

ASYMMETRIC ALL-PAY AUCTIONS WITH INCOMPLETE INFORMATION, THE TWO PLAYER CASE REDUX

JINGLING LU¹ AND SÉRGIO O. PARREIRAS²

ABSTRACT. We re-visit the all-pay auction model of [Amann and Leininger \(1996\)](#) allowing for interdependent values and correlation à la [Lizzeri and Persico \(1998\)](#). But more importantly, we study both monotone and non-monotone (pure strategy) equilibria (henceforth, MPSE and NPSE). Our main result, building on [Araujo et al. \(2008\)](#), characterizes pooling types in NPSE. That allows us to reduce NPSE computation to the monotone case.

We also obtain novel results for the monotone case: First, we characterize the allocation and bidding strategies of MPSE. The equilibrium allocation depends only on the conditional expected values. In the case of correlated private values, it is the same regardless of correlation. In the case of common-values, it sorts players according to their signals' percentiles. Second, we present a local single-crossing condition which is necessary for MPSE, and the standard single crossing, which is sufficient. Lastly, we exhibit families of examples that violate the local single-crossing and thus lack MPSE. In a particular correlated private values example, independence guarantees a MPSE but the slightest amount of correlation breaks down MPSE.

1. INTRODUCTION

In rent-seeking contests (all-pay auctions), distinct individuals may entertain different estimates of the prize (value of the object). Such estimates maybe of varying precision or accuracy; possibly they maybe interdependent and/or correlated.

Our aim here is limited to provide a tractable characterization of pure strategy (monotone or not) equilibria (henceforth MPSE and NPSE) of the (first-price) all-pay auction with two, possibly asymmetric, players with interdependent valuations and correlated signals.

Our model can be viewed either as an extension of [Amann and Leininger \(1996\)](#) as we add correlation and interdependent values and an specialization of [Lizzeri and Persico \(1998\)](#) to the all-pay auction. However, we study non-monotone equilibria which starkly departs from previous studies of the all-pay auction, with the exception of [Araujo et al. \(2008\)](#).

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1. National University Singapore. E-mail: ecsljlf@nus.edu.sg.

2. UNC at Chapel Hill. E-mail: sergiop@unc.edu.

We do not restrict attention neither to affiliated signals, as in [Lizzeri and Persico \(1998\)](#), nor to independent signals as in, [Araujo et al. \(2008\)](#). We allow for positive or negative correlated signals. Speaking plainly, affiliation is mostly an useless assumption – in the context of all-pay auctions with interdependent valuations. As we show in Section 6, there are economically interesting all-pay auctions that lack MPSE when signals are strictly affiliated but do have MPSE when signals are independent. Furthermore, in Lemma 2, we prove that any equilibrium of the all-pay auction with correlated signals is the equilibrium of some all-pay auction with interdependent valuations and independent signals.

In any MPSE, the allocation rule (*i.e.* the assignment of the object given the signal of the players) only depends on the players' expected values for the object conditional on their signals. In particular, for correlated private values, the allocation rule is the same regardless of the nature of the correlation; it coincides with the allocation derived by [Amann and Leininger \(1996\)](#) for the case of independent private values. In the case of pure common-values, it follows the percentiles of the distribution of the agents' signals. That is, the type of agent 1 who gets a signal in the p -percentile bids the same amount of that the type of agent 2 who gets a signal in the p -percentile¹

2. THE MODEL

There are two agents, $i = 1, 2$. Let V_i be the random variable describing the value of the object for player i . Let X_1 and X_2 be the agents' signals. The conditional expected value is $v_i(x, y) = E[V_i | X_1 = x, X_2 = y]$. The cumulative distribution of X_i is F_i and, $F_{i|j}$ is the conditional cumulative distribution of X_i given X_j . The lower-case f denotes the respective probability density function. We also define, $\lambda_i(x, y) \stackrel{\text{def}}{=} E[V_i | X_i = x, X_j = y] \cdot f_{X_j | X_i}(y | x)$ for $i, j = 1, 2$ with $j \neq i$.

We assume:

CONTINUITY: X_i is distributed continuously in a closed (not necessarily bounded) interval of the real line.

FULL SUPPORT: For all $(x, y) \in [0, 1]^2$, $f_{X_1, X_2}(x, y) > 0$.

