled by a single parameter. While both mechanisms can lead to yield differences across assets with identical cash flows, they are socially inefficient mechanisms. See Section 6 for further details. In contrast to those models, our focus will be on socially optimal trading mechanisms.

Mechanism design and normative approaches The idea of using a normative approach to explain the coexistence of fiat money and higher return assets can be traced back to Kocherlakota

(2003).[6] In an economy with competitive markets he establishes that illiquid government bonds have a societal role when agents are subject to idiosyncratic preference shocks.[7] In contrast, in our environment high-return assets are not useful to reallocate liquidity across agents with different marginal utilities of consumption, i.e., the coexistence of money and higher return capital goods is optimal even if buyers have homogeneous preferences and we set the matching probability to one. In Kocherlakota's world illiquid bonds are restricted not to serve as means of payment. In contrast, we have no such restriction on the use of capital: the extent to which capital goods serve as means of payment is determined endogenously as part of an optimal trading mechanism. In fact, whenever the first best is not implementable the optimal mechanism will require buyers to pay for their consumption with all their money and capital goods. Finally, if deflation were feasible in Kocherlakota's world, then the Friedman rule would be optimal and illiquid bonds would play no role. In contrast, we will show that even if the Friedman rule is feasible, it is not necessary to implement a good allocation and there is a range of inflation rates for which the first best can be implemented and there is rate-of-return dominance. Mechanism design has been applied to the Lagos and Wright (2005) environment by Hu, Kennan, and Wallace (2009) to dismiss the usefulness of the Friedman rule. We extend their analysis to have multiple assets and to study the coexistence of fiat money and higher-return capital goods.

2 The environment

The environment is similar to the one in Lagos and Rocheteau (2008). Time is represented by $t \in \mathbb{N}$. Each period, t, is divided into two stages labeled DM (decentralized market) and CM (centralized market). In the first stage, DM, each agent enters a bilateral match with a randomly chosen trading partner with probability $\sigma \in [0, 1]$. In the second stage, CM, agents trade in competitive markets.

[6] Kocherlakota (1998) and Kocherlakota and Wallace (1998) were the first to use implementation theory to prove the essentiality of money. Applications of mechanism design to monetary theory include Cavalcanti and Wallace (1999) and Gu, Mattesini, Monnet, and Wright (2013) on banking and inside money, Cavalcanti and Erosa (2008) on the propagation of shocks in monetary economies, Cavalcanti and Nosal (2009) on cyclical monetary policy, Koeppl, Monnet, and Temzelides (2008) on settlement, Deviatov and Wallace (2001) and Deviatov (2006) on the welfare gains of money creation, Hu, Kennan, and Wallace (2009) on the optimality of the Friedman rule, and Rocheteau (2012) on the cost of inflation. For an overview of this approach, see Wallace (2010).

[7] Shi (2008) also shows that it can be beneficial for a society to restrict the use of nominal bonds as a means of payment for goods when individuals face matching shocks that affect the marginal utility of consumption, but the trading mechanism in the goods market with pairwise meetings is not chosen optimally.

Time starts in the CM of period 0. In each stage there is a perfectly divisible and perishable consumption good.

There is a measure two of infinitely-lived agents divided evenly between two types called *buyers* and *sellers*, where these labels capture agents' roles in the DM: buyers in pairwise meetings in the DM consume the output produced by sellers.[8] The set of buyers is denoted $\mathbb{B}$ and the set of sellers is denoted $\mathbb{S}$. Buyers' preferences are represented by the utility function

$$-h_0 + \mathbb{E}\sum_{t=1}^{\infty}\beta^t\left[u(q_t) - h_t\right],$$

where $\beta \equiv (1+r)^{-1} \in (0,1)$ is the discount factor, q_t is DM consumption, and h_t is CM production (when $h_t < 0$ it is interpreted as consumption). Buyers have the technology to produce the CM good at a linear disutility cost. We shall call the CM good the *numéraire good* henceforth.

Sellers' preferences are given by

$$c_0 + \mathbb{E}\sum_{t=1}^{\infty}\beta^t\left[-v(q_t) + c_t\right],$$

where q_t is DM production and c_t is CM consumption (when $c_t < 0$ it is interpreted as production).[9] The first-stage utility functions, $u(q)$ and $-v(q)$, are increasing and concave, with $u(0) = v(0) = 0$. The surplus function, $u(q) - v(q)$, is strictly concave, with $q^* = \arg\max\left[u(q) - v(q)\right]$. Moreover, both u and v are twice continuously differentiable with $u'(0) = v'(\infty) = \infty$ and $v'(0) = u'(\infty) = 0$.

The numéraire good can be transformed into a capital good one for one. Capital goods accumulated at the end of period t are used by sellers at the beginning of the CM of $t+1$ to produce the numéraire good according to the technology $F(k)$.[10] See Figure 1. We assume that F is twice continuously differentiable, $F' > 0$, $F'' < 0$, $F'(0) = \infty$, $F'(\infty) = 0$, and that $F'(k)k$ is strictly increasing, strictly concave, and has range $\mathbb{R}_+$.[11] An example of a production function satisfying

[8] We assume that an agent's type, buyer or seller, is permanent. This formulation is convenient because the set of incentive-feasible allocations is the same under private information, partial provability, or common knowledge of asset holdings in a match. It would be equivalent to assume that an agent's type is chosen at random at the beginning of the CM, so that all agents are ex-ante identical.

[9] Even though we do not impose nonnegativity constraints on h_t and c_t along the equilibrium path such constraints will be satisfied: buyers will produce the numéraire good and sellers will consume it.

[10] It should be noticed that who operates the technology, F, is irrelevant for our analysis provided that the residual profits, $F(k) - kF'(k)$, are not pledgeable in the DM due to lack of commitment.

[11] In Hu and Rocheteau (2013) we consider the case of assets in fixed supply and we study the implications of an optimal mechanism for asset prices.

these properties is $F(k) = k^\alpha$, with $0 < \alpha < 1$. Capital goods depreciate fully after one period.[12] The rental (or purchase) price of capital in terms of the numéraire good is R_t.

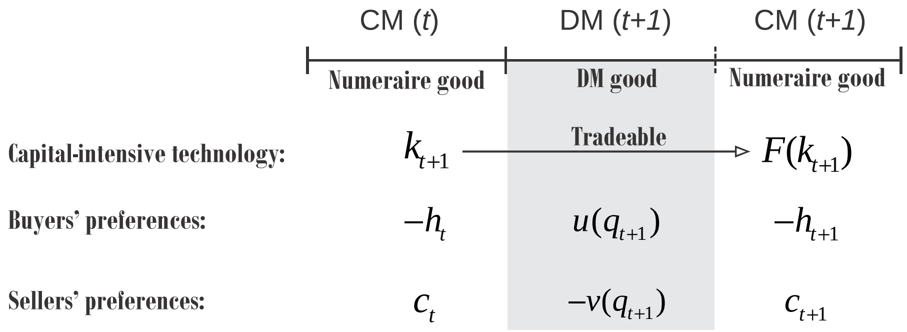


Figure 1: Timing, technology, and preferences

Agents cannot commit to future actions, there is no enforcement technology, and individual histories are private information. These frictions rule out (unsecured) credit arrangements and generate a social role for liquid assets. Capital goods can serve this role. There is also a fixed supply, M, of an intrinsically useless, perfectly divisible asset called fiat money. The price of money in terms of the period-t numéraire good is denoted ϕ_t. In a pairwise meeting in the DM a buyer can transfer any quantity of his asset holdings in exchange for some output. Asset holdings are common knowledge in a match.[13]

3 Implementation

We study equilibrium outcomes that can be implemented by proposals. A *proposal* consists of three objects:

(i) A sequence of functions in bilateral matches, $o_t : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+ \times \mathbb{R}^2$, each of which maps the pair's portfolios, $(\pi_b, \pi_s) = [(z_b, k_b), (z_s, k_s)]$, at period t, where (z_b, k_b) is the asset holdings of the buyer—z_b stands for holdings of real balances and k_b stands for capital holdings—and (z_s, k_s) is

[12] It would be straightforward to introduce partial depreciation by reinterpreting $F(k)$ as output augmented with the capital stock net of depreciation. A functional form that satisfies our assumptions is $F(k) = k^\alpha + (1-\delta)k$ with $\alpha \in (0, 1)$ and $\delta \in [0, 1]$, where δ is interpreted as the depreciation rate.

[13] All our results go through if buyers can hide their asset holdings but cannot overstate them. This private information problem is secondary for the focus of this paper, and for sake of clarity we choose to ignore it.

the asset holdings of the seller, into a proposed trade, $(q, \mathbf{d}) \in \mathbb{R}_+ \times [-z_s, z_b] \times [-k_s, k_b]$, where q is the quantity produced by the seller and consumed by the buyer, $\mathbf{d} = (d_z, d_k)$ is a transfer of assets from the buyer to the seller (d_z is the transfer of real balances and d_k is the transfer of capital goods);

(ii) An initial distribution of money, μ;

(iii) A sequence of prices of money in terms of the numéraire good, $\{\phi_t\}_{t=0}^{\infty}$, and a sequence of rental prices of the capital goods in terms of the numéraire good, $\{R_t\}_{t=0}^{\infty}$, in the CM.

The trading mechanism in the DM is as follows. Given the asset holdings of the pair and the proposed trade associated with those holdings, both the buyer and the seller simultaneously respond with *yes* or *no*. If both respond with *yes*, then the proposed trade is carried out; otherwise, there is no trade. This ensures that trades are individually rational. We also require any proposed DM trade to be in the pairwise core (which we define formally later).[14] This requirement guarantees that there is no room for the two agents in a meeting to renegotiate the proposed terms of trade. We assume that agents in the CM trade competitively against the proposed prices. This is consistent with the core requirement that we impose in the DM due to the equivalence between the core and competitive equilibria.

We denote s_b the strategy of a buyer $b \in \mathbb{B}$, which consists of three components for any given trading history, h^t, at the beginning of period t: (i) $s_b^{h^t,0}(\pi_b) \in \mathbb{R}_+^2$ that maps the buyer's initial asset holdings to his final asset holdings after the CM, conditional on not being matched in the DM; (ii) $s_b^{h^t,1}(\pi_b, \pi_s) \in \{yes, no\}$ that maps the pair's portfolios, (π_b, π_s), to his response in the DM, either *yes* or *no*, conditional on being matched with a seller; (iii) $s_b^{h^t,2}(\pi_b, \pi_s; a_b, a_s) \in \mathbb{R}_+^2$ that maps the buyer's trading history in the DM to his final asset holdings after the CM, where $a_b, a_s \in \{yes, no\}$ are the buyer's and seller's responses, respectively. Similarly, the strategy of a seller $s \in \mathbb{S}$, for any given trading history h^t at the beginning of period t, consists of three functions $s_s^{h^t,0}(\pi_s) \in \mathbb{R}_+^2$, $s_s^{h^t,1}(\pi_b, \pi_s) \in \{yes, no\}$, and $s_s^{h^t,2}(\pi_b, \pi_s; a_b, a_s) \in \mathbb{R}_+^2$ that are defined symmetrically to s_b.

Definition 1 *An equilibrium is a list,* $\langle (s_b : b \in \mathbb{B}), (s_s : s \in \mathbb{S}), \mu, \{(o_t, \phi_t, R_t)\}_{t=0}^{\infty} \rangle$*, composed of*

[14] As in Hu, Kennan, and Wallace (2009), we can incorporate this requirement in the DM trading mechanism by adding a renegotiation stage where, after a "no" response, the buyer makes a take-it-or-leave-it offer to the seller and the planner proposal is carried out if the seller rejects the buyer's offer.

one strategy for each agent in the set $\mathbb{B} \cup \mathbb{S}$ *and the proposals* $(\mu, \{(o_t, \phi_t, R_t)\}_{t=0}^{\infty})$ *such that: (i) Each strategy is sequentially rational given other players' strategies and asset prices; (ii) The centralized market clears at every date.*

Throughout the paper we restrict our attention to equilibria that involve stationary proposals and that use symmetric and stationary strategies in which both the buyer and the seller respond with *yes* in all DM meetings, the initial distribution of money across buyers is degenerate—all buyers hold M units—and money and capital prices are constant over time. We call such equilibria *simple* equilibria. The outcome of a simple equilibrium is characterized by a list, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, where $(q^p, \mathbf{d}^p)$ is the trade in all matches in the DM, $\pi_b^p = (z_b^p, k_b^p)$ and $\pi_s^p = (z_s^p, k_s^p)$ are the buyer's asset holdings and the seller's asset holdings before entering the DM, and hence $\mathbf{d}^p = (d_z^p, d_s^p) \in [-z_s^p, z_b^p] \times [-k_s^p, k_b^p]$. In a simple equilibrium, the price for money is pinned down by $M\phi_t = z^p$, where $z^p \equiv z_s^p + z_b^p$.

Because we only consider stationary proposals, the trading mechanism, o, is constant across all periods, and we may write it as $o(\pi_b, \pi_s) = [q(\pi_b, \pi_s), \mathbf{d}(\pi_b, \pi_s)]$. For a given proposal, o, and rental price, R, let $V^b(\pi_b)$ and $W^b(\pi_b)$ denote the continuation values for a buyer holding $\pi_b = (z_b, k_b)$ upon entering the DM and CM, respectively. Similarly, let $V^s(\pi_s)$ and $W^s(\pi_s)$ denote the continuation values for a seller holding $\pi_s = (z_s, k_s)$ upon entering the DM and the CM, respectively. The Bellman equation for a buyer in the CM solves

$$\begin{aligned} W^b(\pi_b) &= \max_{\hat{z}_b \geq 0, \hat{k}_b \geq 0, h} \left\{ -h + \beta V^b(\hat{z}_b, \hat{k}_b) \right\} && (1) \\ \text{s.t.} \quad \hat{z}_b + \hat{k}_b &= h + z_b + Rk_b, && (2) \end{aligned}$$

where $\hat{z}_b$ and $\hat{k}_b$ denote the real balances and capital taken into the next DM. From the budget identity, (2), the buyer finances his new portfolio, $(\hat{z}_b, \hat{k}_b)$, by supplementing his initial wealth, $z_b + Rk_b$, with h units of numéraire good. After substituting h by its expression given by (2) into (1), the Bellman equation becomes

$$W^b(\pi_b) = z_b + Rk_b + \max_{\hat{z}_b \geq 0, \hat{k}_b \geq 0} \left\{ -\hat{z}_b - \hat{k}_b + \beta V^b(\hat{z}_b, \hat{k}_b) \right\}. \quad (3)$$

Due to the linear preferences in the CM, the buyer's value function is linear in wealth.

The Bellman equation for $V^b(\pi_b)$ is given by

$$V^b(\pi_b) = \sigma \left\{ u\left[q(\pi_b, \pi_s^p)\right] + W^b\left[\pi_b - \mathbf{d}(\pi_b, \pi_s^p)\right] \right\} + (1-\sigma)\, W^b(\pi_b). \tag{4}$$

Equation (4) has the following interpretation. In the DM the buyer meets a seller, whose portfolio is π_s^p in equilibrium, with probability σ, in which case his trade is given by $o(\pi_b, \pi_s^p) = [q(\pi_b, \pi_s^p), \mathbf{d}(\pi_b, \pi_s^p)]$, i.e., he consumes $q(\pi_b, \pi_s^p)$ and spends $d_z(\pi_b, \pi_s^p)$ real balances and $d_k(\pi_b, \pi_s^p)$ units of capital. The buyer enters the CM with $z_b - d_z(\pi_b, \pi_s^p)$ real balances and $k_b - d_k(\pi_b, \pi_s^p)$ units of capital. With probability, $1-\sigma$, the buyer is unmatched and no trade takes place in the DM. Using the linearity of W^b, (4) simplifies to

$$V^b(\pi_b) = \sigma \left\{ u\left[q(\pi_b, \pi_s^p)\right] - d_z(\pi_b, \pi_s^p) - Rd_k(\pi_b, \pi_s^p) \right\} + W^b(\pi_b). \tag{5}$$

The first term on the right side of (5) is the buyer's expected surplus in the DM. The second term is the continuation value in the CM. Substituting $V^b(\pi_b)$ with its expression given by (5) into (3), using the linearity of $W^b(\pi_b)$, and omitting constant terms, the buyer's problem in the CM can be reformulated as

$$\max_{\pi_b=(z_b,k_b)\geq \mathbf{0}} \left\{ -rz_b - (1+r-R)k_b + \sigma \left\{ u\left[q(\pi_b, \pi_s^p)\right] - d_z(\pi_b, \pi_s^p) - Rd_k(\pi_b, \pi_s^p) \right\} \right\}. \tag{6}$$

According to (6) the buyer chooses a portfolio of money and capital in order to maximize his expected surplus in the DM net of the cost of holding assets. The first two terms in the objective function correspond to the costs of holding real balances, $r = 1/\beta - 1$, and capital, $1 + r - R$, while the third term corresponds to the expected surplus in the DM by holding those assets.

Using the same logic as above, the value function of a seller in the CM solves

$$W^s(\pi_s) = z_s + Rk_s + \max_{k'\geq 0} \left\{ F(k') - Rk' \right\} + \max_{\hat{z}_s \geq 0, \hat{k}_s \geq 0} \left\{ -\hat{z}_s - \hat{k}_s + \beta V^s(\hat{z}_s, \hat{k}_s) \right\}, \tag{7}$$

where $\pi_s = (z_s, k_s)$ and k' is the amount of capital rented by the seller, and $(\hat{z}_s, \hat{k}_s)$ is the portfolio chosen for the following period. The value function of a seller in the DM solves

$$\begin{aligned} V^s(\pi_s) &= \sigma \left\{ -v\left[q(\pi_b^p, \pi_s)\right] + W^s\left[\pi_s + \mathbf{d}\left(\pi_b^p, \pi_s\right)\right] \right\} + (1-\sigma)\, W^s(\pi_s) \\ &= \sigma \left\{ -v\left[q(\pi_b^p, \pi_s)\right] + d_z\left(\pi_b^p, \pi_s\right) + Rd_k\left(\pi_b^p, \pi_s\right) \right\} + W^s(\pi_s), \end{aligned} \tag{8}$$

where from the first to the second equality we have used the linearity of W^s. The interpretation of (8) is similar to the interpretation of (4). From (7) and (8) the seller's portfolio problem reduces to

$$\max_{\pi_s=(z_s,k_s)\geq \mathbf{0}} \left\{-rz_s-(1+r-R)k_s+\sigma\left\{-v\left[q(\pi_b^p,\pi_s)\right]+d_z\left(\pi_b^p,\pi_s\right)+Rd_k(\pi_b^p,\pi_s)\right\}\right\}.$$

From the third term on the right side of (7) the seller's optimal choice of input to operate the technology F is such that $F'(k')=R$. By market clearing, $k'=k_s^p+k_b^p\equiv k^p$ and $R_t=R$ for all t with

$$F'(k^p)=R. \tag{9}$$

Lemma 1 *Given the proposed outcome, $(q^p,\mathbf{d}^p,\pi_b^p,\pi_s^p)$, and proposal, o, the value functions that are consistent with the simple equilibrium are:*

$$\begin{aligned}
W^b(\pi_b) &= z_b+F'(k^p)k_b+W^b(0,0), && (10)\\
W^b(0,0) &= -z_b^p-\frac{1+r-F'(k^p)}{r}k_b^p+\frac{\sigma}{r}\left[u(q^p)-d_k^pF'(k^p)-d_z^p\right], && (11)\\
W^s(\pi_s) &= z_s+F'(k^p)k_s+W^s(0,0), && (12)\\
W^s(0,0) &= -z_s^p-\frac{1+r-F'(k^p)}{r}k_s^p && (13)\\
&+ \frac{1}{r}\left\{\sigma\left[-v(q^p)+d_k^pF'(k^p)+d_z^p\right]+F(k^p)-F'(k^p)k^p\right\}, &&\\
V^b(\pi_b) &= \sigma\left\{u\left[q(\pi_b,\pi_s^p)\right]-F'(k^p)d_k\left(\pi_b,\pi_s^p\right)-d_z\left(\pi_b,\pi_s^p\right)\right\}+W^b(\pi_b), && (14)\\
V^s(\pi_s) &= \sigma\left\{-v\left[q(\pi_b^p,\pi_s)\right]+F'(k^p)d_k(\pi_b^p,\pi_s)+d_z(\pi_b^p,\pi_s)\right\}+W^s(\pi_s). && (15)
\end{aligned}$$

As mentioned earlier, we require the proposed trade to be in the pairwise core. The pairwise core is the set of all feasible allocations, $(q,\mathbf{d})\in\mathbb{R}_+\times\left[-z_s^p,z_b^p\right]\times\left[-k_s^p,k_b^p\right]$, such that there exist no alternative feasible trades that would make the buyer and the seller in the match better off, with at least one of the two being strictly better off. In other words, there is no room for renegotiation in pairwise meetings to improve upon the mechanism's proposals. Formally, the set of pairwise core allocations, denoted by $\mathcal{CO}(\pi_b^p,\pi_s^p;R)$, is defined as the set of allocations such that for some

$U^s \geq 0$,

$$(q, \mathbf{d}) \in \arg\max \left[u(q) - d_z - Rd_k\right] \tag{16}$$

$$\text{s.t.} \quad -v(q) + d_z + Rd_k = U^s, \tag{17}$$

$$d_z \in \left[-z_s^p, z_b^p\right], \quad d_k \in \left[-k_s^p, k_b^p\right], \tag{18}$$

$$U^b \equiv u(q) - d_z - Rd_k \geq 0. \tag{19}$$

Here, U^b is the buyer's surplus from the trade, $(q, \mathbf{d})$, and U^s is the seller's surplus. To obtain these expressions, we use the linearity of the value functions, (10) and (12), to compute the continuation values of the transferred assets. The set of output levels in $\mathcal{CO}(\pi_b^p, \pi_s^p; R)$ does not depend on π_s because for any solution, $(q, \mathbf{d})$, to (16)-(19), $d_z + Rd_k \geq 0$ and hence the constraints $d_z \geq -z_s^p$ and $d_k \geq -k_s^p$ are never binding. See Appendix B for a characterization of the pairwise core.

