A Three-State Rational Greater-Fool Bubble With Intertemporal Consumption Smoothing*

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Abstract

We consider a greater-fool “strong” bubble (where everyone knows the asset is overpriced) with two rational agents and three periods, and show that the number of states of the world needed to support this bubble can be reduced, from the five states required in Liu and Conlon (2018), to just three states. We accomplish this by introducing intertemporal consumption smoothing as a motive for trade. Our model is related to models like Allen et al. (1993) and Doblas-Madrid (2016) in that our bubble occurs in a finite-horizon environment and has a greater-fool flavor. It is also related to models like Samuelson (1958) and Doblas-Madrid (2012, 2016) in that our bubble is generated in part by agents’ desire to smooth their consumption over time. Thus, a policy of deflating overpriced assets might hurt agents’ welfare. In fact, such a policy might even hurt the “greater fools” who purchase the overpriced asset, since it interferes with their intertemporal consumption smoothing.

**JEL classification:** D53; D82; D84; E21; G12; G14

**Keywords:** Greater-Fool Bubbles, Asymmetric Information, Intertemporal Consumption Smoothing

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1 Introduction

An asset is said to be in a strong bubble, in a given state of the world, if everyone knows for certain, in that state, that the asset is overpriced (Allen et al., 1993). This can occur because, even though everyone knows that the asset is overpriced, they don’t know that everyone else knows it is overpriced, due to asymmetric information. Thus, rational agents might be willing to hold such an overpriced asset in hopes of selling it to “greater fools” in the future. However, asymmetric information can create a lemons problem since potential buyers are afraid of becoming the ultimate greater fools. Asset holders might then be unable to sell the overpriced asset, causing the bubble to unravel. Therefore, a major challenge in this literature is modeling agents who are rational, but are nevertheless willing to risk becoming these greater fools who purchase the overpriced asset. As a result, sustaining a strong bubble in models with asymmetric information requires that the gains from trading the asset are large enough to overcome the potential lemons problem.

Allen et al. (1993) presented numerical examples of strong bubbles in a finite-horizon model with asymmetric information. Their approach was simplified by Conlon (2004, 2015) and Zheng (2011, 2013), until Liu and Conlon (2018) showed that a rational greater-fool strong bubble can exist in a model with two agents, three periods, and five states of the world.\(^1\) In these models, rational agents are willing to trade an overpriced asset because it helps them smooth consumption across states of the world.\(^2\) That is, the gains from risk-sharing across states overcome the potential lemons problem buyers face of becoming the ultimate victims in greater-fool bubbles.

Consumption in the above models occurs only once, in the last period. Until now, however, no one seems to have noticed the importance of this assumption for models of greater-fool bubbles.

The current paper shows, in fact, that if agents are allowed to consume in each period, then the number of states needed to model a greater-fool bubble can be reduced from the five states required.

\(^1\)This result was also independently obtained by Lien et al. (2015).

\(^2\)To illustrate how cross-state consumption smoothing motivates trade, consider an example with two agents, \(A\) and \(B\), and two states, \(a\) and \(b\). Agents do not know which state will occur, but they know that Agent \(A\)’s wealth in state \(a\) will be large relative to her wealth in state \(b\), while Agent \(B\)’s wealth in state \(b\) will be large relative to his wealth in state \(a\). Suppose that Agent \(A\) owns an asset which pays a dividend in state \(a\) but nothing in state \(b\). By selling this asset to Agent \(B\), Agent \(A\) can transfer wealth to Agent \(B\) in state \(a\) (i.e., the dividend), while Agent \(B\) transfers wealth to Agent \(A\) in state \(b\) (i.e., the purchase price). Hence, asset trading provides both agents with smoother consumption across states \(a\) and \(b\).

Risk-sharing was mentioned as a motive for trade in earlier papers (see, for example, Allen et al., 1993, p. 226-27). However, this motive is most fully explored in Holt (2018), who carefully analyzes the case of risk-averse agents, i.e., agents with downward sloping marginal utility curves. Holt’s model provides an important point of departure for the model developed here.
in Liu and Conlon (2018), to just three states. This is because we can introduce *intertemporal consumption smoothing* as a motive for trade, in addition to consumption smoothing *across states*. This simple three-state bubble model will then hopefully both facilitate applied work and illuminate the essential information structure needed to support greater-fool bubbles.