Notice that due to continuity, without any loss of generality, we take X_i to be uniformly distributed in the unit interval.²

¹In particular, if signals are conditionally independent, the winning probability of agents 1 and 2 are the same. [Einy et al. \(2013\)](#) and [Warneryd \(2013\)](#) independently obtain this corollary as well. They study information structures where agent 1's signal is a sufficient statistic for agent 2's signal and hence, signals are conditionally independent.

²Say S_i is the original signal, re-parametrize signals by taking as the new signal, $X_i = F_{S_i}(S_i)$.

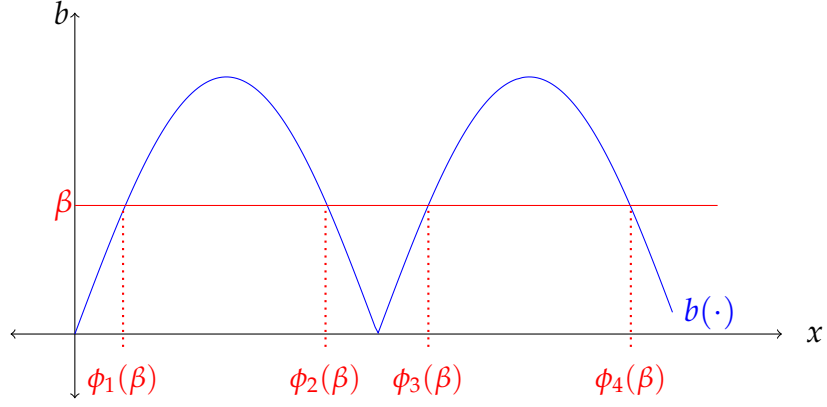


FIG. 1. A piecewise monotone strategy and its local inverse bids.

3. NON-MONOTONE EQUILIBRIA

Consider a pure strategy equilibrium profile in which every bid strategy, $b_i(\cdot)$, is piecewise monotone. Next, partition i 's type space into finite intervals $[0, 1] = \bigcup_{k=1}^{n_i} [s_j^i, t_j^i]$ where $s_1^i = 0 \leq s_{k+1}^i = t_k < t_{k+1}^i \leq t_{n_i} = 1$ such these intervals are maximal with respect the property each restriction $b_i|_{[s_j^i, t_j^i]}$ is monotone. For exposition purposes, let's focus on the case where $b_i(\cdot)$ is increasing (decreasing) in odd (even) intervals, $i = 1, 2$. The cases where one of the strategies is increasing in even intervals are analogous so we omit them here.

Now, let ϕ_k^i be the k th local inverse bid function of player i which takes values on $[s_k^i, t_k^i]$. The payoff of a type x of player i who bids b is:

$$U_i(b|x) = \left(\sum_{k=1}^n \int_{\phi_{2k-2}^j(b)}^{\phi_{2k-1}^j(b)} \lambda_i(x, y) dy \right) - b,$$

where $\phi_0^j(b) \stackrel{\text{def}}{=} 0$, $\phi_{2n-1}^j(b) \stackrel{\text{def}}{=} 1$ if n is odd. Later we shall use the $n_i \times n_j$ - matrices $\Lambda^i(b)$ with entries given by $\Lambda_{k,l}^i(b) = \lambda_i(\phi_k^i(b), \phi_l^j(b))$.

Definition 1. A piecewise monotone equilibrium \mathbf{b} is *regular* if, for almost all b , $n_i(b) = n_j(b) < +\infty$ is constant in a neighborhood³ of b and $\Lambda^i(b)$ is full-rank for $i = 1, 2$.

Definition 2. A piecewise monotone equilibrium \mathbf{b} is *simple* if it is regular and for almost all b , $n_i(b) = n_j(b)$ is constant .

Regularity is a minor requirement as without it, there is little hope of pinning-down the equilibrium using first-order conditions (and boundary conditions) as the system of differential equations would be undetermined. Obviously, any monotone equilibrium is regular.

³Notice the number of types that place a given bid, $n_i(b)$, may vary with b but it can take at most a countable number of values.

Simplicity, is not a minor requisite to ask.. It is plausible that a piecewise equilibrium could have bids that are placed by a single type, other bids by two types, etc...

Our main result only requires regularity but in practice to construct examples it would be convenient to assume simplicity, otherwise we need to patch and glue together different birding regimes.