We study outcomes that can be implemented with simple equilibria and that satisfy the pairwise core requirement. We say that an outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, is *implementable* if it is the outcome of a simple equilibrium for some mechanism proposal, o, and the trade, $(q^p, \mathbf{d}^p)$, is in the pairwise core $\mathcal{CO}(\pi_b^p, \pi_s^p; R)$.

In the following we give necessary conditions for an outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, to be implementable. From (3), a necessary condition for the buyer to follow the equilibrium behavior is

$$-z_b^p - k_b^p + \beta V^b(z_b^p, k_b^p) \geq \beta W^b(0, 0). \tag{20}$$

The left side of (20) is the buyer's equilibrium payoff in the CM while the right side is the payoff for the deviation consisting of not accumulating money or capital in the CM and then responding with *no* in the DM. Using (10) and (14) this condition implies that (recall that $k^p = k_s^p + k_b^p$)

$$-rz_b^p - \left[1 + r - F'(k^p)\right] k_b^p + \sigma \left[u\left(q^p\right) - d_z^p - F'(k^p) d_k^p\right] \geq 0. \tag{21}$$

Symmetrically, from (7), a necessary condition for the seller to follow the equilibrium behavior is

$$-z_s^p - k_s^p + \beta V^s(z_s^p, k_s^p) \geq \beta W^s(0, 0). \tag{22}$$

Using (12) and (15) this implies that

$$-rz_s^p - \left[1 + r - F'(k^p)\right] k_s^p + \sigma \left[-v\left(q^p\right) + d_z^p + F'(k^p) d_k^p\right] \geq 0. \tag{23}$$

Finally, it is necessary that $R \leq 1+r$; for otherwise there will be unbounded production of capital, and perfect competition implies that $R = F'(\infty) < 1+r$, a contradiction. Thus, $R \leq 1+r$ and by (9), it implies that

$$F'(k^p) \leq 1+r. \tag{24}$$

Thus, we have three necessary conditions, (21), (23), and (24), for an outcome to be implementable. The following lemma shows that, together with the pairwise core requirement, those conditions are also sufficient for implementability.

Lemma 2 *An equilibrium outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, that satisfies (21), (23), (24), and the pairwise core requirement, can be implemented by a mechanism, o, that satisfies:*

1. *If $\pi_b = (z_b, k_b) \geq \pi_b^p$ and $\pi_s = (z_s, k_s) \geq \pi_s^p$, then*

$$\begin{aligned} o(\pi_b, \pi_s) \in \arg \max_{q,d_z,d_k} & \left\{ d_z + F'(k^p)d_k - v(q) \right\} \\ s.t. \quad & u(q) - d_z - F'(k^p)d_k \geq u(q^p) - d_z^p - F'(k^p)d_k^p, \\ & q \geq 0,\ d_z \in [-z_s, z_b],\ d_k \in [-k_s, k_b]. \end{aligned} \tag{25}$$

2. *If $\pi_b = (z_b, k_b)$ is such that $z_b < z_b^p$ or $k_b < k_b^p$, then*

$$\begin{aligned} o(\pi_b, \pi_s) \in \arg \max_{q,d_z,d_k} & \left\{ d_z + F'(k^p)d_k - v(q) \right\} \\ s.t. \quad & u(q) - d_z - F'(k^p)d_k = 0, \\ & q \geq 0,\ d_z \in [-z_s, z_b],\ d_k \in [-k_s, k_b]. \end{aligned} \tag{26}$$

3. *If $\pi_b \geq \pi_b^p$ holds but $\pi_s \geq \pi_s^p$ does not hold, then*

$$\begin{aligned} o(\pi_b, \pi_s) \in \arg \max_{q,d_z,d_k} & \left\{ u(q) - d_z - F'(k^p)d_k \right\} \\ s.t. \quad & -v(q) + d_z + F'(k^p)d_k = 0, \\ & q \geq 0,\ d_z \in [-z_s, z_b],\ d_k \in [-k_s, k_b]. \end{aligned} \tag{27}$$

For any given outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, that satisfies (21), (23), (24), and the pairwise core requirement, the program (25)-(27) defines the mapping, o, between the pair's portfolio and the trade in the DM that implements the outcome. The crucial step in proving sufficiency is to show that the buyers and the sellers are willing to leave the CM with the portfolios π_b^p and π_s^p, respectively. The mapping, o, achieves this outcome by punishing agents who leave the CM with insufficient asset holdings. According to (25), if the agents hold at least the equilibrium quantities of both assets, then the mechanism selects the pairwise Pareto-efficient allocation that gives the buyer the same surplus as the one he would obtain under the trade $(q^p, \mathbf{d}^p)$. According to (26), if the buyer holds less than z_b^p real balances or less than k_b^p units of capital, then the mechanism chooses the allocation in the core that generates the lowest utility level for the buyer. Finally, if the seller holds less real balances or less capital than he is supposed to hold at the proposed allocation, and if the buyer holds no less than the equilibrium quantities of both assets, then the mechanism proposes the least preferred trade for the seller in the pairwise core.

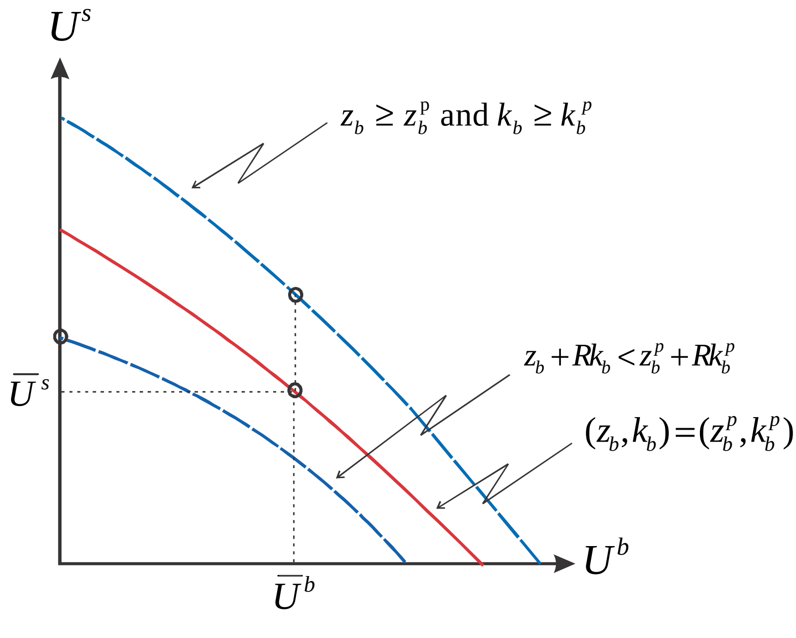


Figure 2: Incentive-feasible mechanism

Figure 2 represents graphically the mechanism in (25)-(27), taking the seller portfolio at π_s^p. For a given aggregate capital stock, k^p, the buyer's surplus is $U^b = u(q) - d_z - Rd_k$, while the seller's surplus is $U^s = -v(q) + d_z + Rd_k$, where $R = F'(k^p)$. The pairwise core (in the utility space) is downward-sloping and concave. The utility levels associated with the proposed trade, $(q^p, \mathbf{d}^p)$, are denoted $\bar{U}^b$ and $\bar{U}^s$. If the buyer holds $z_b \geq z_b^p$ and $k_b \geq k_b^p$, with at least one strict inequality, then the Pareto frontier shifts outward. The mechanism selects the point on the Pareto frontier marked by a circle that assigns the same utility level, $\bar{U}^b$, to the buyer. If the buyer holds less wealth than $z_b^p + Rk_b^p$, the Pareto frontier shifts downward. The mechanism selects the point on the frontier that assigns no surplus to the buyer, $U^b = 0$. Finally, if $z_b + Rk_b \geq z_b^p + Rk_b^p$ (i.e., the Pareto frontier shifts outward) but either $z_b < z_b^p$ or $k_b < k_b^p$, the mechanism will still select the point on the Pareto frontier that gives no surplus to the buyer. By construction, the mechanism is coalition-proof.

Therefore, under o, an agent's surplus in the DM depends not only on the wealth he holds (measured in terms of the numéraire good) but also on the composition of his portfolio. Given that other agents follow equilibrium behavior, an agent enjoys the equilibrium surplus in the DM only if he holds enough units of both assets. Given the constraints (21) and (23), this provides sufficient incentives for agents to leave the CM with their respective equilibrium portfolios.
The argument can be illustrated by Figure 3. For the sake of illustration the buyer's capital stock is fixed at k_b^p and the seller's portfolio is fixed at π_s^p. The top panel represents the buyer's surplus in a match as a function of his real balances. If the buyer holds less than z_b^p then his surplus is 0; otherwise, it is the surplus associated with the proposed allocation. The bottom panel plots the buyer's expected surplus, net of the cost of holding real balances and capital. Given that the buyer accumulates k_b^p units of capital, he will choose to hold z_b^p real balances.

4 Optimal allocation

We consider the problem of choosing a trading mechanism and its associated equilibrium outcome in order to maximize social welfare. Given an outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, social welfare is measured by the discounted sum of buyers' and sellers' utility flows, that is (recall that $k^p = k_s^p + k_b^p$),

$$\mathcal{W}\left(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p\right) = -k^p + \lim_{T \to \infty} \sum_{t=1}^{T} \beta^t \left\{\sigma\left[u(q^p) - v(q^p)\right] + F(k^p) - k^p\right\}, \tag{28}$$

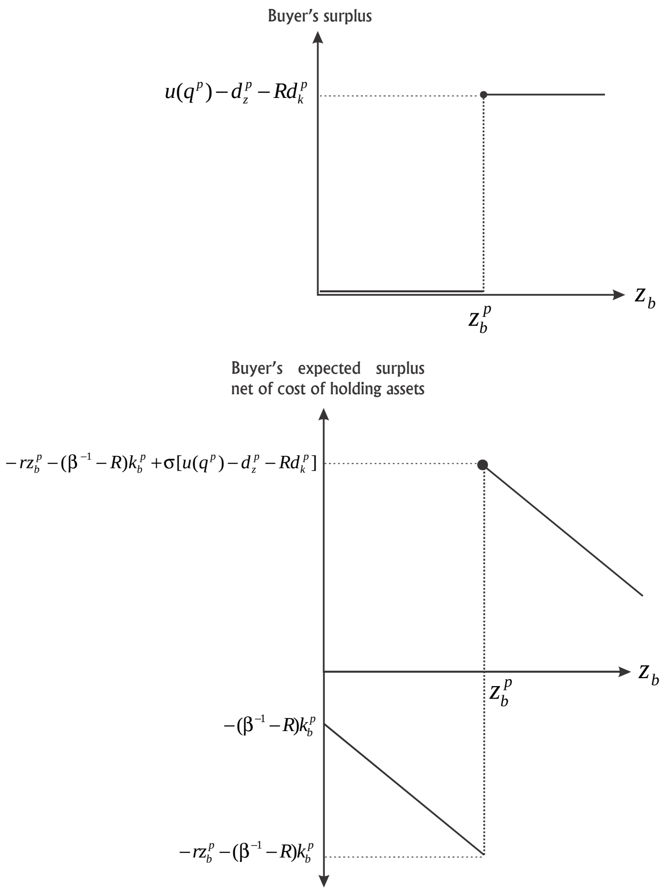


Figure 3: Buyer's surplus under the proposed mechanism

which is equivalent to the following expression:

$$\mathcal{W}\left(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p\right) = \frac{\sigma[u(q^p) - v(q^p)] + F(k^p) - (1+r)k^p}{r}. \tag{29}$$

The first term on the right side of (28) is the utility cost incurred by agents in the initial CM to accumulate the proposed capital stock, k^p. The second term captures utility flows in the subsequent periods. It is composed of the sum of the surpluses in the pairwise meetings, $\sigma\left[u(q^p) - v(q^p)\right]$, and the output from the technology, F, net of the depreciated capital stock, $F(k^p) - k^p$. It can be noticed from (29) that our measure of social welfare is independent of asset transfers in pairwise meetings. The first-best outcome that maximizes (29) but ignores implementability constraints is such that $q^p = q^*$ and $k^p = k^*$, where $u'(q^*) = v'(q^*)$ and $F'(k^*) = 1 + r$.

Definition 2 *An outcome,* $\left(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p\right)$*, is constrained-efficient if it solves*

$$\max_{\left(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p\right)} \left\{\sigma\left[u(q^p) - v(q^p)\right] + F(k^p) - (1+r)k^p\right\} \tag{30}$$

subject to constraints (21), (23), (24), and the pairwise core requirement.

The following lemma provides a convenient characterization of the maximization problem in Definition 2. Let $\bar{k}$ be the value of the capital stock such that $F'(\bar{k})\bar{k} = v(q^*)$. It exists because $F'(k)k$ has range $\mathbb{R}_+$, and it is unique because $F'(k)k$ is strictly increasing. The threshold, $\bar{k}$, is interpreted as the capital stock that is required to compensate sellers for the production of q^* in the DM.

Lemma 3 *Consider the following maximization problem:*

$$\max_{(q,z,k)\in\mathbb{R}_+^3} \left\{\sigma\left[u(q) - v(q)\right] + F(k) - (1+r)k\right\} \tag{31}$$

subject to

$$\sigma[u(q) - v(q)] - [1 + r - F'(k)]k - rz \geq 0 \tag{32}$$

$$-v(q) + F'(k)k + z \geq 0 \tag{33}$$

$$F'(k) \leq 1 + r. \tag{34}$$

1. *A solution,* (q^p, z^p, k^p)*, to (31)-(34) exists and has unique values for* q^p *and* k^p*; moreover, in any solution,* $q^p \leq q^*$*,* $k^p \in [k^*, \max\{\bar{k}, k^*\}]$*, and* $z^p \leq \sigma[u(q^*) - v(q^*)]/r$*.*

2. *If* (q^p, z^p, k^p) *is a solution to (31)-(34), then there exists* $\mathbf{d}^p$ *such that* $\left(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p\right)$ *is a constrained-efficient outcome with* $\pi_b^p = (z^p, k^p)$ *and* $\pi_s^p = (0, 0)$*.*

3. *If* $\left(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p\right)$ *is a constrained-efficient outcome, then* $(q^p, z_b^p, k_b^p + k_s^p)$ *solves the problem (31)-(34).*

Lemma 3 implies that a constrained-efficient outcome exists by the following arguments: by (1) a solution to the problem (31)-(34) exists and by (2) we can construct a constrained-efficient outcome from that solution. Moreover, the solution to (31)-(34) pins down the output level, q^p, and the capital level, k^p, of a constrained-efficient allocation—which are the only two endogenous variables that matter for social welfare according to (29). Finally, we will say that fiat money plays an essential role if and only if any solution to (31)-(34) is such that $z^p > 0$. In what follows, with a slight abuse of language, we call a solution to (31)-(34) a constrained-efficient outcome.

The intuition for the constraints in the simplified problem (31)-(34) are as follows. Inequality (32) states that for an outcome to be implementable, the expected match surplus in the DM must be large enough to cover the buyer's cost of holding capital and real balances. Equivalently, buyers must be willing to participate in the CM if they receive the whole surplus of the match. Inequality (33) requires that there is enough wealth in the form of real balances and capital to compensate sellers for the disutility of producing q^p in the DM, which presumes that sellers do not carry assets across periods and hence incur no cost of carrying them.

Lemma 3 (2) and (3) shows that if there is a constrained-efficient outcome where sellers hold assets, then there is an alternative constrained-efficient outcome with the same level of output and the same capital stock where only buyers hold assets. Therefore, one can restrict sellers not to carry assets across periods without loss in generality. The intuition for this result is as follows. From (23) the seller's expected surplus in the DM net of the cost of holding assets must be non-negative. Hence, if $R < 1+r$, then requiring sellers to hold assets tightens the seller's participation constraint without enlarging the set of incentive-feasible output levels in pairwise meetings. Moreover, the capital held by sellers reduces the rate of return of capital, which tightens the buyer's participation constraint. Consequently, if a first best outcome is not implementable, then allocating no assets to sellers is socially desirable. If a first best is implementable, there can exist outcomes where sellers hold assets because the optimal distribution of capital across buyers and sellers is indeterminate.

In the following proposition we consider a non-monetary economy where z^p is zero. The characterization of constrained-efficient outcomes with this additional constraint gives us a benchmark case for what can be achieved with capital alone.

Proposition 1 *Consider an economy without fiat money, $z^p = 0$. There exists a unique constrained-efficient outcome, $(q^p, 0, k^p)$, and it is such that:*

1. *If $(1 + r)k^* \geq v(q^*)$, then $q^p = q^*$ and $k^p = k^*$.*

2. If $(1+r)k^* < v(q^*)$, then $q^p < q^*$ and $k^p > k^*$ solve

$$F'(k^p) = 1 + r - \sigma \left[u'(q^p) - v'(q^p) \right] g'(k^p) \tag{35}$$

$$q^p = g(k^p) \equiv v^{-1}[F'(k^p)k^p]. \tag{36}$$

According to the first part of Proposition 1 the first-best allocation is implementable when the aggregate stock of capital at the first best provides enough wealth to allow buyers to compensate sellers for their disutility of production, $(1+r)k^* \geq v(q^*)$. According to the second part of Proposition 1, if there is a shortage of capital as means of payment, $(1+r)k^* < v(q^*)$, then the quantities traded in the DM are inefficiently low and the capital stock is inefficiently large. In this case society faces a trade-off between the sizes of two inefficiencies: (i) The shortage of capital for liquidity use: $\bar{k} - k$, where $\bar{k} = g^{-1}(q^*)$; (ii) The overaccumulation of capital for productive use: $k - k^*$, where $k^* = F'^{-1}(1+r) < \bar{k}$. Raising k^p above k^* by a small amount has a second-order negative effect on the term $F(k^p) - (1+r)k^p$ in the social welfare function, (29), but a first-order positive effect on the term $\sigma[u(q^p) - v(q^p)]$. Symmetrically, reducing k^p below $\bar{k}$ has a second-order negative effect on the term $\sigma[u(q^p) - v(q^p)]$ but a first-order positive effect on the term $F(k^p) - (1+r)k^p$. As a result of this trade-off, it is socially optimal to overaccumulate capital in order to mitigate the economywide shortage of liquid assets, and to keep the capital stock lower than the level that maximizes the total surplus in pairwise meetings, $k \in (k^*, \bar{k})$.[15]

From (35) the gross rate of return of capital, $F'(k^p)$, is equal to the gross rate of time preference, $1+r$, minus a liquidity term, $\mathcal{L} \equiv \sigma \left[u'(q^p) - v'(q^p) \right] g'(k^p)$. This liquidity term is defined as the increase in the sum of the surpluses in the DM, $\sigma[u(q^p) - v(q^p)]$, due to a marginal increase in the capital stock.