Intertemporal consumption smoothing has been used as a motive for trade in bubble models dating back to the infinite-horizon overlapping generations (OLG) models of Samuelson (1958), Tirole (1985), and others.³ In the simplest version of these models, agents receive a perishable endowment when young, but nothing when old. The preference for smooth consumption over time motivates agents to use some of the endowment in their youth to purchase an overpriced asset from the previous generation, and then sell this asset in their old age, to the next generation, in exchange for some of that generation’s endowment. This allows each generation to smooth their consumption over their entire lifetime. Also, rational young agents are willing to purchase the overpriced asset because they believe the overpricing can last forever, at least in expected value.

Our model also utilizes intertemporal consumption smoothing to motivate trade. However, our model differs from OLG models in three important respects. First, the horizon in our model is finite, not infinite, so our bubble bursts in a bounded number of periods. Second, our model has a greater-fool flavor, so there is asymmetric information between agents in our model, while there is none in the typical OLG model. Finally, unlike the typical OLG model, our agents’ lifetimes coincide, so agents can trade back and forth with each other to smooth their consumption over time.⁴ Thus, the intertemporal consumption smoothing in our model is closer to that in Bewley (1980) and Townsend (1980) than to that in OLG models like Samuelson (1958).

In short, we present a finite-horizon strong bubble model with two rational agents and three periods, in which agents consume in each period. Rational agents are willing to trade the overpriced asset because it helps them smooth consumption over time, as well as across states of the world. In other words, our model motivates agents to trade in part through intertemporal consumption smoothing.

³Bubble models building on Samuelson (1958) and Tirole (1985) are sometimes referred to as “rational bubble” models. However, note that the greater-fool bubble models listed above – including the current model – also assume perfectly rational agents. See Barlevy (2015) for a recent survey of bubble models.

⁴To illustrate how intertemporal consumption smoothing motivates trade in this context, consider an example with two agents, 1 and 2, and two periods, 1 and 2. Each agent knows his/her current and future wealth with certainty. Agent 1 has wealth that rises over time while Agent 2 has wealth that falls over time. Suppose Agent 1 owns a riskless asset that pays a dividend in period 2. By selling this asset to Agent 2, Agent 1 can receive some wealth from Agent 2 in period 1 (i.e., the purchase price), and Agent 2 can receive some wealth from Agent 1 in period 2 (i.e., the dividend). Thus, asset trading provides both agents with smoother consumption over time.
smoothing as in Samuelson (1958), Bewley (1980), Townsend (1980) and Tirole (1985), and in part through risk-sharing across states, as in Allen et al. (1993), Holt (2018), and others. This then allows us to reduce the number of states needed to support a bubble from the five states in Liu and Conlon (2018) to three states.

Our strong bubble model is therefore simpler than Liu and Conlon’s (2018) model in the sense that it has a smaller state space. However, in regards to the consumption structure, our model is more complex than that of Liu and Conlon.5

Finally, our model is also related to bubble-riding models building on Abreu and Brunnermeier (2003), such as Doblas-Madrid (2012, 2016) and Araujo and Doblas-Madrid (2018). Our model’s intertemporal consumption structure resembles that in these bubble-riding models, in the sense that buyers are unable to distinguish sellers who need liquidity from sellers who wish to exploit buyers. In fact, if we interpret the sellers’ liquidity demand in these models as resulting from temporarily low endowments of consumption goods, then their consumption structure becomes very similar to ours. Thus, our model may help to bridge the gap between greater-fool models based on the Allen et al. (1993) approach and those based on the bubble-riding approach of Abreu and Brunnermeier (2003) and Doblas-Madrid (2012, 2016).6

The rest of the paper is organized as follows. Section 2 describes the model’s information and endowment structures. Section 3 presents a numerical example of a three-state rational strong bubble. Section 4 illuminates the importance of intertemporal consumption smoothing by showing that, in our model, an anti-bubble policy may even reduce the welfare of the greater-fool buyers, since it can interfere with their intertemporal consumption smoothing. Section 5 concludes.