Theorem 1. *Consider a regular equilibrium and let $b \in b_i([0, 1])$ and $\phi_k^i(b)$ for $k = 1, \dots, n_i$ be the corresponding local inverse bids. There exists n_i constants: c_1^i, c_2^i, \dots , and c_n^i such that:*

- (1) $\phi_k^i(b) = (-1)^{k+1} \phi_1^i(b) + c_k^i$ and
- (2) $\sum_{l=1}^{n_j} \lambda_i \left((-1)^{k+1} \phi_1^i(b) + c_k^i, (-1)^{l+1} \phi_1^j(b) + c_l^j \right) = \sum_{l=1}^{n_j} \lambda_i \left(\phi_1^i(b), (-1)^{l+1} \phi_1^j(b) + c_l^j \right)$.

Proof. First we prove the second part of the theorem:

Lemma 1. $\sum_{l=1}^{n_j} \lambda_i \left(\phi_k^i(b), \phi_l^j(b) \right) = \sum_{l=1}^{n_j} \lambda_i \left(\phi_1^i(b), \phi_l^j(b) \right)$

This is similar to Proposition 3 of [Araujo et al. \(2008\)](#). If signals are independent, [Araujo et al. \(2008\)](#) observed the first order-conditions for an optimal bid for type x of player i can be written as $E[V_i | b_j(X_j) = b, X_i = x] \cdot g_j(b) - 1 = 0$ where g_j is the density of the bid distribution of player j (or i 's marginal winning probability) and as result $E[V_i | b_j(Y) = b, X = \phi_k^i(b)] = E[V_i | b_j(Y) = b, X = \phi_1^i(b)]$ for $k = 1, \dots, n_i$. Since marginal cost of bidding and marginal winning probability are same for all pooling types, their marginal marginal benefit of winning must also be the same. However, in our context, signals are not necessarily independent. Nonetheless, we can obtain a similar result but some steps are required.

(THE FICTITIOUS AUCTION) Given an all-pay auction satisfying our assumptions, the fictitious (or auxiliary) auction is the all-pay auction where signals are independently and uniformly distributed on the unit interval, and the (interdependent) expected conditional valuations are $\tilde{v}_i \stackrel{\text{def}}{=} \lambda_i(x, y)$.

Lemma 2. *Any symmetric, pure strategy equilibrium of the fictitious auction is a symmetric, pure strategy equilibrium of the original all-pay auction and vice-versa.*

Proof of lemma 2. assume the player j uses a given pure strategy profile in both actions and let the ϕ_k^j be the local inverse bidding functions of player j that we previously defined. The expected payoff for player i of bidding b is the same in both auctions. ■

By Lemma 2, the marginal marginal benefit of winning in the fictitious auction is the same for all types that place the same bid but this means $\sum_{l=1}^{n_j} \lambda_i(x, \phi_l^j(b))$ is that same for all x such that $b_i(x) = b$. This proves Lemma 1. Later it shall be convenient to express Lemma 2 in

matrix notation as: $\Lambda_i(b) \cdot \mathbf{1} = \kappa_i(b) \cdot \mathbf{1}$ where $\mathbf{1}$ is a n_j -column vector with one in each entry and $\kappa_i(b)$ is a scalar. Regularity implies that $\kappa_i(b) \neq 0$ for almost all b .

Now notice the first-order condition for type x of player i who bids b is:

$$\sum_{l=1}^{n_j} (-1)^{l+1} \lambda_i(x, \phi_l^j(b)) \frac{\partial \phi_l^j}{\partial b}(b) = 1 \quad (\text{FOC})$$

As the FOC must be satisfied for $x = \phi_k^i(b)$ for all k , we obtain an ODE system, which in matrix form reads as, $\Lambda_i(b) \cdot \Phi^j = \mathbf{1}$, where Φ^j is the n_j -column vector $\left((-1)^{l+1} \frac{\partial \phi_l^j}{\partial b}(b) \right)$.

Since Λ_i is full-rank, using Lemma 1 we obtain $\Phi^j = \kappa_i(b)^{-1} \cdot \mathbf{1}$, that is $\frac{\partial \phi_1^j}{\partial b} = (-1)^{l+1}$ so ϕ_1^j

is a constant plus ϕ_1^j or its reflection $-\phi_1^j$. This concludes the proof of the theorem. ■

4. REDUCTION TO THE MONOTONE CASE

In a regular equilibrium $n_i(b) = n_j(b)$ is constant, so the second part of Theorem 1 gives us $2n_i(b)$ equations in the same number of unknowns (the c_k^i and c_l^j constants) for each b ! If one is able to solve this continuum of equations for these finite number of constants, we can reduce the ODE system to a smaller system similar to the monotone case. We use the first part of Theorem 1 to express the system solely in terms of ϕ_1^i and ϕ_1^j which, abusing notation, we write as ϕ_1 and ϕ_2 . After reduction, to solve for the equilibrium, we apply the methods developed in the next section.