From (36) the binding incentive constraint when liquidity is scarce is the seller's participation constraint, $v(q^p) = F'(k^p)k^p$. In contrast, the buyer's participation constraint is slack,

$$-\left[1 + r - F'(k^p)\right] k^p + \sigma \left[u(q^p) - v(q^p) \right] > 0. \tag{37}$$

To see this, use (35) to rewrite the buyer's DM surplus net of the cost of holding capital as

$$\sigma \left[u(q^p) - v(q^p) \right] - \left[1 + r - F'(k^p)\right] k^p = \sigma \left[u(q^p) - v(q^p) \right] - \sigma \left[u'(q^p) - v'(q^p) \right] g'(k^p)k^p.$$

[15] This result is reminiscent of the one in Wallace (1980) in the context of overlapping generation economies and Lagos and Rocheteau (2008) in the context of random-matching economies.

From the concavity of the match surplus as a function of k^p, where $q^p = g(k^p)$, it follows that the expression is strictly positive. So the mechanism designer could induce buyers to hold more capital in exchange for more output, but it is not optimal to do so.

The next proposition characterizes the constrained-efficient outcome of an economy with fiat money. Let $\Delta^* = u(q^*) - v(q^*)$ denote the maximum surplus in pairwise meetings.

Proposition 2 *Consider an economy with a constant supply of fiat money. There exists a constrained-efficient outcome,* (q^p, z^p, k^p)*.*

1. *If* $(1+r)\,k^* \geq v(q^*)$*, then* $q^p = q^*$ *and* $k^p = k^*$*.*

2. *If* $(1+r)k^* \in [v(q^*) - \sigma\Delta^*/r, v(q^*))$*, then* $z^p \in [v(q^*) - (1+r)k^*, \sigma\Delta^*/r]$*,* $q^p = q^*$*, and* $k^p = k^*$*.*

3. *If* $(1+r)k^* < v(q^*) - \sigma\Delta^*/r$*, then* $z^p > 0$*,* $q^p < q^*$ *and* $F'(k^p) \in (1, 1+r]$*. Moreover,* $k^p > k^*$ *if and only if* $r + F''(k^*)k^* > 0$*.*

The first part of Proposition 2 shows that money plays no essential role when the first-best level of the capital stock is larger than buyers' liquidity needs. If the existing capital provides enough wealth to trade the first best, adding an outside asset cannot raise welfare.

The second part of Proposition 2 shows that if there is a liquidity shortage, in the sense that $(1+r)k^* < v(q^*)$, but this shortage is not too large, then the first-best allocation is implementable with a constant money supply. This result obtains from (37), which states that the buyer's participation constraint in the CM in the absence of fiat money is not binding. Because buyers have strict incentives to participate in the CM in a nonmonetary economy, it is incentive-feasible to require them to hold real balances, even if there is an opportunity cost associated with it, in order to mitigate the two inefficiencies described above: the inefficiently low DM output and the inefficiently high capital stock. Indeed, from (33) at equality $v(q^p) = F'(k^p)k^p + z^p$. Therefore an increase in z^p allows for an increase in q^p and/or a decrease in k^p. Moreover, if buyers are willing to hold $z^p = v(q^*) - F'(k^*)k^*$, then the first best is implementable. For this to be the case, the opportunity cost of holding real balances, $r[v(q^*) - (1+r)k^*]$, must not be larger than the expected benefit from trading the first-best output in the DM, $\sigma\,[u\,(q^*) - v\,(q^*)]$. Equalizing these two terms gives the lower bound for the capital stock in Part 2 of Proposition 2 below which the first-best allocation is not implementable.

When the liquidity shortage is large, then the first-best allocation is no longer implementable. The quantity of real balances that would be required to fill the liquidity gap, $v(q^*) - (1+r)k^*$, would make buyers unwilling to participate in the CM, given the cost of holding money. Consequently, the buyer's participation constraint is binding at the constrained optimum.

The third part of Proposition 2 shows that in contrast to the nonmonetary economy it is not always optimal to raise the capital stock above k^*. Indeed, one additional unit of capital beyond the first-best level has two opposite effects on the buyer's participation constraint. On the one hand, from (33) one unit of capital can be substituted for $1 + r$ units of real balances without affecting the level of output traded in the DM. Because capital has a higher return than fiat money, this substitution relaxes the buyer's participation constraint (32). On the other hand, increasing k above k^* reduces R below $1 + r$, which makes it costly to hold the existing capital stock. If $r + F''(k^*)k^* > 0$, then the first effect dominates and it is optimal to accumulate capital beyond the first-best level.[16]

4.1 Rate-of-return dominance

A key insight from Proposition 2 is that irrespective of the size of the liquidity shortage—as measured by $v(q^*) - (1 + r)k^*$—in any constrained-efficient monetary equilibrium the rate of return of capital is greater than the rate of return of money. If the liquidity shortage is not too large, the first best is implementable and the gross rate of return of capital, $R = F'(k^p) = 1 + r$, is larger than the gross rate of return of fiat money, one. If the liquidity shortage is large so that the first best is not implementable, the gross rate of return of capital can be smaller than the

[16] If we interpret $F(k)$ as a storage technology, as in Lagos and Rocheteau (2008), where k units of CM goods stored in period t generate $F(k)$ units of goods at the *beginning* of the following period, before pairwise meetings have taken place, then the buyer's participation constraint in the CM becomes

$$\sigma[u(q^p) - v(q^p)] - [(1 + r)\, k^p - F(k^p)] - rz^p \geq 0.$$

Under this formulation it would always be optimal to increase k^p above k^* when liquidity is scarce. To see this suppose that $k^P = k^*$ and $q^p < q^*$. One can increase k^p by a small $\varepsilon > 0$ and decrease z by $(1 + r)\varepsilon$. This perturbation has a second-order effect on the term $(1 + r)\, k^p - F(k^p)$ and hence it relaxes the buyer's participation constraint allowing for an increase in q^p.

discount rate, but it is always greater than one. Thus, rate-of-return dominance is a property of good allocations in a monetary economy with pairwise meetings. This result is in contrast with the rate-of-return-equality principle that prevails in most monetary models, including Wallace (1980) under price taking and Lagos and Rocheteau (2008) under bargaining.

In order to convey the main intuition for this result, let us do some reasoning by contradiction. Suppose the constrained-efficient outcome, (q^p, z^p, k^p), is such that the rates of return of the two assets are equalized, $F'(k^p) = 1$. The buyer's and seller's participation constraints can be simplified to read

$$\begin{aligned} \sigma[u(q^p) - v(q^p)] &\geq r(z^p + k^p), \\ v(q^p) &\leq z^p + k^p. \end{aligned}$$

In these two constraints, money and capital are perfect substitutes. The mechanism designer could contemplate the alternative outcome, $(\tilde{q}^p, \tilde{z}^p, \tilde{k}^p)$, defined as follows:

$$\tilde{z}^p = z^p + k^p \tag{38}$$

$$\tilde{k}^p = k^* \tag{39}$$

$$v(\tilde{q}^p) = \min\{v(q^*), z^p + k^p + (1+r)k^*\}. \tag{40}$$

From (38) the real balances are equal to the entire wealth at the proposed outcome. From (39) the capital stock is reduced to its efficient level. Therefore, total wealth is larger but the cost of holding this wealth is unchanged. From (40) the output level is the largest level consistent with the pairwise core requirement. By construction all the participation constraints are satisfied. Social welfare is higher because the output in pairwise meetings is higher, and the capital stock is at its efficient level. This proves that (q^p, z^p, k^p) with $F'(k^p) = 1$ is not constrained efficient.

While it is now clear why $F'(k^p) = 1$ is not socially desirable, how can an outcome with $F'(k^p) > 1$ be made incentive compatible? The answer to this question was given by Lemma 2. An optimal trading mechanism specifies an allocation rule that provides incentives for agents to carry the portfolio associated with the constrained-efficient outcome. If buyers substitute capital for real balances in order to reduce the holding cost of their portfolio of assets, the mechanism selects their least-preferred allocation in the pairwise core so that their surplus is driven down to zero.

4.2 Example

In the following we illustrate the results of Proposition 2 for the functional form $F(k) = Ak^{\alpha}$ with $\alpha \in (0,1)$. The first-best capital stock is $k^* = [A\alpha/(1+r)]^{1/(1-\alpha)}$. A first-best allocation is implementable without fiat money if the technology is sufficiently productive, i.e.,

$$A \geq A_1 \equiv \frac{\upsilon(q^*)^{1-\alpha}(1+r)^{\alpha}}{\alpha}.$$

It is implementable with fiat money if

$$A \geq A_2 \equiv \frac{\{\upsilon(q^*) - \sigma\left[u(q^*) - \upsilon(q^*)\right]/r\}^{1-\alpha}(1+r)^{\alpha}}{\alpha}.$$

The threshold, A_2, is decreasing in the frequency of trading opportunities in the DM, σ, since the first best can be implemented for lower productivities when agents have a higher chance of being matched in the DM. If $\sigma = 0$, then $A_2 = A_1$ and for all $\sigma > 0$, $A_2 < A_1$.

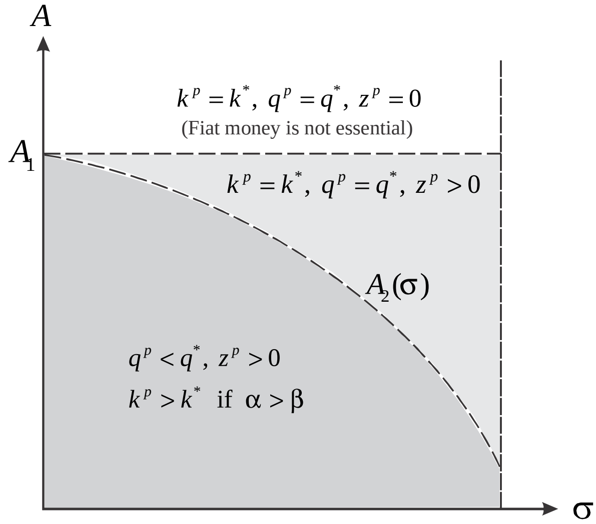


Figure 4: Constrained-efficient outcomes

In Figure 4 we characterize constrained-efficient outcomes for different values for A and σ. The white area corresponds to parameter values for which the first best is implementable without fiat money—fiat money is not essential. This will be the case if A is sufficiently large since the supply of liquidity at the first best, $(1+r)k^* = (A\alpha)^{1/(1-\alpha)}(1+r)^{-\alpha/(1-\alpha)}$, is increasing with the productivity of capital. In the light-grey area the first best is implementable but it requires positive real balances—fiat money plays an essential role. For a given A, the frequency of trading opportunities must be larger than a threshold, $\bar{\sigma}(A)$, for buyers to be willing to hold the required amount of real balances. This threshold is decreasing with A. Finally, in the dark-grey area the first best is not implementable. Fiat money is valued and the output in pairwise meetings is inefficiently low. There is over-accumulation of capital if $\alpha > \beta$.

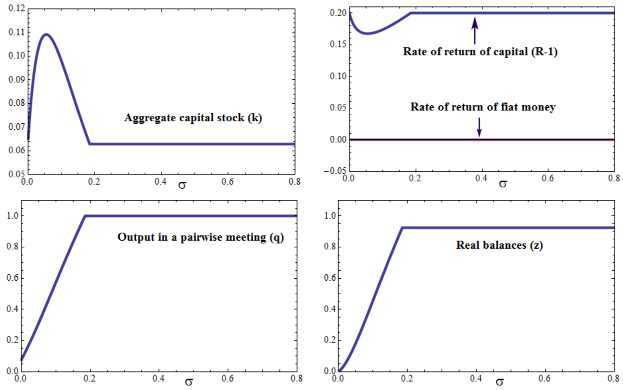


Figure 5: $A = 1.1$, $\alpha = 0.95$, $r = 0.2$

Figure 5 provides a numerical example with overaccumulation of capital for the following functional forms: $v(q) = q$, and $u(q) = 2\sqrt{q}$. When trading frictions are severe, the first-best allocation is not implementable and it is optimal to accumulate capital above k^* (top left panel). The rate of return of capital falls below the rate of time preference, but it is always strictly positive (top right panel). When the trading probability in the DM is sufficiently large, buyers have incentives to hold sufficient real balances to trade the first-best level of output without distorting the capital stock. Figure 6 provides an example where $\alpha < \beta$. Irrespective of the frictions in the DM, the capital stock stays at its efficient level (top left panel), and the real interest rate is equal to the rate of time preference (top right panel). As the frequency of trade increases, output and real balances increase until the first-best allocation is achieved.

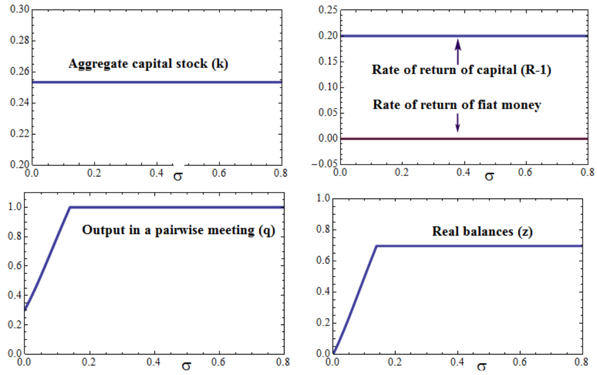


Figure 6: $A = 2$, $\alpha = 0.2$, $r = 0.2$

5 Inflation and capital

In this section we extend the model to allow for growth in the money supply and investigate the relationship between capital and inflation. The money growth rate, $\gamma \geq \beta$, is constant, and new money is injected (withdrawn if $\gamma < 1$) by lump-sum transfers (or taxes if $\gamma < 1$) to buyers in the CM.[17] The quantity of fiat money per buyer at the beginning of period t is $M_t > 0$, with $M_{t+1} = \gamma M_t$. Since we focus on stationary allocations, $M_t\phi_t$ is constant over time and, as a consequence, $\phi_{t+1} = \phi_t/\gamma$.

The only change relative to the case of constant money supply is the buyer's CM problem which becomes

$$W^b(z,k) = \max_{\hat{z}\geq 0, \hat{k}\geq 0} \left\{ z + Rk - \gamma\hat{z} - \hat{k} + T + \beta V^b(\hat{z},\hat{k}) \right\}, \tag{41}$$

where $T = (M_{t+1} - M_t)\phi_t$ is the lump-sum transfer. In order to hold $\hat{z}$ real balances in the next period, the buyer must accumulate $\gamma\hat{z}$ units of current real balances (since the rate of return of fiat money is γ^{-1}). Lemma 2 holds where the buyer's participation constraint is generalized to:

$$-iz^p - \left[(1+r) - F'(k^p)\right]k^p + \sigma\left[u\left(q^p\right) - d_z^p - F'(k^p)d_k^p\right] \geq 0, \tag{42}$$

[17] In the case where $\gamma < 1$, we assume that the government has the power to impose infinite penalties on agents who do not pay taxes. The government, however, does not have the technology to monitor DM and CM trades and cannot observe agents' asset holdings. In contrast, Hu, Kennan, and Wallace (2009) assume that agents can avoid paying taxes by skipping the CM. In this case, there is an upper bound on the rate at which the government can contract the money supply and, in some cases, the Friedman rule is not feasible.

where $i = (\gamma - \beta)/\beta$. The cost of holding real balances in (21), rz^p, is now replaced with iz^p. This cost includes the inflation tax that reduces the value of real balances and the loss of utility due to discounting. An analogous change applies to the seller's participation constraint (23). Similarly, a generalized version of Lemma 3 holds where the constraint (32) is replaced with

$$\sigma[u(q^p) - v(q^p)] - [(1+r) - F'(k^p)]k^p - iz^p \geq 0. \tag{43}$$

The rest of this section specializes to the case of a linear technology, $F(k) = Ak$, as this case allows for closed-form solutions and a straightforward comparison with the existing literature (e.g., Wallace, 1980; Lagos and Rocheteau, 2008).[18] The first-best capital stock is $k^* \in \arg\max [Ak - (1+r)k]$. If $A = 1+r$, then k^* can take any value in $\mathbb{R}_+$. If $A < 1+r$, then $k^* = 0$.

Proposition 3 *Assume $F(k) = Ak$ with $A \leq 1+r$. Let γ^* and $\tilde{\gamma} \geq \gamma^*$ be defined as*

$$\gamma^* = \beta \left\{ 1 + \frac{\sigma\left[u\left(q^*\right) - v(q^*)\right]}{v(q^*)} \right\} \tag{44}$$

$$\tilde{\gamma} = \beta \left\{ 1 + \frac{\sigma\left[u\left(\tilde{q}\right) - v(\tilde{q})\right]}{v(\tilde{q})} \right\}, \tag{45}$$

where $\tilde{q} \leq q^*$ solves

$$u'\left(\tilde{q}\right) = \left[1 + \left(\frac{1+r-A}{\sigma A}\right)\right] v'(\tilde{q}). \tag{46}$$

Then, a constrained-efficient outcome, (q^p, z^p, k^p), is characterized as follows.

1. *If $A = 1+r$, then $q^p = q^*$ and fiat money is inessential.*

2. *If $A < 1+r$ and $\gamma \leq \gamma^*$, then $k^p = 0$, $z^p \geq v(q^*)$ and $q^p = q^*$.*

3. *If $A < 1+r$ and $\gamma \in (\gamma^*, \tilde{\gamma}]$, then $k^p = 0$, and $z^p = v(q^p)$, where $q^p \in [\tilde{q}, q^*)$ is the unique positive solution to*

$$-iv(q^p) + \sigma\left[u\left(q^p\right) - v(q^p)\right] = 0. \tag{47}$$

[18] A formal treatment of the case where $F(k)$ is strictly concave is provided in Rocheteau (2011a) and in the Appendix C. We show that the rate-of-return dominance prevails irrespective of the inflation rate (provided that $i > 0$). Moreover, there is a threshold for the cost of holding money below which the first best is implementable irrespective of the capital stock. For such low money growth rates inflation generates no welfare cost, and there is no Tobin effect.

4. If $A < 1 + r$ and $\gamma > \tilde{\gamma}$, then $q^p = \tilde{q}$ and

$$z^p = \frac{\beta A}{\gamma A - 1}\left\{\sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right] - \left(\frac{1 + r - A}{A}\right)\upsilon(\tilde{q})\right\} > 0 \tag{48}$$

$$k^p = \frac{\beta}{\gamma A - 1}\left\{\upsilon\left(\tilde{q}\right)\left[\gamma(1 + r) - 1\right] - \sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right]\right\} > 0. \tag{49}$$

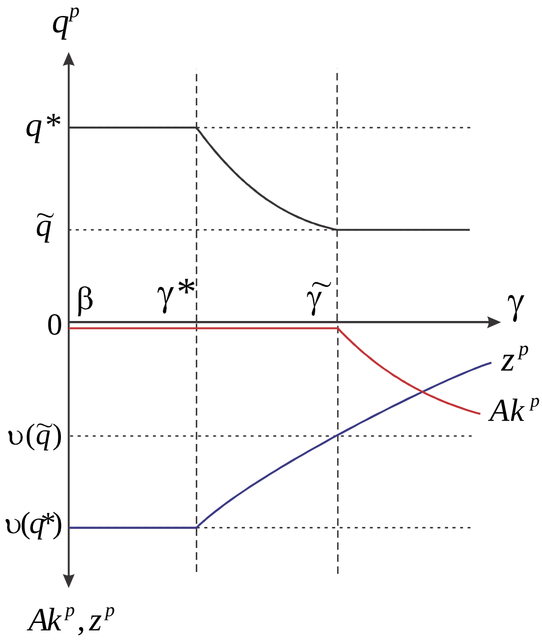


Figure 7: Output, real balances, and capital under a linear technology, $F(k) = Ak$, with $A < 1 + r$.