2 Preliminaries

The current paper builds on a “simple” asset market structure with asymmetric information, as in Allen et al. (1993), Liu and Conlon (2018), and others, but it introduces consumption in every

5In Liu and Conlon (2018), as in earlier papers in the literature, consumption occurs only once, after all asset trading has ended, so intertemporal consumption smoothing has no role in motivating trade. Thus, their result, that at least five states of the world are needed to support a rational greater-fool strong bubble, depends on the standard assumption that consumption occurs only once. However, note that our model builds strongly on the pattern of trade in Liu and Conlon (2018). See footnote 15 below.

6On the other hand, our model contrasts with the risk-shifting models of Allen and Gale (2000) and Barlevy (2014); see footnote 10 below. It also contrasts with Harrison and Kreps (1978) and Scheinkman and Xiong (2003), where trade and overpricing occur because some agents are overconfident.
period. Markets are Walrasian, and the model lasts for three periods, so $t = 1, 2, 3$. There are three possible states of the world, with the actual state, $\omega$, randomly selected from the state space $\Omega = \{b, L, H\}$. In our example below, $b$ is a “strong bubble” state, where every agent knows the asset is overpriced, and $L$ is a “semi-bubble” state, where some agents know the asset is overpriced, but others do not.\(^7\) State $H$ is the only dividend-paying state.

There are two rational agents, Ellen and Frank, indexed by $j = E, F$. Ellen and Frank have the same prior probabilities $\pi(\omega)$ over $\Omega$, and the same increasing, concave utility function in terms of consumption, $U(\cdot)$, for each period, so $U'(\cdot) > 0$ and $U''(\cdot) < 0$.\(^8\) Agents’ individual lifetime utilities are additively separable and given by $U^j = U(C^j_1) + \beta U(C^j_2) + \beta^2 U(C^j_3)$, where $\beta > 0$ is a psychological discount factor.

There is a single consumption good, which is perishable, and there is also a durable asset in the market, which is risky. The risky asset pays a dividend in units of the consumption good.\(^9\) This dividend is positive only in period 3, state $H$, and zero otherwise, so $d(H) > 0$, but $d(b) = d(L) = 0$. Agents have equal access to the market and receive the same dividend for each share they own.\(^10\) Agent $j$ is endowed with $s^j_0(\omega) \geq 0$ shares of the risky asset at the beginning of period 1, which he/she can trade with other agents in each period in exchange for the consumption good. Let $X^j_t(\omega)$ be agent $j$’s net sales of the asset in period $t$, state $\omega$, so the number of shares he/she owns at the end of period $t$ is $s^j_t(\omega) = s^j_{t-1}(\omega) - X^j_t(\omega)$. However, agents cannot short-sell the asset, so $X^j_t(\omega) \leq s^j_{t-1}(\omega)$. That is, the maximum amount of shares they can sell is limited to the number of shares they own. Note that, unlike Liu and Conlon (2018) and others, there is no riskless asset (e.g., money) in our model.\(^11\)

Agent $j$ receives an endowment of $e^j_t(\omega)$ units of the perishable consumption good at the beginning of each period $t$. Let $p_t(\omega)$ be the price of the asset in terms of the consumption good. Agent

\(^7\)Holt (2018) uses the term “weak bubble” instead of “semi-bubble.”

\(^8\)Our model thus builds on Holt (2018), who first introduced concave utility as a major issue for greater-fool bubble models.

\(^9\)The dividend is in the form of the consumption good as opposed to money because there is no money in this model (though see footnote 11 below).

\(^10\)Ellen and Frank therefore have the same investment opportunities, removing the motive for trade used in risk-shifting models such as Allen and Gale (2000) and Barlevy (2014). In those models, gains from trade are generated because safe borrowers’ investments earn higher returns than lenders’ investments. Lenders are thus willing to take the risk of potentially lending to risky borrowers, in their attempt to lend to safe borrowers.

\(^11\)Presumably other assets could be introduced into the model, but it would be necessary to prevent them from completely replacing the bubble asset’s consumption-smoothing role.
j’s consumption is then

\[ C_t^j(\omega) = \begin{cases} 
  e_j^t(\omega) + p_t(\omega)X_t^j(\omega) & \text{for } t = 1, 2, \\
  e_j^3(\omega) + p_3(\omega)X_3^j(\omega) + d(\omega)s_j^3(\omega) & \text{for } t = 3. 
\end{cases} \]

That is, each agent’s consumption in periods 1 and 2 depends on the size of his/her endowment of the good, the prevailing price, and his/her net sales of the asset, while the period-3 consumption also depends on the dividend from his/her asset holdings at the end of period 3.