Applying the procedure described above, we obtain the reduced system:

$$\left(\sum_{l=1}^{n_j} \lambda_i(\phi_i(b), c_l + (-1)^{l+1} \phi_j(b)) \right) \phi_j'(b) = 1 \text{ with } i, j = 1, 2 \text{ and } i \neq j.$$

That is, one just needs to solve for the equilibrium of an auction where signals are independent and the corresponding valuations are $\hat{v}_i(x, y) = \sum_{l=1}^{n_j} \lambda_i(x, c_l + (-1)^{l+1} y)$.

Example 1. Let $v_i(x, y) = x + k \cdot y \cdot (3 - 4x + 2x^2)$ for $i = 1, 2$ where $k > 1$. For $k = 1$, this corresponds to example 4 of [Araujo et al. \(2008, p. 421\)](#).⁴ Furthermore, we assume⁵ signals to be uniformly distributed on $[0, \bar{x}]$ where $\bar{x} = 1 + \sqrt{1 - \frac{1}{k}}$.

⁴For $k = 1$, they show the first-price auction has no monotone eq. The all-pay auction, however, has a monotone equilibrium for any k in $[0, 1]$.

⁵As to ensure a *simple* equilibrium and easy the exposition. See definition 2 and subsequent discussion.

For brevity, we focus on symmetric equilibrium and omit subscripts when convenient. Since we want to restrict attention to non-monotone equilibria, we impose $k > 1$. By Prop. 2, $\lambda_x(x, y)|_{y=x} \geq 0$ is necessary for monotonicity and here that translates to $0 \leq k \leq 1$.

To solve for the pooling types, we must find c such that:

$$\lambda(x, x) + \lambda(x, c - x) - \lambda(c - x, x) - \lambda(c - x, c - x) = 0,$$

for all $x \in [0, 1]$. So, we are dealing with a continuum of equations: $(4c^2k - 8ck + 4)x - 2c^3k + 4c^2k - 2c = 0$ whose solutions are: $\underline{c} = 1 - \sqrt{1 - \frac{1}{k}}$ and $\bar{c} = 1 + \sqrt{1 - \frac{1}{k}}$. We set $c = \bar{c}$ so that $\bar{x} = c - \underline{x}$. Now applying the reduction, we have⁶ $\hat{\lambda}(x) = (\bar{x})^{-1} \cdot (2x + c \cdot k \cdot (3 - 4x + 2x^2))$ and⁷:

$$\begin{aligned} - \left[2\phi_i(b) + c \cdot k \cdot (3 - 4\phi_i(b) + 2\phi_i(b)^2) \right] \phi_j'(b) - \bar{x} &= 0 \quad \text{for } \phi_i(b) \leq \bar{x}/2 \quad \text{and} \\ \left[2\phi_i(b) + c \cdot k \cdot (3 - 4\phi_i(b) + 2\phi_i(b)^2) \right] \phi_j'(b) - \bar{x} &= 0 \quad \text{for } \phi_i(b) \geq \bar{x}/2. \end{aligned}$$

Notice that $\hat{\lambda}$ satisfy a single crossing condition at $x = \frac{\bar{x}}{2}$ since $\hat{\lambda}_x(\hat{x}) < \hat{\lambda}_x(\frac{\bar{x}}{2}) = 0 < \hat{\lambda}_x(\tilde{x})$ for $\hat{x} < \frac{\bar{x}}{2} < \tilde{x}$.

Now we can recover the equilibrium:

$$b(x) = \begin{cases} \bar{b} - \int_0^x \hat{\lambda}(z) dz & \text{if } x \leq \bar{x}/2, \\ \int_{\bar{x}/2}^x \hat{\lambda}(z) dz & \text{otherwise.} \end{cases},$$

$$\text{where } \bar{b} = \int_0^{\bar{x}/2} \hat{\lambda}(z) dz = \frac{1 + 5k}{6} \sqrt{1 - \frac{1}{k}} + \frac{5}{6}k + \frac{1}{2}.$$

The idea is to remove example 1 and replace it with example 2.