Provided that the inflation rate is not too large, the first best can be implemented with fiat money as the only medium of exchange. (See the left part of Figure 7.) The threshold for the money growth rate, γ^*, below which the first best is implementable is the same as the one in a pure currency economy. It can be interpreted as follows. The term $\gamma^*\beta^{-1} - 1$ is the cost of holding real balances due to inflation and discounting. The term on the right side of (44), $\sigma\left[u\left(q^*\right) - \upsilon(q^*)\right]/\upsilon(q^*)$, is the expected nonpecuniary rate of return of money, i.e., the probability that a buyer has an opportunity to trade in the DM, times the first-best surplus expressed as a fraction of the cost to produce q^*. The first best is implementable if the cost of holding real balances is no greater than the nonpecuniary return of money.

In order to understand the role of fiat money when the first best is not implementable, consider the nonmonetary economy, $z = 0$. If $A < 1+r$, then social welfare, $-(1+r-A)k + \sigma\left[u(q) - \upsilon(q)\right]$, is maximum at $q = \tilde{q}$ and $k = \upsilon(\tilde{q})/A$. The introduction of fiat money reduces the inefficiently high capital stock to zero provided that the money growth rate is less than some threshold $\tilde{\gamma}$. Moreover, because the buyer's participation constraint is still slack when the capital stock has been reduced to zero, real balances can be raised further to increase output, $q > \tilde{q}$. It follows that if $\gamma \in (\gamma^*, \tilde{\gamma})$, an increase in inflation has no effect on capital, $k = k^* = 0$, but it reduces DM output.

In contrast, if $\gamma > \tilde{\gamma}$, then the capital stock cannot be reduced to zero without violating the buyer's participation constraint. The optimal allocation is such that buyers hold both money and capital. (See the right part of Figure 7.) As inflation increases, buyers substitute capital for real balances — a Tobin effect — in order to keep their liquid wealth and output constant.

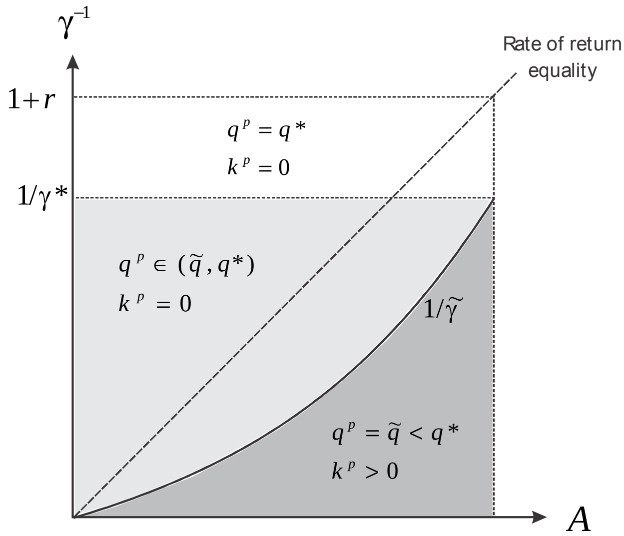


Figure 8: Constrained-efficient allocations under a linear technology: $F(k) = Ak$.

Proposition 3 is illustrated in Figure 8. The rate of return of capital is on the horizontal axis, while the rate of return of fiat money is on the vertical axis. There is rate-of-return equality on the 45^{o} line, and underneath the 45^{o} line there is rate-of-return dominance. In the overlapping generations economy of Wallace (1980) and the random-matching economy of Lagos and Rocheteau (2008) an equilibrium in which fiat money and capital coexist can only occur in the knife-edge case where the two assets have the same rate of return, $A = \gamma^{-1}$. In contrast, under an optimal mechanism, agents never hold capital if there is rate-of-return equality, even if the DM output is inefficiently low. Equilibria in which both fiat money and capital are held (the dark grey area) only exist underneath the 45° line, where capital has a strictly higher rate of return than fiat money.

6 Discussion

In the following we contrast the optimal trading mechanism derived in this paper to alternative trading mechanisms commonly used in the literature. We will distinguish mechanisms where the DM trade is contingent on the buyer's total wealth from more general mechanisms where the DM trade can depend on the composition of the buyer's portfolio. The set of all trading mechanisms that are Pareto-optimal within the pair—they satisfy the pairwise core requirement—and generate incentive-feasible outcomes is represented in Figure 9.

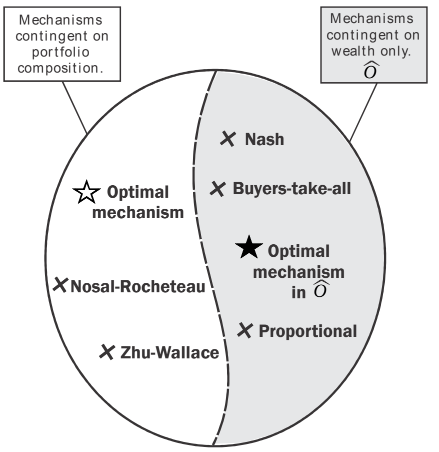


Figure 9: Trading mechanisms in pairwise meetings

Mechanisms contingent on total wealth only. Suppose that we restrict the mechanism in the DM to a function, $\hat{o}_t : \mathbb{R}_+ \to \mathbb{R}^2_+$, that maps the buyer's wealth (expressed in terms of the numéraire good), $\omega \equiv z + Rk$, into a proposed trade, $(q, d_\omega) \in \mathbb{R}_+ \times [0, \omega]$ where q is the quantity of DM output produced by the seller and consumed by the buyer, and d_ω is a transfer of wealth (both money and capital) from the buyer to the seller. (We assume that sellers do not hold assets.) As indicated in Figure 9, this set of mechanisms, denoted $\widehat{O}$, is a strict subset of all incentive-feasible mechanisms, and it includes standard bargaining solutions, such as Nash or proportional (Kalai) bargaining, or buyers-take-all (ultimatum) bargaining games.[19]

Any such mechanism, $\hat{o}_t$, generates an outcome, $(\hat{q}, \hat{\mathbf{d}}, \hat{\pi}, \mathbf{0})$ with $\hat{\pi} = (\hat{z}, \hat{k})$, characterized by rate-of-return equality, $\hat{R} = F'(\hat{k}) = 1$ (assuming the money supply is constant). The proof is by contradiction. Suppose there is an equilibrium with $\hat{z} > 0$, $\hat{k} > 0$, and $\hat{R} = F'(\hat{k}) > 1$. Relative to this proposed outcome the buyer has a profitable deviation that consists in accumulating all his wealth in the form of capital, $k' = \hat{z}/\hat{R} + \hat{k}$. Since the mechanism in the DM is only contingent on the buyer's wealth, $\hat{o}_t(\hat{z}, \hat{k}) = \hat{o}_t(0, k') = (\hat{q}, \hat{d}_\omega)$, where $\hat{d}_\omega = \hat{d}_z + \hat{R}\hat{d}_k$. Therefore the buyer's expected surplus in the DM net of the cost of holding assets is larger than the one he would obtained under the proposed equilibrium,

$$-\left(1 + r - \hat{R}\right) k' + \sigma \left[u\left(\hat{q}\right) - \hat{d}_\omega\right] > -r\hat{z} - \left(1 + r - \hat{R}\right) \hat{k} + \sigma \left[u\left(\hat{q}\right) - \hat{d}_\omega\right].$$

So a necessary condition for rate-of-return dominance is that the mechanism in pairwise meetings is not restricted to treat all forms of wealth as perfect substitutes for payment purposes.

The bargaining solutions used in the literature are not only suboptimal among all incentive-feasible mechanisms, they are also suboptimal in the restricted set $\widehat{O}$. Indeed, any outcome that satisfies the following conditions is implementable,

$$-[1 + r - F'(\hat{k})]\hat{k} - r\hat{z} + \sigma \left[u\left(\hat{q}\right) - \hat{d}_z - \hat{d}_k\right] \geq 0 \tag{50}$$

$$-v\left(\hat{q}\right) + \hat{d}_z + \hat{d}_k \geq 0 \tag{51}$$

$$F'(\hat{k}) \in [1, 1 + r], \text{ and } F'(\hat{k}) = 1 \text{ if } \hat{z} > 0 \tag{52}$$

$$\hat{d}_z \in [0, \hat{z}], \quad \hat{d}_k \in [0, \hat{k}] \tag{53}$$

Inequalities (50) and (51) are the buyer's and the seller's participation constraints. The restriction that the mechanism, $\hat{o}$, cannot discriminate among different assets is taken into account by (52).

[19] Most of the literature on dual-asset economies with random, pairwise meetings has been using Nash-Kalai or buyers-take-all bargaining. Recent examples include Geromichalos, Licari, and Suarez-Lledo (2007), Lagos and Rocheteau (2008), Lagos (2010), and Lester, Postlewaite, and Wright (2012).

If fiat money is held, $\hat{z} > 0$, then capital and fiat money have the same rate of return, $\hat{R} = 1$. The optimal mechanism among all mechanisms in $\widehat{O}$ maximizes social welfare subject to (50)-(53). To see that standard bargaining solutions do not correspond to this optimal mechanism it is sufficient to notice that in any monetary equilibrium, $q < q^*$. In contrast, from (50) and (51), the first-best level of trade, $\hat{q} = q^*$, is implementable in a monetary equilibrium whenever $-rv\left(q^*\right) + \sigma\left[u\left(q^*\right) - v\left(q^*\right)\right] \geq 0$.

Mechanisms where the portfolio composition matters. First, there are mechanisms where the composition of the portfolio is made relevant by imposing exogenous restrictions on asset transfers in pairwise meetings. For instance, in Aruoba, Waller, and Wright (2010) a buyer holding a portfolio of fiat money and capital can only transfer fiat money in a bilateral match. In Lagos (2010) and Lester, Postlewaite, and Wright (2012) a buyer holding fiat money (or risk-free bonds) and stocks can transfer their stocks only in a fraction $\theta \leq 1$ of all matches.

Second, and closer to our approach, there are (pairwise) Pareto-optimal mechanisms that generate trades that depend on the buyer's portfolio composition. Zhu and Wallace (2007) constructed a mechanism that, in the context of our model, works as follows. Consider a buyer holding z real balances and k units of capital. The trade in the DM assigns to the buyer the same payoff he would obtain in an economy where fiat money is the only means of payment. Formally, the buyer's surplus is

$$U^b(z) = \max_{\tilde{q},\tilde{d}_z \leq z} \left\{u(\tilde{q}) - \tilde{d}_z\right\} \quad \text{s.t.} \quad -v\left(\tilde{q}\right) + \tilde{d}_z = 0, \tag{54}$$

where we denote the solution to this problem by $(\tilde{q}, \tilde{\mathbf{d}})$ to distinguish it from the actual trade. From (54) the buyer's surplus is obtained from a take-it-or-leave-it offer to the seller, but with the restriction that the buyer can only transfer real balances, $\tilde{d}_z \leq z$ and $\tilde{d}_k = 0$. Assuming $z \leq v(q^*)$, the buyer's payoff is $U^b(z) = u\left[v^{-1}(z)\right] - z$, and it is independent of his holdings of capital. Given this surplus, the actual trade is determined by the following program:

$$\max_{q,d_z,d_k} \left\{-v\left(q\right) + d_z + Rd_k\right\} \tag{55}$$

$$\text{s.t.} \quad u(q) - d_z - Rd_k \geq U^b(z) \tag{56}$$

$$(d_z, d_k) \in [0, z] \times [0, k]\,. \tag{57}$$

From (55)-(56), the trade is in the pairwise core, and from (56) it is such that the buyer enjoys a surplus equal to $U^b(z)$. The feasibility constraints in (57) make it clear that there are no restrictions on the use of assets as means of payment.

The buyer's choice of asset holdings maximizes his expected surplus in the DM, $\sigma U^b(z)$, net of the cost of holding assets,

$$\max_{z\geq 0, k\geq 0}\left\{-rz-(1+r-R)k+\sigma U^b(z)\right\}. \tag{58}$$

It follows from (58) that $R = 1+r$ and $k = k^*$ in any equilibrium. (Since buyers are indifferent between holding capital or not, we focus on symmetric equilibria where all buyers hold k^* and sellers hold no capital.) Given that $U^{b\prime}(0) = +\infty$, the value of (58) is strictly positive at the optimum. The choice of real balances is $z = v(\tilde{q})$ where $\tilde{q}$ solves $r/\sigma = u'(\tilde{q})/v'(\tilde{q}) - 1$. For all $r > 0$, $U^b(z) = u(\tilde{q}) - v(\tilde{q}) < u(q^*) - v(q^*)$, i.e., the buyer's surplus in the DM is less than the match surplus at the first best. The fact that $\tilde{q} < q^*$ implies that the solution to (55)-(57) with $z = v(\tilde{q})$ is such that $d_k > 0$, i.e., it is Pareto-optimal to use capital goods as means of payment. Moreover, $d_k > 0$ implies that the seller enjoys a positive surplus. Consequently, any allocation under the Zhu-Wallace mechanism is such that

$$-rz+\sigma\left[u\left(q\right)-d_z-Rd_k\right]>0 \tag{59}$$

$$-v\left(q\right)+d_z+Rd_k>0. \tag{60}$$

From (59) and (60) both the buyer's and the seller's participation constraints are slack.

From (56) the Zhu-Wallace mechanism can implement the first-best outcome when $(1+r)k^* \geq u(q^*) - u\left[v^{-1}(z)\right]$. In contrast, according to Proposition 2 under the optimal mechanism the first best is implementable whenever $(1+r)k^* \geq v(q^*) - \sigma\left[u(q^*) - v(q^*)\right]/r$, and it can be shown that $v(q^*) - \sigma\left[u(q^*) - v(q^*)\right]/r < u(q^*) - u(\tilde{q})$. Therefore, the Zhu-Wallace mechanism fails to implement the first best in situations in which it is implementable. Moreover, when the Zhu-Wallace mechanism fails to implement the first best, it also fails to deliver a constrained-efficient allocation. First, the capital stock is always chosen to be at its first-best level. In contrast, we derived a condition in Proposition 2 under which it is optimal to accumulate capital above the first best in order to mitigate the economy's liquidity shortage. Second, and also in contradiction with the constrained-efficient outcome, when the first best is not implementable both the buyer's and the seller's participation constraints are slack.

Nosal and Rocheteau (2013) extend the Zhu-Wallace mechanism so that the buyer can receive a share of the surplus that capital goods generate in the DM. Formally, the buyer's payoff is the one he would get in an economy where he can transfer all his real balances and a fraction $\theta \leq 1$ of his capital stock, i.e., $U^b(z,k) = u\left[v^{-1}(z + \theta Rk)\right] - (z + \theta Rk)$. Following the same logic as (58), in any monetary equilibrium the rate of return of capital is $R = (1+r)/(1+\theta r)$. So provided that $\theta < 1$, the trading mechanism generates rate-of-return dominance. Moreover, there is a $\theta \in [0,1)$ such that the mechanism implements the constrained-efficient capital stock. But the mechanism is socially inefficient for similar reasons as the ones mentioned above.

7 Conclusion

This paper was about an empirical puzzle raised by Hicks (1935), the coexistence of money and higher return assets. We addressed this empirical puzzle with a normative analysis asking whether a socially optimal trading arrangement was consistent with the puzzle. Implicitly this approach is based on the premise that socially inefficient arrangements cannot perdure and that societies tend to adopt optimal ones. We obtained a positive answer: the coexistence of money and higher-return capital is a property of good allocations in monetary economies. We showed that the use of high-return assets as media of exchange is socially desirable to increase agents' incentives to hold assets in situations in which (unsecured) credit arrangements are not incentive feasible. While it can be optimal to increase the capital stock above its first-best level to mitigate a shortage of liquid wealth, it is never beneficial from the society's view point to drive the rate of return of capital down to the rate of return of fiat money.

References

[1] Aiyagari, S. R, Wallace, N., and Wright, R., 1996, Coexistence of money and interest-bearing securities. Journal of Monetary Economics. 37, 397–419.

[2] Aruoba, B., and Wright, R., 2003, Search, money and capital: a neoclassical dichotomy. Journal of Money, Credit, and Banking. 35, 1085-1105.

[3] Aruoba, B., Waller, C., and Wright, R., 2011, Money and capital. Journal of Monetary Economics. 58, 98-116.

[4] Avriel, M., 2003, Nonlinear Programming: Analysis and Methods. Dover.

[5] Banerjee, A. and Maskin, E., 1996, A Walrasian theory of money and barter. The Quarterly Journal of Economics. 111, 955–1005.

[6] Caballero, R., 2006, On the macroeconomics of asset shortages. NBER Working Paper, 12753.

[7] Cavalcanti, R., and Erosa, A., 2008, Efficient propagation of shocks and the optimal return on money. Journal of Economic Theory. 142, 128-148

[8] Cavalcanti, R., and Nosal, E. 2009, Some benefits of cyclical monetary policy. Economic Theory. 39, 195-216.

[9] Cavalcanti, R., and Wallace, N., 1999, Inside and outside money as alternative media of exchange. Journal of Money, Credit, and Banking. 31, 443–457.

[10] Deviatov, A., 2006, Money creation in a random matching model. Topics in Macroeconomics 6.

[11] Deviatov, A., and Wallace, N., 2001, Another example in which money creation is beneficial. Advances in Macroeconomics 1.

[12] Freeman, S., 1985, Transactions costs and the optimal quantity of money. Journal of Political Economy. 93, 146-57.

[13] Fudenberg, D., and Tirole, J., 1991, Game Theory. MIT press.

[14] Geanakoplos, J., and Zame, W., 2010, Collateral equilibrium. Working Paper.

[15] Geromichalos, A., Licari, J. M., and Suarez-Lledo, J., 2007, Monetary policy and asset prices. Review of Economic Dynamics. 10, 761-779.

[16] Gu, C., Mattesini, F., Monnet, C., and Wright, R., 2013, Banking: a mechanism design approach. Review of Economic Studies (Forthcoming).

[17] Hicks, J., 1935, A suggestion for simplifying the theory of money. Economica. 2, 1–19.

[18] Hu, T.-W., 2013, Imperfect recognizability and coexistence of money and higher-return assets. Economic Theory (forthcoming).

[19] Hu, T.-W., Kennan, J., and Wallace, N., 2009, Coalition-proof trade and the Friedman rule in the Lagos-Wright model. Journal of Political Economy. 117, 116-137.

[20] Hu, T.-W., and Rocheteau, G., 2013, Monetary policy and asset prices: A mechanism design approach. Working Paper of the University of California, Irvine.

[21] Kiyotaki, N. and Wright, R., 1989, On money as a medium of exchange. Journal of Political Economy. 97, 927–54.

[22] Kocherlakota, N., 1998, Money is memory. Journal of Economic Theory. 81, 232–251.

[23] Kocherlakota, N., 2003, Societal benefits of illiquid bonds. Journal of Economic Theory. 108, 179-193.

[24] Kocherlakota, N., and Wallace, N., 1998, Incomplete record-keeping and optimal payment arrangements. Journal of Economic Theory. 81, 272–289.

[25] Koeppl, T., Monnet, C., and Temzelides, T., 2008, A dynamic model of settlement. Journal of Economic Theory. 142, 233–246.

[26] Lagos, R., 2010, Asset prices and liquidity in an exchange economy. Journal of Monetary Economics. 57, 913-930.

[27] Lagos, R., 2013, Moneyspots: extraneous attributes and the coexistence of money and interest-bearing nominal bonds. Journal of Political Economy (Forthcoming).

[28] Lagos, R. and Rocheteau, G., 2008, Money and capital as competing media of exchange. Journal of Economic Theory. 142, 247-258.