As in Liu and Conlon (2018) and earlier papers, our model uses information sets and information partitions to describe agents’ information structures. An information set, or cell, consists of states that are indistinguishable to an agent, while an information partition is a collection of information sets that are disjoint but cover the state space \( \Omega \). Thus, let \( I_{t,i}^j \) be the \( i \)th information set of agent \( j \) in period \( t \). Then agent \( j \)’s period-\( t \) information partition is \( P_t^j = \{ I_{t,1}^j, I_{t,2}^j, \ldots \} \), where \( I_{t,i}^j \cap I_{t,k}^j = \emptyset \), \( \forall i \neq k \) and \( I_{t,1}^j \cup I_{t,2}^j \cup \ldots = \Omega \).

In period 1 of our model, Ellen’s information partition is \( P_1^E = \{ \{b, L\}, \{H\} \} \), so \( I_{1,1}^E = \{b, L\} \) and \( I_{1,2}^E = \{H\} \). That is, Ellen can distinguish state \( H \) from states \( b \) and \( L \), but she cannot distinguish between states \( b \) and \( L \). Frank’s period-1 information partition is \( P_1^F = \{ \{b\}, \{L, H\} \} \), so \( I_{1,1}^F = \{b\} \) and \( I_{1,2}^F = \{L, H\} \). That is, Frank can distinguish state \( b \) from states \( L \) and \( H \), but he cannot distinguish between states \( L \) and \( H \). Figure 1 illustrates Ellen’s and Frank’s information partitions in period 1.

We call state \( b \) the \textit{strong bubble state}, since this is the state where a strong bubble may form. In period 1, state \( b \), Frank knows the true state is \( b \), so he knows that the asset will pay nothing, while Ellen knows the true state is either \( b \) or \( L \), so she also knows the asset will pay nothing. Thus, if \( p_1(b) > 0 \), the asset is in a strong bubble in period 1. We call state \( L \) the \textit{semi-bubble state},
since Ellen knows the asset is worthless, but Frank thinks the asset might pay a dividend. Thus, if \( p_t(L) > 0 \), we may say that the asset is in a semi-bubble in period \( t \). Lastly, we call state \( H \) the dividend-paying state since this is the state where the asset pays a positive dividend in period 3.\(^{12}\)

The agents’ information structures are exogenously refined at the beginning of each period as new information arrives. In period 2, Ellen learns the true state with certainty, so her period-2 information partition becomes \( \mathbb{P}_2^E = \{ \{ b \}, \{ L \}, \{ H \} \} \). Frank, however, learns nothing between periods 1 and 2, so his period-2 information partition remains \( \mathbb{P}_2^F = \mathbb{P}_1^F = \{ \{ b \}, \{ L, H \} \} \). Figure 2 illustrates the information partitions of Ellen and Frank in period 2. In period 3, Frank also learns the true state, so \( \mathbb{P}_3^F = \mathbb{P}_3^E = \{ \{ b \}, \{ L \}, \{ H \} \} \). Since the true state becomes common knowledge by period 3, the period-3 asset price will equal the dividend, so \( p_3(\omega) = d(\omega) \). As a result, there is no motive for trade in period 3, so we can assume that \( X_3^E(\omega) = 0 \), though this does not affect any of our results.

Figure 2: Period 2 Information Structure. Ellen’s information sets are denoted by solid ovals, and Frank’s information sets are denoted by dashed ovals.

In general, information partitions can also be endogenously refined through the information revealed by market trades and prices, though this will not happen in our example below.

Note that the endowment \( e_t^j(\omega) \) of the consumption good must conform to Agent \( j \)’s information structure. In other words, \( e_t^j(\omega) \) must be constant on each of Agent \( j \)’s period-\( t \) information sets. For instance, \( e_2^F(L) = e_2^F(H) \) must be true for Frank on his period-2 information set \( \{ L, H \} \). Otherwise, he could use the information revealed by his endowment to refine his information structure. The same is also true for the initial endowment of shares, \( s_0^j(\omega) \).