Example 2. Let $v_i(x, y) = x + k \cdot y \cdot (3 - 4x + 2x^2)$ for $i = 1, 2$ where $1 \leq k \leq \frac{9}{8}$. For $k = 1$, this corresponds to example 4 of Araujo et al. (2008, p. 421).⁸ Signals are independent and uniformly distributed on $[0, 1]$.

For brevity, we focus on symmetric equilibrium and omit subscripts when convenient. Again, since we want to restrict attention to non-monotone equilibria, we impose $k \geq 1$. By Prop. 2, $\lambda_x(x, y)|_{y=x} \geq 0$ is necessary for monotonicity and here that translates to $0 \leq k \leq 1$.

To solve for the pooling types, we must find c such that:

$$\lambda(x, x) + \lambda(x, c - x) - \lambda(c - x, x) - \lambda(c - x, c - x) = 0,$$

⁶In principle, $\hat{\lambda}$ may depend on both signals but here it only depends only on the player's own signal.

⁷The minus sign in the FOC arises here because as Araujo et al. (2008) we are looking for a U-shaped equilibrium so the first local inverse bid function is decreasing (not increasing as in the previous section).

⁸For $k = 1$, they show the first-price auction has no monotone eq. The all-pay auction, however, has a monotone equilibrium for any k in $[0, 1]$.

for all $x \in [0, 1]$. So, we are dealing with a continuum of equations: $(4c^2k - 8ck + 4)x - 2c^3k + 4c^2k - 2c = 0$ whose solutions are: $\underline{c} = 1 - \sqrt{1 - \frac{1}{k}}$ and $\bar{c} = 1 + \sqrt{1 - \frac{1}{k}}$. We set $c = \underline{c}$; and look for a continuous equilibrium that is U-shaped in $[0, c]$ and strictly increasing in the range $[c, 1]$. Now applying the reduction, we have⁹ $\hat{\lambda}(x) = (\bar{x})^{-1} \cdot (2x + c \cdot k \cdot (3 - 4x + 2x^2))$ and¹⁰:

$$\begin{aligned} - \left[2\phi_i(b) + c \cdot k \cdot (3 - 4\phi_i(b) + 2\phi_i(b)^2) \right] \phi_j'(b) - 1 &= 0 && \text{for } \phi_i(b) \in [0, c/2), \\ \left[2\phi_i(b) + c \cdot k \cdot (3 - 4\phi_i(b) + 2\phi_i(b)^2) \right] \phi_j'(b) - 1 &= 0 && \text{for } \phi_i(b) \in (c/2, c) \text{ and} \\ \left[\phi_i(b) + \phi_j(b) \cdot k \cdot (3 - 4\phi_i(b) + 2\phi_i(b)^2) \right] \phi_j'(b) - 1 &= 0 && \text{for } \phi_i(b) \in (c, 1]. \end{aligned}$$

Notice that $\hat{\lambda}$ satisfy a single crossing condition since $\hat{\lambda}_x(\hat{x}) < \hat{\lambda}_x(\frac{c}{2}) = 0 < \hat{\lambda}_x(\bar{x})$ for $\hat{x} < \frac{c}{2} < \bar{x}$. That is, higher types in $[0, c/2]$ want to bid lower and, higher types in $[c/2, c]$ want to bid higher. As for the types in $[c, 1]$, we have that $\lambda_x(x, y)|_{y=x} \geq 0$ if $x > \frac{\bar{c}}{2}$ but $c = \underline{c} \geq \frac{\bar{c}}{2}$ provided $k \leq \frac{9}{8}$.

Now we can recover the equilibrium:

$$b(x) = \begin{cases} \hat{b} - \int_0^x \hat{\lambda}(z) dz & \text{if } 0 \leq x \leq c/2, \\ \int_{\bar{c}}^x \hat{\lambda}(z) dz & \text{if } c/2 \leq x \leq c, \\ \hat{b} + \int_c^x \lambda(x, x) dx & \text{if } c \leq x \leq 1. \end{cases}$$

where $\hat{b} = \int_0^{c/2} \hat{\lambda}(z) dz$.

5. MONOTONE EQUILIBRIUM

TO DO: Related this section to [Siegel \(2013\)](#)...