[29] Lagos, R. and Wright, R., 2005, A unified framework for monetary theory and policy analysis. Journal of Political Economy. 113, 463-484.

[30] Lester, B., Postlewaite, A., and Wright, R., 2012, Information, liquidity and asset prices. Review of Economic Studies. 79, 1209-38.

[31] Li, Y., Rocheteau, G., Weill, P.-O., 2012, Liquidity and the threat of fraudulent assets. Journal of Political Economy. 120, 815-846.

[32] Li, Y., and Wright, R., 1998, Government transaction policy, media of exchange, and prices. Journal of Economic Theory. 81, 290–313.

[33] Molico, M. and Zhang, Y., 2006, Monetary policy and the distribution of money and capital. Computing in Economics and Finance. 2006, 136.

[34] Nosal, E. and Rocheteau, G., 2011, Money, Payments, and Liquidity. MIT Press, Cambridge.

[35] Nosal, E. and Rocheteau, G., 2013, Pairwise trades, asset prices, and monetary policy. Journal of Economic Dynamics and Control. 31, 1-17.

[36] Renero, J.-M., 1999, Does and should a commodity medium of exchange have relatively low storage costs? International Economic Review. 40, 251-264.

[37] Rocheteau, G., 2011a, On the coexistence of money and higher-return assets and its social role. Working Paper of the Federal Reserve Bank of Cleveland 11-04.

[38] Rocheteau, G., 2011b, Payments and liquidity under adverse selection. Journal of Monetary Economics. 58, 191-205.

[39] Rocheteau, G., 2012, The cost of inflation: A mechanism design approach. Journal of Economic Theory. 147, 1261–1279.

[40] Shi, S., 1997, A divisible search model of fiat money. Econometrica. 65, 75–102.

[41] Shi, S., 1999, Search, inflation and capital accumulation. Journal of Monetary Economics. 44, 81-103.

[42] Shi, S., 2008, Efficiency improvement from restricting the liquidity of nominal bonds. Journal of Monetary Economics. 55, 1025-1037.

[43] Vayanos, D., and Weill, P.-O., 2008, A search-based theory of the on-the-run phenomenon. Journal of Finance. 63, 1351-1389.

[44] Wallace, N., 1980, The overlapping generations model of fiat money. in Models of Monetary Economies, Eds, Kareken, J. and Wallace, N., Federal Reserve Bank of Minneapolis, pp. 49-82.

[45] Wallace, N., 1983, A legal restrictions theory of the demand for money and the role of monetary policy. Federal Reserve Bank of Minneapolis Quarterly Review. Winter, 1-7.

[46] Wallace, N., 1998, A dictum for monetary theory. Federal Reserve Bank of Minneapolis Quarterly Review. 22, 20–26.

[47] Wallace, N., 2000, A model of the liquidity structure based on asset indivisibility. Journal of Monetary Economics. 45, 55-68.

[48] Wallace, N., 2010, The mechanism-design approach to monetary theory. Handbook of Monetary Economics, Volume 3, Friedman, B. M., and Woodford, M., (Eds), Chapter 1, 3-23.

[49] Williamson, S. and Wright, R., 2010, New monetarist economics: models. Handbook of Monetary Economics, Volume 3, Friedman, B. M., and Woodford, M., (Eds), Chapter 2, 25-96.

[50] Zhu, T. and Wallace, N., 2007, Pairwise trade and coexistence of money and higher-return assets. Journal of Economic Theory. 133, 524-535.

Appendix A: Proofs

Proof of Lemma 1: The linearity in (10) follows directly from (3). Now consider (11). Following the equilibrium behavior in a simple equilibrium, the buyer is supposed to leave the CM with the portfolio (z_b^p, k_b^p). Thus, in equilibrium, the solution to the maximization problem in (3) is $(\hat{z}_b, \hat{k}_b) = (z_b^p, k_b^p)$ and hence

$$W^b(0,0) = -z_b^p - k_b^p + \beta V^b(z_b^p, k_b^p).$$

Substituting $V^b(z_b^p, k_b^p)$ with its expression given by (4), noting that in equilibrium $o(\pi_b^p, \pi_s^p) = (q^p, \mathbf{d}^p)$, and using the linearity of W^b, we obtain

$$W^b(0,0) = -z_b^p - k_b^p + \beta\{\sigma[u(q^p) - d_z^p - Rd_k^p] + z_b^p + Rk_b^p + W^b(0,0)\},$$

which gives the expression for $W^b(0,0)$ in (11). From equation (9), $R = F'(k^p)$. The expression for V^b given by (14) is directly obtained from (4) and the linearity of W^b.

The linearity in (12) follows from (7). In equilibrium, the optimal solution for the maximization problem in (7) is $k_s' = k^p$ and $(\hat{z}_s, \hat{k}_s) = (z_s^p, k_s^p)$. Substituting $V^s(z_s^p, k_s^p)$ with its expression given by (8), noting that in equilibrium $o(\pi_b^p, \pi_s^p) = (q^p, \mathbf{d}^p)$, and using the linearity of W^s, we obtain

$$W^s(0,0) = -z_s^p - k_s^p + \beta\{\sigma[-v(q^p) + d_z^p + Rd_k^p] + z_s^p + Rk_s^p + F(k^p) - Rk^p + W^s(0,0)\},$$

which gives the expression for $W^s(0,0)$ in (13). Finally, the expression for V^b given by (15) is directly obtained from (8) and the linearity of W^s. ■

Proof of Lemma 2: The solutions to the maximization problems (25)-(27) exist and each solution has a unique $q(\pi_b, \pi_s)$. Although d_z and d_k may not be uniquely determined, we will select the solution such that $d_z(\pi_b, \pi_s) = z_b$ if it exists and $d_k(\pi_b, \pi_s) = 0$ otherwise for any $(\pi_b, \pi_s) \neq (\pi_b^p, \pi_s^p)$. To show that $(q^p, \mathbf{d}^p)$ is a solution to (25) for $(\pi_b, \pi_s) = (\pi_b^p, \pi_s^p)$, notice that (25) is the dual problem that defines the core of a pairwise meeting, (16)-(19). Because $(q^p, \mathbf{d}^p) \in \mathcal{CO}(\pi_b^p, \pi_s^p; R)$, it is also a solution to (25). This gives us a well-defined mechanism, o. Moreover, because for any (π_b, π_s), $q(\pi_b, \pi_s) \leq q^*$, it follows from the buyer's participation constraint that $d_z(\pi_b, \pi_s) \leq u(q^*)$ and $Rd_k(\pi_b, \pi_s) \leq u(q^*)$.

The following observation will be useful to introduce a bound on asset holdings. For any $\pi_b = (z_b, k_b) \geq \pi_b^p$ and $\pi_b' = (z_b', k_b') \geq \pi_b^p$, if $z_b + Rk_b \geq u(q^*)$ and $z_b' + Rk_b' \geq u(q^*)$, then $o(\pi_b, \pi_s^p) = o(\pi_b', \pi_s^p)$ with $q(\pi_b, \pi_s^p) = q^*$. Similarly, for any $\pi_s = (z_s, k_s) \geq \pi_s^p$ and $\pi_s' = (z_s', k_s') \geq \pi_s^p$, $o(\pi_b^p, \pi_s) = o(\pi_b^p, \pi_s')$.

Now we show that the following strategy profile, (s_b^*, s_s^*), form a simple equilibrium: for all t and for all h^t, $(s_b^*)^{h^t,0}(\pi_b) = \pi_b^p$, $(s_b^*)^{h^t,1}(\pi_b, \pi_s) = yes$, $(s_b^*)^{h^t,2}(\pi_b, \pi_s, a_b, a_s) = \pi_b^p$, and $(s_s^*)^{h^t,0}(\pi_s) = \pi_s^p$, $(s_s^*)^{h^t,1}(\pi_b, \pi_s) = yes$, and $(s_s^*)^{h^t,2}(\pi_b, \pi_s, a_b, a_s) = \pi_s^p$. In words, irrespective of their portfolios when entering the CM, buyers and sellers exit the CM with their proposed portfolios, π_b^p and π_s^p. In the DM they always say *yes* to the proposals. We show that s_b^* and s_s^* are optimal strategies following any history, given that all other agents follow (s_b^*, s_s^*). To this end we first show that, given that all other agents follow (s_b^*, s_s^*), any strategy s_b or s_s that specifies large asset-holdings after the CM is weakly dominated by another strategy with bounded asset holdings.

Specifically, we will use the following bound on asset holdings, expressed in the numéraire good:

$$\bar{a} = \max\{Rk_b^p, z_b^p, u(q^*)\}.$$

Assuming that all other agents follow (s_b^*, s_s^*), we claim that for any strategy s_b, specifying the buyer to leave the CM with a portfolio $\pi_b = (z_b, k_b)$ such that $z_b > \bar{a}$ or $Rk_b > \bar{a}$ following some history, is weakly dominated by another strategy s_b' constructed by induction as follows.

(Period-0) At period 0 (at which only the CM is open), if the portfolio choice under s_b is (z_b, k_b), then the portfolio choice under s_b' is $(\min\{\bar{a}, z_b\}, \min\{\bar{a}/R, k_b\})$.

(Period-t) Suppose that s_b' are defined for all histories before period t. At period t, for each history up to t, h^t, that is generated by s_b and by others following (s_b^*, s_s^*) (and some given realization of matching opportunities), there is a corresponding history, $(h')^t$, that is generated by s_b' and others following (s_b^*, s_s^*). For such histories, h^t and $(h')^t$, we require the alternative strategy s_b' to specify the same DM response under $(h')^t$ as those specified by the strategy s_b under the corresponding history h^t, conditional on being matched with a seller. In the period-t CM following the corresponding histories and responses, if the protfolio choice under s_b is (z_b, k_b), then the portfolio choice under s_b' is $(\min\{\bar{a}, z_b\}, \min\{\bar{a}/R, k_b\})$. For all other histories (which will not be reached under either s_b or s_b' if others follow (s_b^*, s_s^*)), s_b' specifies the same actions as s_b^*.

Given the construction, for any period t and for any realization of matching opportunities, the buyer's portfolio when entering period-t DM differ under s'_b and s_b only if the buyer holds more than $\bar{a}$ (in terms of its value in the CM) of one asset under s_b. When the portfolios differ under the two strategies upon entering the DM, the portfolios under both strategies are such that $\pi_b = (z_b, k_b) \geq \pi_b^p$ and $z_b + Rk_b \geq u(q^*)$, and hence the proposed trades are identical, with $q = q^*$ and with the same transfers. Thus, the DM outcomes coincide under the two strategies. However, because the asset holdings are weakly lower under s'_b than those under s_b and because $R \leq 1 + r$, the discounted sum of CM payoffs is weakly lower under s_b than that under s'_b. Formally, for a given realization of matching opportunities, let (z_t, k_t) and (z'_t, k'_t) be the buyer's portfolio when leaving the period-t CM under s_b and s'_b, respectively. We have shown that $(z'_t, k'_t) \leq (z_t, k_t)$ for all t. Because the buyer has exactly the same transfers in the DM, the difference between the discounted sum of the CM consumption under s'_b and that under s_b up to date T is given by (notice that $R \leq 1 + r = 1/\beta$)

$$
\begin{aligned}
\Delta_T \equiv\ & (-z'_0 - k'_0) - (-z_0 - k_0) + \\
& \sum_{t=1}^{T} \beta^t [(z'_{t-1} - z'_t) - (z_{t-1} - z_t) + (Rk'_{t-1} - k'_t) - (Rk_{t-1} - k_t)] \\
=\ & \sum_{t=0}^{T-1} \beta^t [(-z'_t + \beta z'_t) - (-z_t + \beta z_t) + (-k'_t + \beta Rk'_t) - (-k_t + \beta Rk_t)] \\
& + \beta^T [(z_T - z'_T) + (k_T - k'_T)] \\
=\ & \sum_{t=0}^{T-1} \beta^t (1 - \beta)(z_t - z'_t) + \sum_{t=0}^{T-1} \beta^t (1 - \beta R)(k_t - k'_t) + \beta^T [(z_T - z'_T) + (k_T - k'_T)].
\end{aligned}
$$

Because $z_t \geq z'_t$ and $k_t \geq k'_t$ for all t, $\Delta_T \geq 0$ for all T. The limit of Δ_T may not exist as T goes to infinity, however. The limits for the two terms $\sum_{t=0}^{T-1} \beta^t (1-\beta)(z_t - z'_t)$ and $\sum_{t=0}^{T-1} \beta^t (1 - \beta R)(k_t - k'_t)$ always exist (although they may be positive infinity), but the term $\beta^T [(z_T - z'_T) + (k_T - k'_T)]$ may not converge. Because the per-period payoffs under s'_b are bounded and hence its aggregate payoff is well-defined, the problem comes from the original strategy, s_b, whose aggregate payoff (against (s_b^*, s_s^*)) may not be well-defined. When that happens, we assume that the payoff criterion is the limit inferior of the discounted sum of per-period payoffs. Because $\Delta_T \geq 0$ for all T, the difference of the aggregate payoffs (w.r.t. limit inferior) between s'_b and s_b is no less than $\limsup_{T \to \infty} \Delta_T \geq 0$.

Thus, s_b' weakly dominates s_b.

To check sequential rationality, we need to show that such weak dominance holds following any private trading history of a buyer. Because our mechanism is stationary and equilibrium strategies in a simple equilibrium are stationary, following any private trading history, the same construction works. Hence, any continuation strategy with asset holdings exceeding $\bar{a}$ is weakly dominated by another strategy that does not exceed $\bar{a}$.

Thus, to check that s_b^* is optimal among all strategies following any history, it is sufficient to show that it is optimal among all strategies s_b which specify the buyer to leave the CM with both assets less than $\bar{a}$ following all histories (notice that (s_b^*, s_s^*) satisfy this requirement as well). Because the per-period payoffs are bounded for such strategies, by Fudenberg and Tirole (1991), Theorem 4.2, we can use the one deviation property. A symmetric argument holds for sellers as well.

Now we use the one deviation property to show that the strategies (s_b^*, s_s^*) are optimal. First consider buyers. From (25) and (26), for any π_b, if $o(\pi_b, \pi_s^p) = (q, d_z, d_k)$, then $u(q) - d_z - F'(k^p)d_k \geq 0$. Thus, the buyer is willing to respond with *yes* in the DM. Moreover, from (6) and the conditions for o in (25)-(26), the buyer's problem in the CM is

$$\max_{z_b \geq 0, k_b \geq 0} -rz_b - [1 + r - F'(k^p)]k_b + \sigma \left[u(q^p) - d_z^p - F'(k^p)d_k^p\right] \mathbb{I}_{\{z_b \geq z_b^p, k_b \geq k_b^p\}},$$

where $\mathbb{I}_{\{z_b \geq z_b^p, k_b \geq k_b^p\}} = 1$ if $z_b \geq z_b^p$ and $k_b \geq k_b^p$, and $\mathbb{I}_{\{z_b \geq z_b^p, k_b \geq k_b^p\}} = 0$ otherwise. From (21) it is optimal for buyers to leave the CM with π_b^p. By the one deviation property, the strategy s_b^* is optimal.

Next consider sellers. From (25) and (27), for any π_s, if $o(\pi_b^p, \pi_s) = (q, d_z, d_k)$, then $-v(q) + d_z + F'(k^p)d_k \geq 0$. Thus, the seller is willing to respond with *yes* in the DM. Moreover, from (6) and the conditions for o in (25) and (27), the seller's problem in the CM is

$$\max_{z_b \geq 0, k_b \geq 0} -rz_s - [1 + r - F'(k^p)]k_s + \sigma \left[-v(q^p) + d_z^p + F'(k^p)d_k^p\right] \mathbb{I}_{\{z_s \geq z_s^p, k_s \geq k_s^p\}}.$$

From (23) it is optimal for sellers to leave the CM with π_s^p. Thus, by the one deviation property, the seller strategy s_s^* is optimal. ■

Proof of Lemma 3:

(1) Let $U = \{(q^p, z^p, k^p) \in \mathbb{R}^3_+ : q^p \leq q^*, k^* \leq k^p \leq \max\{k^*, \bar{k}\}, z^p \leq \sigma[u(q^*) - v(q^*)]/r\}$. We show that for any $(q^p, z^p, k^p) \notin U$ but satisfies (32) to (34), there exists another $(q', z', k') \in U$ that also satisfies (32) to (34) but has a strictly higher value to (31). Because U is a compact set and the objective function (31) is continuous, this implies that a solution to the problem (31)-(34) exists. Moreover, because the constraints are all convex sets, the solutions have unique values for k^p and q^p.

Let $(q^p, z^p, k^p) \notin U$ but satisfies (32) to (34). We will construct a triple $(q', z', k') \in U$ that also satisfies (32) to (34) but has a strictly higher value to (31). We distinguish two cases.

(1.a) Suppose that $k^* \geq \bar{k}$. Consider an optimal solution $(q', z', k') = (q^*, 0, k^*) \in U$. It satisfies (32) to (34): (34) holds because $k^p = k^*$ and hence $F'(k^p) = 1 + r$; (32) holds because $k^p = k^*$ and hence $[1 + r - F'(k^p)]k^p = 0$, and because $z^p = 0$; by the definition of $\bar{k}$ and by the fact that $F'(k)k$ is strictly increasing,

$$v(q') = v(q^*) = F'(\bar{k})\bar{k} \leq F'(k^*)k^* = F'(k^p)k^p,$$

and hence (33) holds. Thus, for any $(q^p, z^p, k^p) \notin U$ that satisfies (32) to (34), either $q^p > q^*$ or $k^p > k^*$, and in either case it is dominated by (q', z', k').

(1.b) Suppose that $k^* < \bar{k}$. Let $(q', z', k') = (\min\{q^*, q^p\}, z^p, \min\{\bar{k}, k^p\})$. Because $(q^p, z^p, k^p) \notin U$, either $q^p > q^*$ or $k^p > \bar{k}$, and hence (q', z', k') has a strictly higher value than (q^p, z^p, k^p).

We show that (q', z', k') satisfies (32) to (34). The constraint (34) is satisfied because $k^p \geq k^*$ and hence $k' \geq k^*$. Now consider (32). Because $k' \leq k^p$ and $q' = \min\{q^*, q^p\}$, $[1 + r - F'(k')]k' \leq [1 + r - F'(k^p)]k^p$ and $u(q') - v(q') \geq u(q^p) - v(q^p)$. Thus,

$$\sigma[u(q') - v(q')] - rz' - [1 + r - F'(k')]k' \geq \sigma[u(q^p) - v(q^p)] - rz^p - [1 + r - F'(k^p)]k^p \geq 0.$$

Next, consider (33). If $k^p \leq \bar{k}$, then $k' = k^p$. Because $q' \leq q^p$, (33) is satisfied by (q', z', k') because it is satisfied by (q^p, z^p, k^p). Assume now that $k^p > \bar{k}$. Then, $k' = \bar{k}$. But by the definition of $\bar{k}$, we have

$$-v(q') + F'(\bar{k})\bar{k} + z' \geq -v(q^*) + F'(\bar{k})\bar{k} = 0.$$

Thus, we may add the constraint $(q, z, k) \in U$ to the problem (31)-(34) without affecting its solutions. Hence, any solution, (q^p, z^p, k^p), to that problem satisfies $(q^p, z^p, k^p) \in U$.

(2) We distinguish two cases: (ii.a) for some $z_0 \geq 0$, (q^*, z_0, k^*) satisfies (32)-(34); (ii.b) (q^*, z_0, k^*) does not satisfy (32)-(34) for any $z_0 \geq 0$.

(2.a) For some $z_0 \geq 0$, (q^*, z_0, k^*) satisfies (32) to (34). Then, (q^p, z^p, k^p) solves the problem (31)-(34) if and only if it satisfies (32) to (34) and $q^p = q^*$, $k^p = k^*$. Let (q^p, z^p, k^p) be a solution (and hence $q^p = q^*$ and $k^p = k^*$). We construct a payment $\mathbf{d}^p$ so that $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$ with $\pi_b^p = (z^p, k^p)$ and $\pi_s^p = \mathbf{0}$ is a constrained-efficient outcome. If $v(q^*) \leq z^p$, then let $d_z^p = v(q^*)$ and let $d_k^p = 0$. Otherwise, let $d_z^p = z^p$ and let $d_k^p = v(q^*) - z^p/F'(k^p)$. Because (q^p, z^p, k^p) satisfies (32)-(34), in either case $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$ satisfies (21), (23), and (24). The pairwise core requirement is satisfied because $q^p = q^*$.