As in Liu and Conlon (2018), a competitive equilibrium in our model consists of state-and-time dependent asset prices, \( p_t(\omega) \), and net sales, \( X_t^E(\omega) \) and \( X_t^F(\omega) \), such that

\(^{12}\)Note that state \( b \) in our model corresponds to state \( b \) in Liu and Conlon (2018), while our state \( L \) corresponds to their state \( e^B \), and our state \( H \) corresponds to their state \( e^G \). We don’t need any states corresponding to states \( \omega^1 \) or \( \omega^2 \) in Liu and Conlon (2018).
information and the prices, \( p_t(\omega) \);

(ii) \( p_t(\omega) \), \( X_t^E(\omega) \), and \( X_t^F(\omega) \) depend only on the join (coarsest common refinement) of the exogenous information partitions, \( \mathcal{P}_t^E \) and \( \mathcal{P}_t^F \);

(iii) \( X_t^j(\omega), j = E, F \), depends only on agent \( j \)'s endogenous price-and-trade refined information partitions; and

(iv) the asset market clears, so that \( X_t^E(\omega) + X_t^F(\omega) = 0 \).

To simplify the equilibrium, we choose parameters such that asset sellers will sell all the shares in their possession, so they will always be at least weakly short-sale constrained. That is, we set up our equilibrium so that, graphically, demand curves will always intersect supply curves at or above the kink that corresponds to the total number of shares the sellers own at the beginning of the period, \( s_{t-1}^j(\omega) \). This is illustrated in Graph I and Graph II in Figure 3. Consequently, asset sellers’ total number of shares will determine the net sales, so \( X_t^j(\omega) = s_{t-1}^j(\omega) \), and asset buyers’ marginal willingness-to-pay (WTP) will determine the equilibrium price, \( p_t(\omega) \). In our numerical example below, we also choose parameters such that there is no endogenous information refining.

\[ \text{Graph I} \]

\[ \text{Graph II} \]

**Figure 3:** Supply and Demand for the risky asset. Our strong bubble example will involve asset sellers selling all the shares they possess, \( s_{t-1}^j(\omega) \).

Consider Agent \( j \)'s choice conditional on his/her period-\( t \) cell, \( I_{t,i}^j \). In equilibrium, \( C_t^j(\omega), X_t^j(\omega), s_{t-1}^j(\omega), \) and \( p_t(\omega) \) will be constant on \( I_{t,i}^j \). By abuse of notation, we can therefore write these quantities as \( C_t^j(I_{t,i}^j), X_t^j(I_{t,i}^j), s_{t-1}^j(I_{t,i}^j), \) and \( p_t(I_{t,i}^j) \). If Agent \( j \) is a buyer in his/her cell \( I_{t,i}^j \),
then his/her first order condition for his/her choice of $C^j_t(I^j_t,\omega)$ is given by the usual Euler Equation:

$$U'(C^j_t(I^j_t,\omega)) p_t(I^j_t,\omega) = \beta E \left[ U'(C^j_{t+1}(\omega)) p_{t+1}(\omega) \mid \omega \in I^j_t \right],$$

(2)

where $C^j_{t+1}(\omega)$ is given by the period-$(t+1)$ version of (1). The Euler Equation in (2) will determine the equilibrium asset price, $p_t(\omega)$, given expectations of the future.

Next, if Agent $j$ is a seller in $I^j_t$, he/she will sell all the shares he/she owns in period $t$, so $X^j_t(I^j_t,\omega) = s^j_{t-1}(I^j_t,\omega)$. He/she will do this as long as

$$U'(C^j_t(I^j_t,\omega)) p_t(I^j_t,\omega) \geq \beta E \left[ U'(C^j_{t+1}(\omega)) p_{t+1}(\omega) \mid \omega \in I^j_t \right].$$

(3)

Finally, for the purpose of this paper, we say that a strong bubble exists in state $\omega$, period $t$ if the asset has a positive price even though every agent knows with certainty, at that point, that the asset is worthless. This is more restrictive than the definition of a strong bubble in Allen et al. (1993), where the asset need not be worthless, but instead must only have a price higher than the largest possible dividend.13