Let ϕ_1 and ϕ_2 denote the inverse bidding functions of some monotone equilibrium. Define, as in [Amann and Leininger \(1996\)](#) or [Parreiras \(2006\)](#), the tying function Q (*i.e.* the equilibrium allocation rule) as the function that maps the type of player 1 to the type of player 2 that bids the same in equilibrium, that is $Q(\phi_1(b)) \stackrel{\text{def}}{=} \phi_2(b)$.

Proposition 1. *The tying function is the solution of the differential equation,*

$$Q'(x) = \frac{v_2(x, Q(x))}{v_1(x, Q(x))} \quad \text{and} \quad Q(1) = 1.$$

Also $b_1(x) = \int_0^{Q(z)} v_1(z, Q(z)) dz$.

⁹In principle, $\hat{\lambda}$ may depend on both signals but for quadratic valuations, it depends only on the player's own signal.

¹⁰The minus sign in the FOC arises here because as [Araujo et al. \(2008\)](#) we are looking for a U-shaped equilibrium so the first local inverse bid function is decreasing (not increasing as in the previous section).

Proof. First-order conditions for an optimal bid are

$$v_1(x, \phi_2(b))f_{2|1}(\phi_2(b)|x)\phi_2'(b) - 1 = 0 \quad \text{and} \quad v_2(\phi_1(b), y)f_{1|2}(\phi_1(b)|y)\phi_1'(b) - 1 = 0. \quad (5.1)$$

Combine the first-order conditions with the identity $Q' \cdot \phi_1' = \phi_2'$ and remember that, since wlog. signals are uniformly distributed in the unit interval, the conditional density coincides with the joint density by Baye's rule. ■

The proposition says that the interdependence of valuations, as opposed to the correlation between the signals, is the only factor that matters for determining the tying function. We illustrate this remark in a couple of interesting environments:

Corollary 1. CORRELATED PRIVATE VALUES. *In any monotone equilibrium, the tying function is the identical to the tying function when signals are independent.*

Corollary 2. COMMON-VALUES. *In any monotone equilibrium, the tying function is the identity and bid functions are $b_i(x) = \int_0^x v(z, z) f_{1,2}(z, z) dz$ for $i = 1, 2$.*

Without re-scaling signals, in the case of pure-common values, the tying function is $Q(x) = F_2^{-1}(F_1(x))$. In the statistical literature, the tying function is also known as the quantile-quantile plot, or simply $Q-Q$ plot.

Let's define the local single crossing and the single crossing conditions:

(LOCAL SINGLE CROSSING) At $(x, y) = (x, Q(x))$, $\lambda_1(x, y)$ is non-decreasing in x and $\lambda_2(x, y)$ is non-decreasing in y , for all x .

(SINGLE CROSSING) For all $\hat{x} < x < \tilde{x}$: $\lambda_1(\hat{x}, Q(x)) < \lambda_1(x, Q(x)) < \lambda_1(\tilde{x}, Q(x))$ and analogously, for all $\hat{y} < y < \tilde{y}$, $\lambda_2(\hat{y}, Q^{-1}(y)) < \lambda_2(y, Q^{-1}(y)) < \lambda_2(\tilde{y}, Q^{-1}(y))$.

Proposition 2. *The local single-crossing is necessary and the single-crossing is sufficient for $b_1(x) = \int_0^{Q(x)} v_1(z, Q(z)) dz$ and $b_2(y) = b_1(Q^{-1}(y))$ to be an (monotone) equilibrium.*

Proof. The function $v_i(z, \phi_j(b))f_{j|i}(\phi_j(b)|z)\phi_j'(b) - 1$ satisfies the local single-crossing condition with respect to z if and only if $v_i(x, \phi_j(b))f_{1,2}(z, \phi_j(b))$ is non-decreasing in z . Again remember that $f_1 = f_2 = 1$ in their respective supports. Differentiating the identity, $v_i(\phi_i(b), \phi_j(b))f_{j|i}(\phi_j(b)|\phi_i(b))\phi_j'(b) - 1 = 0$, with respect to b , and assuming $\phi_i' > 0$, the local single-crossing in z at $\phi_i(b)$ is equivalent to the second-order condition for i 's optimal bid. Local single crossing is thus clearly necessary. The argument to establish single-crossing is sufficient for a monotone eq. is standard:¹¹ single-crossing implies that, at $b = b_1(x)$,

$$\frac{\partial U_i}{\partial b}(b|\hat{x}) < \frac{\partial U_i}{\partial b}(b|x) = 0 < \frac{\partial U_i}{\partial b}(b|\tilde{x}). \quad \blacksquare$$

¹¹See Krishna and Morgan (1997, p. 351), or Lizzeri and Persico (1998, p. 104), or Athey (2001).