(2.b) For any $z \geq 0$, (q^*, z, k^*) does not satisfy (32)-(34). We need four steps.

(b.1) First, we show that if (q^p, z^p, k^p) solves (31)-(34), then for some $\mathbf{d}^p$, $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ with $\pi_b^p = (z^p, k^p)$ satisfies (21), (23), (24), and the pairwise core requirement. We distinguish two cases according to the magnitude of q^p.

(b.1.1) Suppose that $q^p = q^*$. If $v(q^*) \leq z^p$, then let $d_z^p = v(q^*)$ and let $d_k^p = 0$. Otherwise, let $d_z^p = z^p$ and let $d_k^p = v(q^*) - z^p/F'(k^p)$. In either case $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ is implementable. The pairwise core requirement is satisfied because $q^p = q^*$.

(b.1.2) Suppose that $q^p < q^*$. Then, (33) is binding. Suppose, by contradiction, that (33) is not binding. By continuity, for some $q' \in (q^p, q^*)$, (33) holds for (q', z^p, k^p). Then, $u(q') - v(q') > u(q^p) - v(q^p)$ and hence (32) holds for (q', z^p, k^p) as well. However, (q', z^p, k^p) is strictly better than (q^p, z^p, k^p). Thus, $v(q^p) = F'(k^p)k^p + z^p$. Let $d_k^p = k^p$ and let $d_z^p = z^p$. Then, $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ satisfies (21), (23), and (24). Moreover, it satisfies the pairwise core requirement because $d_k^p = k^p$, $d_z^p = z^p$, and $q^p < q^*$ (see Appendix B for the characterization of the pairwise core).

(b.2) Next, we show that for any outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, with $\pi_s^p \neq \mathbf{0}$ that satisfies (21), (23), and (24), and the pairwise core requirement, there exists another outcome, $(q', \mathbf{d}', \pi_b', \mathbf{0})$, that satisfies (21), (23), and (24), and the pairwise core requirement but has higher welfare. As a result, in any constrained-efficient outcome, sellers' asset holdings are zero.

We prove the claim by two steps: (b.2.1) if $k_s^p > 0$, we construct another outcome $(q', \mathbf{d}', \pi_b', \pi_s')$ with $\pi_s' = (z_s^p, 0)$ that has higher welfare; (b.2.2) if $k_s^p = 0$ but $z_s^p > 0$, we construct another outcome $(q', \mathbf{d}', \pi_b', \mathbf{0})$ that has higher welfare.

(b.2.1) $k_s^p > 0$. We distinguish two subcases: $k_s^p \geq k^p - k^*$ and $k_s^p < k^p - k^*$.

Suppose that $k_s^p \geq k^p - k^*$. Let $\pi'_b = (z_b^p, k^*)$ and $\pi'_s = (z_s^p, 0)$. If $k^p > k^*$, then the new outcome, $(q^p, \mathbf{d}^p, \pi'_b, \pi'_s)$, satisfies (21), (23), and (24), but has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$ because it has the first-best level of capital stock while the DM transfer remains the same. Otherwise, we have $k^p = k^*$ and $q^p < q^*$. Let $q' \in (q^p, q^*]$ be such that

$$v(q') - v(q^p) < (1+r)(k^* - k_b^p).$$

Notice that $k^* - k_b^p > 0$ because $k_s^p > 0$. Then, the new outcome, $(q', \mathbf{d}', \pi'_b, \pi'_s)$, with $d'_z = d_z^p$ and $d'_k = d_k^p + [v(q') - v(q^p)]/(1+r)$, also satisfies (21), (23), and (24), but has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$. The constraint (24) is satisfied because the capital stock in the new outcome is k^*. (23) holds for $(q', \mathbf{d}', \pi'_b, \pi'_s)$ because it holds for $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$:

$$\begin{aligned}
& -rz_s^p + \sigma[(1+r)d'_k + d_z^p - v(q')] \\
= \; & -rz_s^p + \sigma\left\{(1+r)\{d_k^p + [v(q') - v(q^p)]/(1+r)\} + d_z^p - v(q')\right\}] \\
= \; & -rz_s^p + \sigma[(1+r)d_k^p + d_z^p - v(q^p)] \geq 0.
\end{aligned}$$

We turn to (21). Because $q' \in (q^p, q^*]$, $u(q') - v(q') > u(q^p) - v(q^p)$,

$$\begin{aligned}
& -rz_b^p + \sigma[u(q') - (1+r)d'_k - d_z^p] \\
= \; & -rz_b^p + \sigma[u(q^p) - (1+r)d_k^p - d_z^p] + \sigma\{u(q') - u(q^p) - (1+r)[v(q') - v(q^p)]/(1+r)\} \\
= \; & -rz_b^p + \sigma[u(q^p) - (1+r)d_k^p - d_z^p] + \sigma\{[u(q') - u(q^p)] - [v(q') - v(q^p)]\} > 0.
\end{aligned}$$

Suppose that $k_s^p < k^p - k^*$. Let $\pi'_s = (z_s^p, 0)$. Then, the total capital stock in the new outcome will be $k' = k_b^p \in [k^*, k^p)$. Because $F'(k') > F'(k^p)$, we can find $\mathbf{d}' \geq \mathbf{0}$ such that $\mathbf{d}' \leq \pi_b^p$ and $F'(k')d'_k + d'_z = F'(k^p)d_k^p + d_z^p$. We show that the new outcome, $(q^p, \mathbf{d}', \pi_b^p, \pi'_s)$, also satisfies (21) and (23) but has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$. The constraint (21) holds because $[1 + r - F'(k')] < [1 + r - F'(k^p)]$, and the continuation value of the transferred wealth remains the same; the constraint (23) holds because $k'_s = 0$ and the continuation value of the transferred wealth remains the same.

(b.2.2) $z_s^p > 0$ and $k_s^p = 0$. We distinguish two subcases: $q^p < q^*$ or $k^p > k^*$.

Suppose that $q^p < q^*$. Let $q' \in (q^p, q^*]$ be such that

$$v(q') - v(q^p) \leq r z_s^p / \sigma.$$

We claim that the new outcome, $(q', \mathbf{d}^p, \pi_b^p, \mathbf{0})$, satisfies (21) and (23), but has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$. The constraint (21) holds for the new outcome because we increase the output level while keeping the buyer portfolio and payments unchanged. As for (23):

$$\begin{aligned} & \sigma[-v(q') + F'(k^p)d_k^p + d_z^p] = -\sigma[v(q') - v(q^p)] + \sigma[-v(q^p) + F'(k^p)d_k^p + d_z^p] \\ \geq \quad & -r z_s^p + \sigma[-v(q^p) + F'(k^p)d_k^p + d_z^p] \geq 0. \end{aligned}$$

Suppose that $k^p > k^*$. Then, $k_b^p = k^p > k^*$. If $d_k^p = 0$, then let $(q', \mathbf{d}', \pi_b', \mathbf{0}) = (q^p, \mathbf{d}^p, (z_b^p, k^*), \mathbf{0})$, which has a strictly higher welfare. It is easy to verify that $(q', \mathbf{d}', \pi_b', \mathbf{0})$ satisfies (21) to (24). Suppose that $d_k^p > 0$. Let $k' \in [k^*, k^p)$ be such that

$$F'(k^p)k^p - F'(k')k' \leq (r k^p / \sigma d_k^p) z_s^p.$$

Such k' exists because $F'(k)k$ is strictly increasing. Let $d_k' = k'(d_k^p / k^p)$. We claim that the new outcome, $(q', \mathbf{d}', \pi_b', \mathbf{0})$ with $\pi_b' = (z_b^p, k')$ and $\mathbf{d}' = (d_z^p, d_k')$, satisfies (21) and (23) but has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$. The constraint (21) holds because $[1+r-F'(k')] < [1+r-F'(k^p)]$ and because $F'(k')d_k' = F'(k')k'(d_k^p/k^p) < F'(k^p)d_k^p$:

$$\begin{aligned} & -r z_b^p - [1 + r - F'(k')]k' + \sigma[u(q^p) - F'(k')d_k' - d_z^p] \\ > \quad & -r z_b^p - [1 + r - F'(k^p)]k^p + \sigma[u(q^p) - F'(k^p)d_k^p - d_z^p] \geq 0. \end{aligned}$$

As for (23) (recall that $k_s^p = 0$):

$$\begin{aligned} & \sigma[-v(q^p) + F'(k')d_k' + d_z^p] \\ = \quad & \sigma[-v(q^p) + F'(k^p)d_k^p + d_z^p] + (d_k^p/k^p)[F'(k')k' - F'(k^p)k^p] \\ \geq \quad & -r z_s^p + \sigma\{-v(q^p) + F'(k^p)d_k^p + d_z^p\} \geq 0. \end{aligned}$$

(b.3) Next, we show that if $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ satisfies (21), (23), and (24), then (q^p, z_b^p, k_b^p) satisfies (32)-(34). The constraint (24) is the same as (34). Suppose that $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ satisfies (21) and (23). Because $d_z^p \leq z_b^p$ and $d_k^p \leq k_b^p$,

$$\sigma\left[-v(q^p) + F'(k_b^p)k_b^p + z_b^p\right] \geq \sigma[-v(q^p) + F'(k_b^p)d_k^p + d_z^p] \geq 0.$$

Thus, (q^p, z_b^p, k_b^p) satisfies (33). Now consider (32). Because $-v(q^p) + F'(k_b^p)d_k^p + d_z^p \geq 0$,

$$\sigma[u(q^p) - v(q^p)] - [1 + r - F'(k_b^p)]k_b^p - rz_b^p \geq \sigma[u(q^p) - F'(k_b^p)d_k^p - d_z^p] - [1 + r - F'(k_b^p)]k_b^p - rz_b^p \geq 0.$$

Thus, (q^p, z_b^p, k_b^p) satisfies (32)-(34).

(b.4) Now we show part (ii) using (b.1)-(b.3). Let (q^p, z^p, k^p) be a solution to the problem (31)-(34). By (b.1) we can find a payment $\mathbf{d}^p$ such that $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ with $\pi_b^p = (z^p, k^p)$ satisfies (21), (23), (24), and the pairwise core requirement. We claim that $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ is constrained-efficient. Instead, assume that $(q', \mathbf{d}', \pi_b', \pi_s')$ has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$. By (b.2) we may assume that $\pi_s' = \mathbf{0}$. By (b.3) we know that (q', z_b', k_b') satisfies (32)-(34). However, because $(q', \mathbf{d}', \pi_b', \pi_s')$ gives a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$, (q', z_b', k_b') gives a strictly higher value to (31) than (q^p, z^p, k^p), a contradiction. Thus, $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ is constrained-efficient.

(3) As in (2), we distinguish two cases: (3.a) for some $z_0 \geq 0$, (q^*, z_0, k^*) satisfies (32)-(34); (3.b) (q^*, z_0, k^*) does not satisfy (32)-(34) for any $z_0 \geq 0$.

(3.a) By (2.a), in this case the first-best allocation is implementable and hence in any constrained-efficient outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, $q^p = q^*$ and $k^p \equiv k_b^p + k_s^p = k^*$.

Let $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$ be a constrained-efficient outcome. We show that (q^p, z_b^p, k^p) solves the problem (31)-(34). Because $q^p = q^*$ and $k^p = k^*$, it suffices to show that (q^p, z_b^p, k^p) satisfies (32)-(34). (34) is satisfied because $k^p = k^*$. Because $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$ satisfies (23),

$$\sigma[-v(q^p) + d_z^p + (1 + r)d_k^p] - rz_s^p \geq 0,$$

and hence, noting that $d_z^p \leq z_b^p$ and $d_k^p \leq k_b^p \leq k^p$,

$$-v(q^p) + z_b^p + (1 + r)k^p \geq -v(q^p) + d_z^p + (1 + r)d_k^p \geq rz_s^p/\sigma \geq 0.$$

This takes care of (33). Because $-v(q^p) + d_z^p + (1 + r)d_k^p \geq 0$ and because $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$ satisfies (21), (q^p, z_b^p, k^p) satisfies (32) as well:

$$\sigma[u(q^p) - v(q^p)] - rz_b^p \geq \sigma[u(q^p) - d_z^p - (1 + r)d_k^p] - rz_b^p \geq 0.$$

(3.b) For any $z_0 \geq 0$, the outcome (q^*, z_0, k^*) does not satisfy (32)-(34). By (b.2) in (2), in any constrained-efficient outcome, $(q^p, \mathbf{d}^p, \pi_b^p, \pi_s^p)$, $\pi_s^p = \mathbf{0}$.

Let $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$ be a constrained-efficient outcome. We show that (q^p, z_b^p, k_b^p) solves the problem (31)-(34). Instead, assume that (q', z', k') solves the problem (31)-(34) and gives a strictly higher value to (31) than (q^p, z_b^p, k_b^p). By (b.1) in (2) we can construct a payment $\mathbf{d}'$ so that $(q', \mathbf{d}', \pi_b', \mathbf{0})$ with $\pi_b' = (z', k')$ satisfies (21)-(24) and the pairwise core requirement and hence is implementable. However, $(q', \mathbf{d}', \pi_b', \mathbf{0})$ has a strictly higher welfare than $(q^p, \mathbf{d}^p, \pi_b^p, \mathbf{0})$, a contradiction. ■

Proof of Proposition 1: From Lemma 3, we restrict attention to outcomes of the form $(q^p, 0, k^p)$ with $q^p \leq q^*$ and $k^p \leq \bar{k}$. Notice the logic for these upper bounds hold for the case with the additional restriction $z^p = 0$ as well. This reduces the problem (31)-(34) to the maximization of a continuous function over a compact set, and hence a solution exists.

(1) Suppose that $k^* \geq v(q^*)/(1+r)$. Consider the first-best outcome, $(q^p, z^p, k^p) = (q^*, 0, k^*)$. By definition, $F'(k^*) = 1+r$, and hence both (32) and (34) hold. The inequality $k^* \geq v(q^*)/(1+r)$ implies that (33) holds. Therefore, the first-best outcome satisfies (32)-(34) and hence is the solution to the maximization problem (31).

(2) Suppose that $k^* < v(q^*)/(1+r)$. In this case $(q^*, 0, k^*)$ violates (33) and hence it is not feasible. Moreover, it follows that for any $(q, 0, k)$ that satisfies (33), either $q < q^*$ or $k > k^*$ or both. Indeed, if $q \geq q^*$ and $k \leq k^*$, then

$$v(q) \geq v(q^*) > (1+r)k^* \geq F'(k)k,$$

where the first inequality follows from v being an increasing function, the second inequality follows from the assumption $k^* < v(q^*)/(1+r)$, and the third inequality follows from $F'(k)k$ being an increasing function. Thus, $(q, 0, k)$ violates (33).

At the optimum, the constraint, (33), is binding. Suppose, by contradiction, that $(q^p, 0, k^p)$ is a constrained-efficient outcome with $v(q^p) < F'(k^p)k^p$. Moreover, because the first-best is not implementable in this case, either $q^p < q^*$ or $k > k^*$ or both. We consider these two possibilities separately. (i) $k^p > k^*$. By continuity, for some $\epsilon > 0$ sufficiently small, $v(q^p) < F'(k^p - \epsilon)(k^p - \epsilon)$ and $F'(k^p - \epsilon) < 1 + r$. Because $F'(k)$ is decreasing in k, it follows from (32) that

$$\sigma[u(q^p) - v(q^p)] \geq [r + 1 - F'(k^p)]k^p > [r + 1 - F'(k^p - \epsilon)](k^p - \epsilon).$$

Thus, $(q^p, 0, k^p - \epsilon)$ satisfies (32)-(34) but has a strictly higher welfare than $(q^p, 0, k^p)$, a contradiction. (ii) $q^p < q^*$. By continuity, for some $\epsilon > 0$ sufficiently small, we have $v(q^p + \epsilon) < F'(k^p)(k^p)$

and $q^p+\epsilon<q^*$. Moreover, because $u(q^p+\epsilon)-v(q^p+\epsilon)>u(q^p)-v(q^p)$, it follows that $(q^p+\epsilon,0,k^p)$ also satisfies (32). Hence, $(q^p+\epsilon,0,k^p)$ satisfies (32)-(34) but has a strictly higher welfare than $(q^p,0,k^p)$, a contradiction. Consequently, (33) is binding at the optimum.

Now, we claim that the solution to the following maximization problem is the unique constrained-efficient outcome:

$$\max_{(q,k)\in[0,q^*]\times\left[0,\bar{k}\right]} \sigma[u(q)-v(q)]+F(k)-(r+1)k$$
$$\text{s.t. } v(q)=F'(k)k.$$

We prove the claim by showing that the solution to the above problem also satisfies (32) and (34). We first reduce the above problem to the following:

$$\max_{k\in\left[0,\bar{k}\right]} \sigma\{u[g(k)]-v[g(k)]\}+F(k)-(r+1)k,$$

where $g(k)=v^{-1}[F'(k)k]$ for any $k\in\left[0,\bar{k}\right]$ (notice that, by definition, $g(\bar{k})=q^*$). Because v is strictly increasing and convex and because $F'(k)k$ is strictly increasing and concave, it follows that g is strictly increasing and concave. Hence, this maximization problem of a strictly concave function over a compact set has a unique solution, $\tilde{k}$, that satisfies the following first-order condition:

$$\sigma\left\{u'[g(\tilde{k})]-v'[g(\tilde{k})]\right\}g'(\tilde{k})+F'(\tilde{k})-(r+1) \begin{array}{c}\le\\=\\\ge\end{array} 0 \text{ if } \tilde{k} \begin{array}{c}=0\\\in\left(0,\bar{k}\right)\\=\bar{k}\end{array}. \tag{61}$$

From the properties of u, v, and F, it is easy to check that the solution is interior, $\tilde{k}\in\left(0,\bar{k}\right)$.

Next we show that $(g(\tilde{k}),0,\tilde{k})$ satisfies (32). By strict concavity of $u[g(k)]-v[g(k)]$ and by the fact that $u[g(0)]-v[g(0)]=0$, we have

$$\left\{u'[g(\tilde{k})]-v'[g(\tilde{k})]\right\}g'(\tilde{k})\tilde{k}<u[g(\tilde{k})]-v[g(\tilde{k})],$$

and hence, from (61),

$$\begin{aligned}&\sigma\{u[g(\tilde{k})]-v[g(\tilde{k})]\}-[r+1-F'(\tilde{k})]\tilde{k}\\ =\;&\sigma\{u[g(\tilde{k})]-v[g(\tilde{k})]\}-\sigma\left\{u'[g(\tilde{k})]-v'[g(\tilde{k})]\right\}g'(\tilde{k})\tilde{k}>0.\end{aligned}$$

This takes care of (32).

Next, we prove that $\tilde{k} > k^*$. Suppose, by contradiction, that $\tilde{k} \leq k^*$. Then $F'(\tilde{k}) \geq 1 + r$ and hence by (61) and by the fact that $g'(k) > 0$ for all k,

$$\sigma \left\{ u'[g(\tilde{k})] - v'[g(\tilde{k})] \right\} \leq 0,$$

which implies that $g(\tilde{k}) \geq q^*$. However, $[g(\tilde{k}), 0, \tilde{k}]$ satisfies (33), which implies that $g(\tilde{k}) < q^*$ or $\tilde{k} > k^*$, a contradiction. Thus, $\tilde{k} > k^*$ and hence $[g(\tilde{k}), 0, \tilde{k}]$ satisfies (34). Therefore, $[g(\tilde{k}), 0, \tilde{k}]$ is the constrained-efficient outcome. Finally, using condition (61) again, a symmetric argument shows that $g(\tilde{k}) < q^*$ and hence $q^p < q^*$. ■

Proof of Proposition 2:

(1) Suppose that $k^* \geq v(q^*)/(1 + r)$. By Proposition 1, $(q^*, 0, k^*)$ is implementable, and hence it is a constrained-efficient outcome.