3 Numerical Example

Assume the prior probability for each state is $\pi(b) = \pi(L) = \pi(H) = 1/3$. Thus, Ellen’s period-1 probability that the true state is $L$, given that it is in $\{b, L\}$, is 1/2. Similarly, Frank’s period-1 and period-2 probabilities that the true state is $L$, given that it is in $\{L, H\}$, are also 1/2. Each agent possesses one share of the risky asset at the beginning of period 1, so $s^E_0(\omega) = s^F_0(\omega) = 1$ for all $\omega$. The asset pays a dividend of 4 units of the consumption good in period 3, state $H$, and 0

13The definition of a strong bubble in our model must be more restrictive than that in Allen et al. (1993) because, in our model, an asset may have a price higher than the largest possible dividend simply because agents greatly value the asset’s future dividend in states where their future endowment is very low. Thus, even if the price equals the expected discounted value of future dividends, the price can exceed those dividends if the discount factors for some states are greater than one. For instance, consider a two-period model with no uncertainty and Walrasian asset markets, where the asset buyer has an additively separable logarithmic utility, $U(C_1, C_2) = \ln C_1 + \ln C_2$. The buyer initially owns no shares, but her period-1 endowment of the consumption good is 10, and her period-2 endowment is 1. If there exists one share of the asset, which pays a dividend of 2 in period 2, then the buyer is willing to pay a price of $p = 4$ for that share in period 1, since her discount factor would then be $\beta [U'(C_2)/U'(C_1)] = (1) (C_1/C_2) = (10 - 4)/(1 + 2) = 2$. That is, $p = 4$ is a fundamental value to the buyer. The buyer is willing to pay such a high price because her current endowment is much larger than her future endowment. Thus, this asset is not overpriced by any reasonable definition of overpricing.
otherwise, so the period-3 asset price is \( p_3(H) = 4 \) in state \( H \) and \( p_3(b) = p_3(L) = 0 \) in states \( b \) and \( L \). The endowments of the consumption good for Ellen and Frank, \( e^E_t(\omega) \) and \( e^F_t(\omega) \), are given in Tables 1 and 2, respectively.\(^{14}\) We also assume logarithmic utility, so \( U(C) = \ln C \), and let \( \beta = 1 \), so there is no discounting of utility.

<table>
<thead>
<tr>
<th>Table 1: Ellen’s Endowments</th>
<th>Table 2: Frank’s Endowments</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>( b )</td>
</tr>
<tr>
<td>Period 1</td>
<td>6</td>
</tr>
<tr>
<td>Period 2</td>
<td>1</td>
</tr>
<tr>
<td>Period 3</td>
<td>1</td>
</tr>
</tbody>
</table>

3.1 The Overall Structure of the Bubble Equilibrium

In our strong bubble equilibrium, it will turn out that Frank will be a seller in period 1 in his information sets \( \{b\} \) and \( \{L,H\} \), and Ellen will be a seller in period 2 in her cells \( \{L\} \) and \( \{H\} \). Frank will sell all his shares to Ellen in period 1, in all states, \( b \), \( L \), and \( H \). In period 2, whether or not the true state is \( b \) becomes common knowledge, so there are no more gains from trade in state \( b \). In states \( L \) and \( H \), however, Ellen will sell all her shares to Frank in period 2. In period 3, everyone learns the true state, and the dividend is paid.\(^{15}\) Also, our strong bubble equilibrium requires \( p_1(b) = p_1(L) = p_1(H) \) and \( p_2(L) = p_2(H) \). First, \( p_1(b) = p_1(L) \) must hold since Ellen is the period-1 buyer, whose WTP will determine these two prices, and she cannot distinguish \( b \) from \( L \). Similarly, \( p_2(L) = p_2(H) \) must hold since Frank’s WTP will determine the period-2 prices in these two states, and they are indistinguishable to him. However, a bubble equilibrium also requires \( p_1(L) = p_1(H) \) because, if \( p_1(L) \neq p_1(H) \), then Frank could distinguish \( L \) from \( H \), which would leave him no reason to purchase the worthless asset in period 2, state \( L \), so Ellen would not purchase the worthless asset in \( \{b,L\} \) in period 1, thereby unraveling the bubble.\(^{16}\)

\(^{14}\)Only the values in the unshaded areas of Tables 1 and 2 affect our bubble equilibrium. The values in the shaded areas can be any positive numbers without influencing the equilibrium. This is because the asset prices and dividends will be zero in state \( b \), periods 2 and 3, and state \( L \), period 3, so asset trades don’t affect consumption in these states and periods.