6. NON-EXISTENCE OF MONOTONE EQUILIBRIUM

When signals are correlated, the all-pay auction may lack monotone equilibria.

Example 3. (CORRELATED PRIVATE VALUES) The signals (X_1, X_2) follow a truncated, symmetric, bivariate normal distribution specified by (μ, σ^2, ρ) and truncation points $\mu - M$ and $\mu + M$. The expected value of the object for player i is $\exp(h(X_i))$ where h is a given increasing function and $i = 1, 2$.

Proposition 3. *Example 3 does not have a monotone pure strategy equilibrium if $h'(x) \geq \frac{2\rho}{\sigma^2(1+\rho)}(\mu - x)$ for some x .*

Proof. As players are symmetric, by Proposition 1, if a monotone, pure strategy equilibrium exists then it must be symmetric. However, using the fact that $X_j|X_i \sim \mathcal{N}((1-\rho)\mu + \rho X_i, (1-\rho^2)\sigma^2)$ we obtain $\frac{\partial}{\partial x} v_i(x, y) \cdot f_{X_j|X_i}(y|x) \Big|_{y=x} > 0 \Leftrightarrow h'(x) > \frac{2\rho}{\sigma^2(1+\rho)}(\mu - x)$. ■

As a result, for a large class of examples, the symmetric monotone equilibrium is not robust to the introduction of a small degree of correlation:

Corollary 3. *Assume $\|h'\|_\infty < K$ then for any $\rho > 0$ there is $M > 0$ such that the private values model of example 1 has no monotone equilibrium.*

Example 4. (COMMON-VALUE) In all three families given by the table below: signals and the value are affiliated; the parameter θ measures the precision of the players' information; and there is no monotone equilibrium.

| V | $S_i V$ | $E[V S_i = x, S_j = y]$ | $f_{S_j S_i}(y x)$ |
|------------------------------------|-------------------------------|---|--|
| $\ln \mathcal{N}(\mu, \tau^{-1})$ | $\mathcal{N}(V, \theta^{-1})$ | $\exp\left(\frac{\tau\mu + \theta x + \theta y + \frac{1}{2}}{\tau + 2\theta}\right)$ | $\mathcal{N}\left(\frac{\tau\mu + \theta x}{\tau + \theta}, \frac{\tau + 2\theta}{\theta(\tau + \theta)}\right)$ |
| $\text{Pareto}(\omega, \alpha)$ | $V \cdot \text{B}(\theta, 1)$ | $\frac{\alpha + 2\theta}{\alpha + 2\theta - 1} \max(\omega, x, y)$ | $\frac{(\alpha + \theta)\theta}{\alpha + 2\theta} \frac{\omega^{(\alpha + \theta)\mathbf{1}_{[x < \omega]}} x^{(\alpha + \theta - 2)\mathbf{1}_{[x > \omega]}} y^{\theta - 1}}{\max(\omega, x, y)^{\alpha + 2\theta}}$ |
| $\text{Inv-}\Gamma(\alpha, \beta)$ | $\Gamma(\theta V^{-1})$ | $\frac{\Gamma(\alpha + 2\theta - 1)}{\Gamma(\alpha + 2\theta)}(x + y + \beta)$ | $\frac{\Gamma(\alpha + 2\theta)}{\Gamma(\alpha + \theta)\Gamma(\theta)} \frac{y^{\theta - 1}(x + \beta)^{\alpha + \theta}}{(x + y + \beta)^{\alpha + 2\theta}}$ |

TABLE 1. Common-value models without MPSE.

7. CONCLUSIONS

Motivated by the non-robustness of the monotone equilibrium to small degree of correlation, we study non-monotone pure strategy equilibrium.

The existence of pure strategy equilibria (monotone or not), however, remains an open question [Rentschler and Turocy \(2012\)](#)'s results suggest that some models may have mixed

strategy equilibrium (where each type mixes). In this paper, we do not discuss mixed strategy equilibrium.

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