(2) Suppose that $k^* \in \left[\frac{(r+\sigma)v(q^*) - \sigma u(q^*)}{r(1+r)}, \frac{v(q^*)}{1+r} \right)$. We show $(q^p, z^p, k^p) = (q^*, \sigma[u(q^*) - v(q^*)]/r, k^*)$ satisfies (32) to (34). First,

$$\sigma[u(q^p) - v(q^p)] - [1 + r - F'(k^p)]k^p - rz^p = 0.$$

This takes care of (32). Now consider (33):

$$-v(q^p) + F'(k^p)k^p + z^p = -v(q^*) + (1 + r)k^* + \frac{\sigma[u(q^*) - v(q^*)]}{r} \geq 0,$$

where the last inequality follows from the assumption that $k^* \geq [(r + \sigma)v(q^*) - \sigma u(q^*)] / r(1 + r)$. Last, (34) holds since $F'(k^p) = F'(k^*) = 1 + r$.

Finally, if (q^*, z^p, k^*) satisfies (32) and (33), then

$$\sigma[u(q^*) - v(q^*)] \geq rz^p \geq r[v(q^*) - (1 + r)k^*],$$

which implies that

$$\frac{\sigma[u(q^*) - v(q^*)]}{r} \geq z^p \geq v(q^*) - (1 + r)k^* > 0.$$

(3) Suppose that $k^* < [(r + \sigma)v(q^*) - \sigma u(q^*)] / r(1 + r)$. A first-best outcome with $q^p = q^*$ and $k^p = k^*$ is not implementable. Indeed, for any z, (q^*, z, k^*) satisfies (32) and (33) only if

$$\sigma[u(q^*) - v(q^*)] - [1 + r - F'(k^*)]k^* \geq rz \geq r[v(q^*) - (1 + r)k^*],$$

which implies that

$$\sigma[u(q^*) - v(q^*)] \geq r[v(q^*) - (1+r)k^*],$$

a contradiction to $k^* < [(r+\sigma)v(q^*) - \sigma u(q^*)]/r(1+r)$.

(3.1) Here we show that $q^p < q^*$. The Lagrangian associated with (31)-(34) is:

$$\begin{aligned}
\mathcal{L}(q,k,z;\lambda,\mu,\nu_k,\nu_z) &= \sigma\left[u(q)-v(q)\right]+F(k)-(1+r)k \\
&\quad +\lambda\left\{\sigma\left[u\left(q\right)-v(q)\right]-rz-\left[1+r-F'(k)\right]k\right\} \\
&\quad +\mu\left[F'(k)k+z-v(q)\right] \\
&\quad +\nu_k(k-k^*) \\
&\quad +\nu_z z,
\end{aligned}$$

where $\lambda \geq 0$, $\mu \geq 0$, $\nu_k \geq 0$, $\nu_z \geq 0$ denote the Lagrange multipliers associated with the constraints (32), (33), (34), and $z \geq 0$, respectively. Because the objective function in (31) and the function in the constraints (32)-(34) are concave, by Theorems 4.38 and 4.39 in Avriel (2003), the following Kuhn-Tucker conditions are necessary and sufficient for an outcome, (q^p, z^p, k^p), to be constrained-efficient: There exist $\lambda \geq 0$, $\mu \geq 0$, $\nu_k \geq 0$, and $\nu_z \geq 0$ such that

$$\begin{aligned}
(1+\lambda)\,\sigma\left[u'(q^p)-v'(q^p)\right]-\mu v'(q^p) &= 0 \quad (62)\\
F'(k^p)-(1+r)-\lambda\left[1+r-F'(k^p)-F''(k^p)k^p\right]+\mu\left[F''(k^p)k^p+F'(k^p)\right]+\nu_k &= 0 \quad (63)\\
-\lambda r+\mu+\nu_z &= 0 \quad (64)\\
\lambda\left\{\sigma\left[u\left(q^p\right)-v(q^p)\right]-rz^p-\left[1+r-F'(k^p)\right]k^p\right\} &= 0 \quad (65)\\
\mu\left[F'(k^p)k^p+z^p-v(q^p)\right] &= 0 \quad (66)\\
\nu_k(k^p-k^*) &= 0 \quad (67)\\
\nu_z z^p &= 0. \quad (68)
\end{aligned}$$

First, we show that $z^p > 0$. Suppose, by contradiction, that $z^p = 0$. By the proof of Proposition 1, $q^p = g(\tilde{k}) < q^*$ and $k^p = \tilde{k}$ satisfying (61). Moreover, it implies that (32) holds with a strict inequality for (q^p, z^p, k^p), and hence, by (65), $\lambda = 0$. By (64) and the fact that $\mu \geq 0$ and $\nu_z \geq 0$, this implies that $\mu = \nu_z = 0$. Thus, by (62) and (63), $q^p = q^*$ and $k^p = k^*$, a contradiction. Thus, $z^p > 0$ and by (68), $\nu_z = 0$ and hence, by (64), $\mu = r\lambda$.

Next, we show that $\mu > 0$. Suppose, by contradiction, that $\mu = 0$ and hence $\lambda = 0$. By (62), this implies that $q^p = q^*$. By (63) this implies that $k^p \leq k^*$ and hence $k^p = k^*$. This leads to a contradiction to the fact that a first-best outcome is not implementable. Thus, $\mu > 0$ and hence $\lambda > 0$. Then, by (62),

$$\frac{u'(q^p)}{v'(q^p)} = 1 + \frac{\mu}{\sigma\left(1 + \frac{\mu}{r}\right)},$$

which, together with the fact that $\mu > 0$, implies that $q^p < q^*$.

(3.2) We show that $k^p > k^*$ if and only if $r + F''(k^*)k^* > 0$. We first show that $k^p > k^*$ if $r + F''(k^*)k^* > 0$. Suppose, by contradiction, that $k^p = k^*$. Substituting k^p by k^* and μ by λr, the left side of (63) becomes

$$\begin{aligned} & -\lambda\left[1 + r - F'(k^*) - F''(k^*)k^*\right] + \lambda r\left[F''(k^*)k^* + F'(k^*)\right] + \nu_k \\ = \; & \lambda[(1+r)F''(k^*)k^* + rF'(k^*)] + \nu_k \\ = \; & \lambda(1+r)[F''(k^*)k^* + r] + \nu_k > 0. \end{aligned}$$

The last inequality follows from the assumption that $F''(k^*)k^* + r > 0$ and the fact that $\nu_k \geq 0$ and $\lambda > 0$. However, this is a contradiction to (63). Thus, $k^p > k^*$.

Next we show that $k^p = k^*$ if $r + F''(k^*)k^* \leq 0$. The Kuhn-Tucker conditions are necessary and sufficient for an optimum, and (q^p, k^p) is uniquely determined due to the strict concavity of the objective function. Therefore, it is sufficient to find a (q^p, z^p, k^*) that solves (62)-(68) for some $\lambda \geq 0$, $\mu \geq 0$, $\nu_k \geq 0$, and $\nu_z \geq 0$. From (66) and the fact that $\mu > 0$,

$$z^p = v(q^p) - (1+r)k^*. \tag{69}$$

From (65), the fact that $\lambda > 0$, and (69) q^p is the unique positive solution to

$$\frac{\sigma}{r}[u(q^p) - v(q^p)] - v(q^p) + (1+r)k^* = 0, \tag{70}$$

where the left side of (70) is a strictly concave function that is positive at $q^p = 0$. Hence, $\sigma[u'(q^p) - v'(q^p)]/r - v'(q^p) < 0$. Moreover, from the assumption $(1+r)k^* < [(r+\sigma)v(q^*) - \sigma u(q^*)]/r$, it follows that $q^p < q^*$, i.e., $u'(q^p) - v'(q^p) > 0$. Hence, from (62) and $\mu = r\lambda$,

$$\mu = r\lambda = \frac{r\sigma\left[u'(q^p) - v'(q^p)\right]}{(r+\sigma)\, v'(q^p) - \sigma u'(q^p)} > 0. \tag{71}$$

The fact that $u(q^p) - v(q^p) > 0$ implies from (69) and (70) that $z^p > 0$. From (68), $\nu_z = 0$. Substituting $k^p = k^*$ and $\mu = \lambda r$ into (63) we obtain

$$\nu_k = -\lambda(1+r)[F''(k^*)k^* + r] \geq 0 \tag{72}$$

since $r + F''(k^*)k^* \leq 0$. This shows that (q^p, z^p, k^*), which is uniquely determined by (69) and (70), is a solution to (62)-(68) for some $\mu = r\lambda > 0$ solution to (71), $\nu_k \geq 0$ solution to (72), and $\nu_z = 0$, and hence it is a constrained-efficient outcome.

(3.3) Here we show that $F'(k^p) > 1$. Suppose, by contradiction, that $F'(k^p) \leq 1$ and hence $k^p > k^*$. Then, by (67), $\nu_k = 0$. Moreover, by the arguments in (3.1), $\nu_z = 0$ and $\mu = \lambda r$. Substituting $\nu_k = 0$ and $\mu = \lambda r$ into (63) we obtain

$$\lambda(1+r)[F'(k^p) - 1] + [F'(k^p) - (1+r)] + \lambda(1+r)F''(k^p)k^p = 0.$$

However, the left side is negative since $\lambda \geq 0$ and $F'(k^p) - 1 \leq 0$ by assumption, $F'(k^p) - (1+r) < 0$ because $k^p > k^*$, and $F''(k^p)k^p < 0$ because $F(k)$ is strictly concave. This leads to a contradiction. Hence, $F'(k^p) > 1$. ■

Proof of Proposition 3: From the generalized Lemma 3 that allows inflation with $F(k) = Ak$, an equilibrium outcome, (q^p, z^p, k^p), is constrained efficient if and only if it solves

$$\max_{q,z,k} \{\sigma[u(q) - v(q)] + Ak - (1+r)k\} \tag{73}$$

$$\text{s.t. } \sigma[u(q) - v(q)] - iz - (1 + r - A)\, k \geq 0 \tag{74}$$

$$-v(q) + z + Ak \geq 0 \tag{75}$$

$$k \geq 0. \tag{76}$$

(1) If $A = 1+r$, then $(q^p, z^p, k^p) = (q^*, 0, v(q^*)/A)$ satisfies the (74)-(76) and hence it is constrained efficient. Money is inessential because the first-best outcome is implementable without it.

(2) $A < 1+r$ and $\gamma \leq \gamma^*$. We show that the first-best outcome, $(q^p, z^p, k^p) = (q^*, v(q^*), 0)$, satisfies the constraints (74)-(76) and hence it is a solution to (73)-(76). Constraints (75) and (76) hold by construction. Now consider (74):

$$\sigma[u(q^p) - v(q^p)] - iz^p \geq 0 \Leftrightarrow \ i \leq \frac{\sigma[u(q^*) - v(q^*)]}{v(q^*)},$$

which, from the definition of γ^*, (44), is equivalent to $\gamma \leq \gamma^*$. Since a first-best outcome is implementable, any constrained-efficient outcome must be such that $k^p = 0$ and $q^p = q^*$, and hence $z^p \geq v(q^*)$.

(3) $A < 1 + r$ and $\gamma \in (\gamma^*, \tilde{\gamma}]$. First, we show a first-best outcome with $q^p = q^*$ and $k^p = 0$ is not implementable. A first-best outcome, $(q^*, z, 0)$, satisfies (74) and (75) if and only if

$$\sigma[u(q^*) - v(q^*)] \geq iz \geq iv(q^*),$$

which is equivalent to $\gamma \leq \gamma^*$.

Next we show that at the optimum both (74) and (75) are binding. Suppose that (74) holds as a strict inequality for (q^p, z^p, k^p). Consider two cases. (i) $k^p > 0$. By continuity, for $\epsilon > 0$ sufficiently small, $(q^p, z^p + A\epsilon, k^p - \epsilon)$ also satisfies (74) to (76) and, from (73), it generates a higher social welfare than (q^p, z^p, k^p) since $A < 1 + r$. A contradiction. (ii) $k^p = 0$ and $q^p < q^*$. By continuity, for $\epsilon > 0$ sufficiently small, $(q^p + \epsilon, z^p + v(q^p + \epsilon) - v(q^p), 0)$ is also implementable and it generates a higher welfare. A contradiction.

Let us turn to constraint (75). Suppose that it holds as a strict inequality for (q^p, z^p, k^p). Consider two cases. (i) $k^p > 0$. By continuity, for $\epsilon > 0$ sufficiently small, $(q^p, z^p, k^p - \epsilon)$ satisfies (74)-(76) and, from (73), it generates a higher social welfare than (q^p, z^p, k^p) since $A < 1 + r$. A contradiction. (ii) $k^p = 0$ and $q^p < q^*$. By continuity, for $\epsilon > 0$ sufficiently small, $(q^p + \epsilon, z^p, 0)$ is also implementable and it generates a higher welfare, a contradiction.

The fact that we can restrict our attention to outcomes that satisfy both (74) and (75) at equality will simplify the mechanism designer's problem by reducing it to a choice of q. If $A\gamma \neq 1$, then for any given q the two constraints (74) and (75) with equalities have a unique solution given by:

$$k(q) = \beta \left\{ \frac{iv(q) - \sigma[u(q) - v(q)]}{A\gamma - 1} \right\} \tag{77}$$

$$z(q) = \beta \left\{ \frac{\sigma A[u(q) - v(q)] - (1 + r - A)\, v(q)}{A\gamma - 1} \right\}. \tag{78}$$

The outcome $(q, z(q), k(q))$ is feasible only if $z(q) \geq 0$ and $k(q) \geq 0$. Consider next the case $A\gamma = 1$ and denote $\underline{q} > 0$ the unique positive root to

$$\sigma[u(q) - v(q)] - iv(q) = 0. \tag{79}$$

The two constraints (74) and (75) with equalities imply that $q \in \{0, \underline{q}\}$, and $z(q)$ and $k(q)$ (which may not be unique for a given q) solve

$$z(q) + Ak(q) = v(q), \quad q \in \{0, \underline{q}\}. \tag{80}$$

Finally, from (74)

$$iz(q) = \sigma[u(q) - v(q)] + Ak(q) - (1+r)k(q),$$

which is the expression for the mechanism designer's objective function. Thus, the mechanism designer's maximization problem can be rewritten as

$$\max_{q \geq 0} \{iz(q)\} \quad \text{s.t.} \quad k(q) \geq 0. \tag{81}$$

The constraint $z(q) \geq 0$ is unnecessary because $z(0) = 0$.

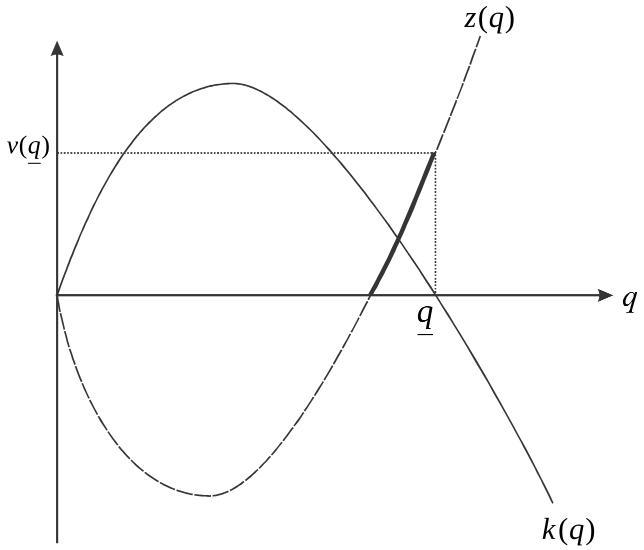
We distinguish two cases.

(3.1) $A\gamma \leq 1$. If $A\gamma = 1$, then $q \in \{0, \underline{q}\}$. From (80) $z(0) = k(0) = v(0) = 0$; if $q = \underline{q}$, then $z + Ak = v(\underline{q}) > 0$. From (81) the solution to the mechanism designer's problem maximizes real balances, and hence $q^p = \underline{q}$, $z^p = v(\underline{q})$, and $k^p = 0$.

Let us turn to the case $A\gamma < 1$. The function, $z(q)$, given by (78) is convex with $z(0) = 0$,

$$\begin{aligned} z(\underline{q}) &= \frac{A\beta}{A\gamma - 1}\left[i - \frac{(1+r) - A}{A}\right] v\left(\underline{q}\right) \\ &= v\left(\underline{q}\right) > v\left(0\right) = 0, \end{aligned}$$

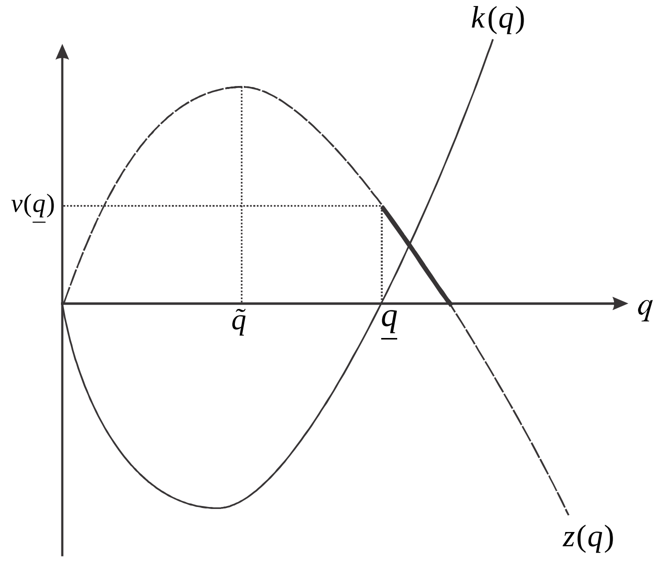
and $z'(0) < 0$. Here $\underline{q}$ is also given by (79), which is implicitly a function of i. The function, $k(q)$, given by (77) is concave with $k(0) = 0$, $k(\underline{q}) = 0$, and $k'(0) > 0$. See the figure below for a graphical representation of $z(q)$ and $k(q)$. The mechanism designer's problem, (81), can be reexpressed as $\max_{q \in [0, \underline{q}]} \{iz(q)\}$. The solution is $q^p = \underline{q}$, $z^p = z(\underline{q}) = v\left(\underline{q}\right)$, and $k^p = k(\underline{q}) = 0$. In the following figure the part of $z(q)$ that is consistent with both $z(q) \geq 0$ and $k(q) \geq 0$ is indicated by a thick line. The solution of the mechanism designer's problem corresponds to the highest point on this thick line.



(3.2) $\gamma A > 1$. From (77) $k(q)$ is strictly convex with $k(0) = k(\underline{q}) = 0$. Therefore, $k(q) \geq 0$ if and only if $q \geq \underline{q}$ or $q = 0$. But we already showed that $q = 0$ is never optimal. So the mechanism designer's problem, (81), becomes $\max_{q \geq \underline{q}} \{iz(q)\}$. From (78) $z(q)$ is strictly concave. Therefore the solution to the mechanism designer's problem is the unique solution to the following first-order condition:

$$\sigma[u'(q) - v'(q)] - \frac{1 + r - A}{A} v'(q) \leq 0, \quad \text{“} = \text{”} \text{ if } q > \underline{q}. \tag{82}$$

From (45) and (79) the condition $\gamma \leq \tilde{\gamma}$ implies $\underline{q} \geq \tilde{q}$, where from (46) $\tilde{q}$ solves (82) at equality. Thus, the solution to (82) is $q^p = \underline{q}$. By (77), $k^p = k(\underline{q}) = 0$. Finally, $z^p = z(\underline{q}) > 0$. In the following figure we represent $k(q)$ by a plain curve and $z(q)$ by a dashed curve. The part of $z(q)$ that is consistent with both $z(q) \geq 0$ and $k(q) \geq 0$ is indicated by a thick line. The solution of the mechanism designer's problem corresponds to the highest point on this thick line.