Note also that the current example is not intended to be quantitatively realistic, but only to underscore the basic structure of a greater-fool bubble model which is as simple as possible, in terms of the state space. Thus, the endowments are chosen to make the calculations easy, rather than to mimic the quantitative features of a real-world economy.

\(^{15}\)Note that this trading pattern builds on Liu and Conlon (2018), where it is also the case that Frank sells all his shares to Ellen in period 1, and Ellen sells all her shares back to Frank in certain states in period 2.

\(^{16}\)Recall that prices are determined by the buyers’ WTP. Thus, \( p_1(L) = p_1(H) \) requires Ellen to coincidentally have the same WTP in her two period-1 information sets, \( \{b\} \) and \( \{H\} \). Because of the need for this coincidence,
3.2 The Bubble Equilibrium

Tables 3 and 4 describe our bubble equilibrium. Table 3 provides the equilibrium prices, \( p_t(\omega) \), and Table 4 provides Ellen’s net sales, \( X_t^E(\omega) \). The asset market clears, so Frank’s net sales are \( X_t^F(\omega) = -X_t^E(\omega) \). Note from Table 3 that the period-1 asset price in state \( b \) is \( p_1(b) = 1 \), even though both Ellen and Frank know the asset is worthless. Hence, a strong bubble is present in period 1, state \( b \). Furthermore, the period-1 and period-2 asset prices in state \( L \) are \( p_1(L) = 1 \) and \( p_2(L) = 2 \), even though Ellen knows the asset is worthless. Hence, a semi-bubble is present in periods 1 and 2, state \( L \).

Table 3: Equilibrium Asset Prices

<table>
<thead>
<tr>
<th>State</th>
<th>( b )</th>
<th>( L )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Period 2</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Period 3</td>
<td>0</td>
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<td>4</td>
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</table>

Table 4: Net sales for Ellen

<table>
<thead>
<tr>
<th>State</th>
<th>( b )</th>
<th>( L )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Period 2</td>
<td>0</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>Period 3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

To check that the equilibrium conditions are all met, note first that \( p_1(\omega) \) is determined by Ellen’s WTP. Assuming Ellen purchases all of Frank’s shares in period 1, we get \( X_t^E(\omega) = -s_0^F(\omega) = -1 \). Thus, in her cell \( I_{1,1}^E = \{ b, L \} \), Ellen’s period-1 consumption is \( C_1^E(I_{1,1}^E) = 6 - p_1(I_{1,1}^E) \), using (1). Her period-2 consumption in state \( b \) is \( C_2^E(b) = 1 \), since there is no trade in that state. Her period-2 consumption in state \( L \) is \( C_2^E(L) = 1 + (2)(2) = 5 \), since she sells \( X_2^E(L) = 2 \) shares at a per-share-price of \( p_2(L) = 2 \). Using all this in (2), together with \( U'(C_t^E(\omega)) = 1/C_t^E(\omega) \) for the logarithmic utility function, Ellen’s period-1 Euler Equation given \( I_{1,1}^E \) is

\[
\frac{1}{6 - p_1(I_{1,1}^E)} p_1(I_{1,1}^E) = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (0) + \left( \frac{1}{2} \right) \left( \frac{1}{5} \right) (2) = \frac{1}{5},
\]

so the price that solves this equation is \( p_1(b) = p_1(L) = p_1(I_{1,1}^E) = 1 \), as shown in Table 3.