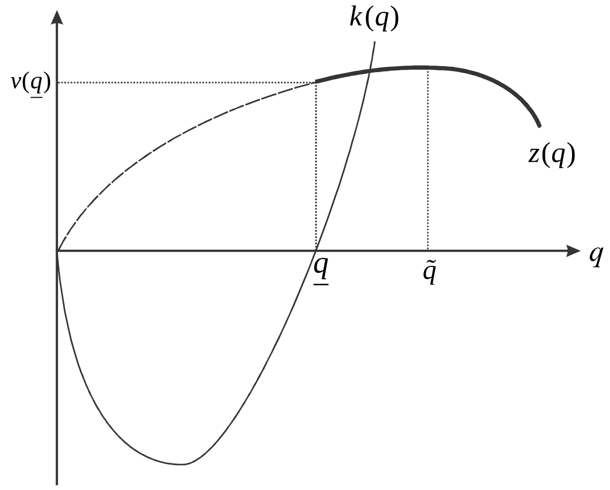
(4) $A < 1 + r$ and $\gamma > \tilde{\gamma}$. It follows from (46) that

$$\sigma[u(\tilde{q}) - v(\tilde{q})] - \left(\frac{1+r}{A} - 1\right) v(\tilde{q}) > 0.$$

By the definition of $\tilde{\gamma}$, we have

$$\sigma[u(\tilde{q}) - v(\tilde{q})] - \left(\frac{1+r}{\tilde{\gamma}^{-1}} - 1\right) v(\tilde{q}) = 0.$$

Thus, $\tilde{\gamma}^{-1} < A$ and hence $\gamma > \tilde{\gamma} > 1/A$. Following the same reasoning as in Part (3.2) of this proof the condition $\gamma > \tilde{\gamma}$ implies $\underline{q} < \tilde{q}$. Thus, the solution to (82) is $q^p = \tilde{q}$. Moreover, $z^p = z(\tilde{q}) > 0$, $k^p = k(q^p) > k(\underline{q}) = 0$. In the following figure we represent $k(q)$ by a plain curve and $z(q)$ by a dashed curve. The part of $z(q)$ that is consistent with both $z(q) \geq 0$ and $k(q) \geq 0$ is indicated by a thick line. The solution of the mechanism designer's problem corresponds to the highest point on this thick line.



k(q)
v(q)
z(q)
q
q
q̃

■

Online Appendix B: Pairwise core

Consider a match between a buyer holding a portfolio $\pi_b = (z_b, k_b)$ and a seller holding a portfolio $\pi_s = (z_s, k_s)$. The pairwise core, $\mathcal{CO}(\pi_b, \pi_s; R)$, is defined as the set of allocations such that for some $U^s \geq 0$,

$$(q, d_z, d_k) \in \arg\max \left[u(q) - d_z - Rd_k\right] \tag{83}$$
$$\text{s.t. } d_z \in \left[-z^s, z^b\right], \quad d_k \in \left[-k^s, k^b\right] \tag{84}$$
$$-v(q) + d_z + Rd_k \geq U^s \tag{85}$$
$$U^b = u(q) - d_z - Rd_k \geq 0. \tag{86}$$

If none of the constraints (84)-(86) is binding, then

$$q = q^* \tag{87}$$
$$d_z + Rd_k = U^s + v(q^*) \tag{88}$$
$$u(q^*) - v(q^*) \geq U^s \tag{89}$$
$$z^b + Rk^b \geq U^s + v(q^*). \tag{90}$$

If (84) binds, then

$$v(q) = z_b + Rk_b - U^s \tag{91}$$
$$(d_z, d_k) = (z_b, k_b) \tag{92}$$
$$u(q) - v(q) \geq U^s \tag{93}$$
$$z_b + Rk_b < U^s + v(q^*). \tag{94}$$

The results above can be summarized into three cases.

1. $z_b + Rk_b \geq u(q^*)$

   For all U^s that satisfy (89), the feasibility constraint, (90), holds. Therefore, from (87) and (88)

$$\mathcal{CO}(\pi_b, \pi_s; R) = \{q^*\} \times \left\{(d_z, d_k) \in [-z_s, z_b] \times [-k_s, k_b] : d_z + Rd_k \in [v(q^*), u(q^*)]\right\}.$$

If the buyer's wealth is larger than his willingness to pay for the first-best level of output, $u(q^*)$, then any allocation in the pairwise core implements the efficient level of output and the transfer of wealth is between the seller's cost and the buyer's willingness to pay.

2. $z_b + Rk_b \in [v(q^*), u(q^*))$

   For all U^s such that $U^s \leq z_b + Rk_b - v(q^*)$, (q, d_z, d_k) solves (87)-(88). For all $U^s \in (z_b + Rk_b - v(q^*), z_b + Rk_b - v \circ u^{-1}(z_b + Rk_b)]$, (q, d_z, d_k) solves (91)-(92). We have used that, from (93), the largest feasible surplus for the seller is when $u(q) - v(q) = U^s$, which from (91) implies $q = u^{-1}(z_b + Rk_b)$ and hence $U^s = z_b + Rk_b - v \circ u^{-1}(z_b + Rk_b)$. This gives:

$$\mathcal{CO}(\pi_b, \pi_s; R) = \{q^*\} \times \{(d_z, d_k) \in [-z_s, z_b] \times [-k_s, k_b] : d_z + Rd_k \in [v(q^*), z_b + Rk_b]\} \\ \cup \left[u^{-1}(z_b + Rk_b), q^*\right] \times \{z_b\} \times \{k_b\}.$$

   If the buyer's wealth is less than his willingness to pay for the first-best level of output, $u(q^*)$, but greater than the seller's cost, $v(q^*)$, then the first-best allocation is achieved provided that the seller's surplus is not too large; otherwise, the buyer transfers all his wealth and output is less than the efficient level.

3. $z_b + Rk_b < v(q^*)$

   For all $U^s \in [0, z_b + Rk_b - v \circ u^{-1}(z_b + Rk_b))$, (q, d_z, d_k) solves (91)-(92). This gives:

$$\mathcal{CO}(\pi_b, \pi_s; R) = \left[u^{-1}(z_b + Rk_b), v^{-1}(z_b + Rk_b)\right] \times \{z_b\} \times \{k_b\}.$$

   If the buyer's wealth is not large enough to compensate the seller for the cost of producing the first-best level of output, then any allocation in the pairwise core is such that the buyer transfers all his wealth and the output level is inefficiently low.

Online Appendix C: Inflation

Here we allow money to grow at the gross rate $\gamma > \beta$ through lump-sum transfers/taxes. As in the case with constant money supply, it is without loss of generality to assume that sellers do not carry assets across periods. Therefore, an equilibrium outcome can be written as $(q^p, \mathbf{d}^p, \pi^p)$, where $\pi^p = (z^p, k^p)$ refers to the buyer portfolio.

Lemma 4 *An equilibrium outcome $(q^p, \mathbf{d}^p, \pi^p)$ is implementable if and only if it satisfies*

$$-iz^p - \left[1 + r - F'(k^p)\right] k^p + \left[\sigma u\left(q^p\right) - d_z^p - F'(k^p)d_k^p\right] \geq 0; \tag{95}$$

$$-v\left(q^p\right) + d_z^p + F'(k^p)d_k^p \geq 0; \tag{96}$$

$$F'(k^p) \leq 1 + r, \text{ and} \tag{97}$$

$$(q^p, d_z^p, d_k^p) \in \mathcal{CO}(\pi_b, \mathbf{0}; R), \tag{98}$$

where $R = F'(k^p)$ and $i = \frac{\gamma - \beta}{\beta}$.

Given an outcome, $(q^p, \mathbf{d}^p, \pi^p)$, the social welfare is still given by the same expression as the constant-money-supply case:

$$\mathcal{W}(q^p, \mathbf{d}^p, \pi^p) = \frac{\sigma[u(q^p) - v(q^p)] + F(k^p) - (1 + r)k^p}{r}. \tag{99}$$

The definition of a constrained-efficient outcome is the same as before, except for that the implementability constraints are now given by (95)-(98). The following lemma, which generalizes Lemma 3 but follows the exactly same logic, gives a convenient reduced form for the maximization problem in the above definition. Let $\bar{k}$ be such that $v(q^*) = F'(\bar{k})\bar{k}$, as before.

Lemma 5 *A triple (q^p, z^p, k^p) is part of a constrained efficient outcome if and only if it solves*

$$\max\left\{\sigma\left[u(q^p) - v(q^p)\right] + F(k^p) - (r + \delta)k^p\right\} \tag{100}$$

subject to

$$\sigma[u(q^p) - v(q^p)] - [1 + r - F'(k^p)]k^p - iz^p \geq 0; \tag{101}$$

$$-v(q^p) + F'(k^p)k^p + z^p \geq 0; \tag{102}$$

$$F'(k^p) \leq 1 + r. \tag{103}$$

Moreover, in any solution (q^p, z^p, k^p) *to the above problem, we have* $q^p \leq q^*$, $k^p \in [k^*, \max\{k^*, \bar{k}\}]$, *and* $z^p \leq \sigma[u(q^*) - v(q^*)]/i$.

The following proposition generalizes Proposition 2 in the main text.

Proposition 4 *Consider an economy where the supply of fiat money grows at gross growth rate* $\gamma > \beta$. *Let* (q^p, z^p, k^p) *be a constrained efficient outcome. Suppose that* $k^* < v(q^*)/(1+r)$.

1. *If* $\gamma \leq \gamma^* \equiv \beta \left\{ \frac{\sigma[u(q^*)-v(q^*)]}{v(q^*)-(1+r)k^*} + 1 \right\}$, *then* $z^p > 0$, $q^p = q^*$ *and* $k^p = k^*$.

2. *If* $\gamma > \gamma^*$, *then* $z^p > 0$, $q^p < q^*$ *and* k^p *satisfies* $F'(k^p) \in (1/\gamma, 1+r]$. *Moreover,* $k^p > k^*$ *if and only if* $\gamma > \frac{1}{r+1+F''(k^*)k^*}$.

Proof. **(1)** Suppose that $k^* < v(q^*)/(1+r)$ and that $\gamma \leq \gamma^*$. We show that $(q^p, z^p, k^p) = (q^*, \sigma[u(q^*) - v(q^*)]/i, k^*)$ satisfies (101) to (103). First,

$$\sigma[u(q^p) - v(q^p)] - [1 + r - F'(k^p)]k^p - iz^p = 0.$$

This takes care of (101). Now consider (102):

$$-v(q^p) + F'(k^p)k^p + z^p = -v(q^*) + (1+r)k^* + \frac{\sigma[u(q^*) - v(q^*)]}{i} \geq 0,$$

where the last inequality follows from the assumption that $\gamma \leq \gamma^*$:

$$-v(q^*) + (1+r)k^* + \frac{\sigma[u(q^*) - v(q^*)]}{i} \geq 0 \text{ if and only if } i \leq \frac{\sigma[u(q^*) - v(q^*)]}{v(q^*) - (1+r)k^*},$$

which is equivalent to $\gamma \geq \gamma^*$. Because the first-best is implementable, by Proposition 1 we know that $z^p > 0$.

(2) Suppose that $k^* < v(q^*)/(1+r)$ and that $\gamma > \gamma^*$. Here a first-best outcome, (q^*, z, k^*), is not implementable: for any z, the outcome (q^*, z, k^*) satisfies (101) and (102) only if

$$\sigma[u(q^*) - v(q^*)] - [1 + r - F'(k^*)]k^* \geq iz \geq i[v(q^*) - (1+r)k^*],$$

which implies that

$$\sigma[u(q^*) - v(q^*)] \geq i[v(q^*) - (1+r)k^*],$$

a contradiction to $\gamma > \gamma^*$.

(2.1) Here we show that $q^p < q^*$. The Lagrangian associated with (100)-(103) is:

$$\begin{aligned}\mathcal{L}(q,k,z;\lambda,\mu,\nu_k,\nu_z) &= \sigma\left[u(q)-v(q)\right]+F(k)-(1+r)k \\ &\quad +\lambda\left\{\sigma\left[u\left(q\right)-v(q)\right]-iz-\left[1+r-F'(k)\right]k\right\} \\ &\quad +\mu\left[F'(k)k+z-v(q)\right] \\ &\quad +\nu_k(k-k^*) \\ &\quad +\nu_z z,\end{aligned}$$

where $\lambda \geq 0$, $\mu \geq 0$, $\nu_k \geq 0$, $\nu_z \geq 0$ denote the Lagrange multipliers associated with the problem (100)-(103). Notice that the constraint (103) is equivalent to $k^p \geq k^*$.

Because the objective function in (100) and the function in the constraints (101)-(103) are concave, by Theorems 4.38 and 4.39 in Avriel (2003), the following Kuhn-Tucker conditions are necessary and sufficient for an outcome, (q^p, z^p, k^p), to be constrained-efficient: there exist $\lambda \geq 0$, $\mu \geq 0$, $\nu_k \geq 0$, and $\nu_z \geq 0$ such that

$$(1+\lambda)\,\sigma\left[u'(q^p)-v'(q^p)\right]-\mu v'(q^p) = 0, \quad (104)$$

$$F'(k^p)-(1+r)-\lambda\left[1+r-F'(k^p)-F''(k^p)k^p\right]+\mu\left[F''(k^p)k^p+F'(k^p)\right]+\nu_k = 0, \quad (105)$$

$$-\lambda i+\mu+\nu_z = 0, \quad (106)$$

$$\lambda\left\{\sigma\left[u\left(q^p\right)-v(q^p)\right]-iz^p-\left[1+r-F'(k^p)\right]k^p\right\} = 0, \quad (107)$$

$$\mu\left[F'(k^p)k^p+z^p-v(q^p)\right] = 0, \quad (108)$$

$$\nu_k(k^p-k^*) = 0, \quad (109)$$

$$\nu_z z^p = 0. \quad (110)$$

Let (q^p, z^p, k^p) be a constrained-efficient outcome. First we show that $z^p > 0$. Suppose, by contradiction, that $z^p = 0$. Then, by the proof of Proposition 1, $q^p = g(\tilde{k}) < q^*$ and $k^p = \tilde{k}$ satisfying (61). Moreover, it implies that (101) is not binding for (q^p, z^p, k^p), and hence, by (107), $\lambda = 0$. By (106) and the fact that $\mu \geq 0$ and $\nu_z \geq 0$, this implies that $\mu = \nu_z = 0$. Thus, by (104) and (105), $q^p = q^*$ and $k^p = k^*$, a contradiction. Thus, $z^p > 0$ and by (110), $\nu_z = 0$ and hence, by (106), $\mu = i\lambda$.

Next, we show that $\mu > 0$. Suppose, by contradiction, that $\mu = 0$ and hence $\lambda = 0$. By (104), this implies that $q^p = q^*$. By (105) this implies that $k^p \leq k^*$ and hence $k^p = k^*$. This leads to a contradiction to the fact that the first-best is not implementable. Thus, $\mu > 0$ and hence $\lambda > 0$. Then, by (104),

$$\frac{u'(q^p)}{v'(q^p)} = 1 + \frac{\mu}{\sigma\left(1 + \frac{\mu}{i}\right)},$$

which, together with the fact that $\mu > 0$, implies that $q^p < q^*$.

(2.2) We show that $k^p > k^*$ if and only if $\gamma > \frac{1}{r+1+F''(k^*)k^*}$.

We first show that $k^p > k^*$ if $\gamma > \frac{1}{r+1+F''(k^*)k^*}$. Suppose, by contradiction, that $k^p = k^*$. Substituting k^p by k^* and μ by λi, the left-hand side of (105) becomes

$$\begin{aligned} & -\lambda\left[1 + r - F'(k^*) - F''(k^*)k^*\right] + \lambda i\left[F''(k^*)k^* + F'(k^*)\right] \\ = \; & \lambda[(1+i)F''(k^*)k^* + i(1+r)] > 0. \end{aligned}$$

The last inequality follows from the assumption that $\gamma > \frac{1}{r+1+F''(k^*)k^*}$ and the fact that $\lambda > 0$ (notice that $\frac{1+i}{1+r} = \gamma$):

$$\begin{aligned} & (1+i)F''(k^*)k^* + i(1+r) = (1+r)\left[\frac{1+i}{1+r}F''(k^*)k^* + i\right] \\ = \; & (1+r)\left[\gamma F''(k^*)k^* + \gamma(1+r) - 1\right] \\ = \; & (1+r)\gamma\left[F''(k^*)k^* + 1 + r - \frac{1}{\gamma}\right] > 0. \end{aligned}$$

However, this is a contradiction to (105). Thus, $k^p > k^*$.

Next we show that $k^p = k^*$ if $\gamma \leq \frac{1}{r+1+F''(k^*)k^*}$. Because (104)-(110) are also sufficient conditions for an outcome to be constrained-efficient and because the constrained outcome is unique up to the (q^p, k^p)-component due to strict concavity of the objective function, we prove the result by showing that $(\tilde{q}, \tilde{z}, k^*)$ satisfies (104)-(110) for some $\lambda \geq 0$ $\mu \geq 0$, $\nu_k \geq 0$, and $\nu_z \geq 0$, where $\tilde{q}$ is the unique $q > 0$ that solves

$$\sigma[u(q) - v(q)] = i[v(q) - (1+r)k^*], \tag{111}$$

and

$$\tilde{z} = v(\tilde{q}) - (1+r)k^* > 0. \tag{112}$$

Equations (111) and (112) ensures that $(\tilde{q}, \tilde{z}, k^*)$ satisfies (101) and (102). Because the first best is not implementable and because $u(q) - v(q)$ is concave and $v(q)$ is convex, it follows that $\tilde{q} < q^*$ and

$$u'(\tilde{q}) - v'(\tilde{q}) > 0 \tag{113}$$

and

$$iv'(\tilde{q}) - \sigma[u'(\tilde{q}) - v'(\tilde{q})] > 0. \tag{114}$$

Now we verify that there exist $\lambda \geq 0$, $\mu \geq 0$, $\nu_k \geq 0$, and $\nu_z \geq 0$ such that $(\tilde{q}, \tilde{z}, k^*)$ satisfies (104)-(110).

By arguments in (3.1) it must be the case that $\nu_z = 0$ and hence $\mu = \lambda i$. Substituting $\mu = \lambda i$ and $q^p = \tilde{q}$ into (104) we obtain

$$\mu = \lambda i = \frac{i\sigma[u'(\tilde{q}) - v'(\tilde{q})]}{iv'(\tilde{q}) - \sigma[u'(\tilde{q}) - v'(\tilde{q})]}.$$

By (113) and (114), $\mu > 0$ and $\lambda > 0$. This shows that $(\tilde{q}, \tilde{z}, k^*)$, together with μ, λ, and ν_z, satisfies all the multiplier conditions except for (105). Substituting $k^p = k^*$ and $\mu = \lambda r$ into (105) we obtain

$$\nu_k = -\lambda[(1+i)F''(k^*)k^* + i(1+r)].$$

Because $\gamma \leq \frac{1}{r+1+F''(k^*)k^*}$, $\nu_k \geq 0$. Thus, $(\tilde{q}, \tilde{z}, k^*)$ is a constrained-efficient outcome.

(2.3) Here we show that $F'(k^p) > \frac{1}{\gamma}$. Suppose, by contradiction, that $F'(k^p) \leq \frac{1}{\gamma} < 1 + r$ and hence $k^p > k^*$ (recall that $\gamma > \gamma^* > \beta$). Then, by (109), $\nu_k = 0$. Moreover, by the arguments in (2.1), $\nu_z = 0$ and $\mu = \lambda i$. Substituting $\nu_k = 0$ and $\mu = \lambda i$ into (105) we obtain

$$\lambda[(1+i)F'(k^p) - (1+r)] + [F'(k^p) - (1+r)] + \lambda(1+i)F''(k^p)k^p = 0.$$

However, the left-hand side is necessarily negative: $\lambda \geq 0$ and $(1+i)F'(k^p) - (1+r) = (1+i)[F'(k^p) - 1/\gamma] \leq 0$ by assumption; $F'(k^p) - (1+r) < 0$ because $k^p > k^*$; $F''(k^p)k^p < 0$ because $F(k)$ is strictly concave. This leads to a contradiction. Hence, $F'(k^p) > 1/\gamma$. ■