Ellen is also a buyer in period 1, state \( H \), so her period-1 consumption in that state is \( C_1^E(H) = 6 - p_1(H) \). In period 2, Ellen sells \( X_2^E(H) = 2 \) shares at a per-share-price of \( p_2(H) = 2 \), so her period-2 consumption is \( C_2^E(H) = 6 + (2)(2) = 10 \). Using (2) once more, Ellen’s period-1 Euler

\[\text{our model is not robust to changing parameters, but see Doblas-Madrid (2012), Zheng (2011), and Zhang and Zheng (2017).}\]
Equation given $I_{1,2}^E = \{H\}$ is

$$\frac{1}{6 - p_1(H)} p_1(H) = (1) \left( \frac{1}{10} \right) (2) = \frac{1}{5},$$

and the price that solves this equation is also $p_1(H) = 1$, again as in Table 3. The equilibrium prices and trades in other states and periods follow similar calculations. Tables 5 and 6 provide the resulting consumption for Ellen and Frank, respectively.

**Table 5: Ellen’s Consumption**

<table>
<thead>
<tr>
<th>State</th>
<th>b</th>
<th>L</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Period 2</td>
<td>1</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>Period 3</td>
<td>1</td>
<td>1</td>
<td>28</td>
</tr>
</tbody>
</table>

**Table 6: Frank’s Consumption**

<table>
<thead>
<tr>
<th>State</th>
<th>b</th>
<th>L</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Period 2</td>
<td>24</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Period 3</td>
<td>12</td>
<td>12</td>
<td>20</td>
</tr>
</tbody>
</table>

Intuitively, in period 1, Frank sells his one share in his information set $\{b\}$ because he knows it is worthless. Frank sells his one share in $\{L, H\}$ because he has a large liquidity demand – his period-1 endowment (i.e., 4 units, as in Table 2) is only one sixth of his period-2 endowment (i.e., 24 units). Frank therefore wants to liquidate his asset, even at a low price. Ellen, however, cannot distinguish $b$ from $L$, so she does not know whether Frank knows the asset is worthless, i.e., she does not know whether Frank is a “bad” seller, exploiting her by selling her a worthless asset, or a “good” seller, selling her an asset he thinks is potentially valuable, out of a need for liquidity. In spite of this, Ellen is willing to purchase Frank’s share in $\{b, L\}$, since she has a large willingness to supply liquidity in state $L$ in period 1 (her endowment is 6 units in period 1 but only 1 unit in period 2). Thus, Ellen is betting that the true state is $L$, so that she can resell the asset to Frank in period 2 in exchange for the consumption good, and thus smooth her state-$L$ consumption between periods 1 and 2. In state $H$, Ellen has a lower period-1 willingness to supply liquidity compared to $\{b, L\}$ (her state-$H$ endowment is 6 units in both periods 1 and 2), but she knows for sure that the price will rise, so she is willing to pay the same price for the asset in state $H$ as in $\{b, L\}$.

In period 2 state $b$, Ellen learns that the true state is $b$, and the asset price crashes to 0. If the true state is not $b$, then Ellen sells her two shares to Frank in state $L$ because she knows they

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17 Some features of this numerical example, such as rising asset prices, are not necessary for a strong bubble to exist. Asset prices can move up or down over time. For instance, if Ellen has very high endowments in period 1 but very low endowments in period 2, she might be willing to buy at a high price but sell at a low price later. In this case, the bubble equilibrium prices might very well satisfy $p_1(b) = p_1(L) = p_1(H) > p_2(L) = p_2(H)$. 

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are worthless. Ellen also sells in state $H$, at a discount, because of her large liquidity demand. As a buyer, Frank faces a potential lemons problem in $\{L, H\}$. He does not know whether Ellen is selling him the asset because it is worthless or because she has a large liquidity demand. However, the gains from trade in state $H$ are large enough to overcome the potential lemons problem created by state $L$, so Frank is willing to buy in $\{L, H\}$.

4 Intertemporal Consumption Smoothing and Welfare

A complete welfare analysis is beyond the scope of this paper. Nevertheless, to illustrate how intertemporal consumption smoothing motivates trade in our model, we make some brief comments about the effects of this consumption smoothing on agents’ welfare. As Tables 1 and 2 show, Frank has a large liquidity demand in period 1, in his information set $\{L, H\}$, while Ellen has a strong incentive to supply liquidity in period 1, in her information set $\{b, L\}$. The gains from smoothing her consumption in state $L$ between periods 1 and 2 are sufficiently large that Ellen is willing to run the risk that the true state is $b$. In state $H$, Ellen also has a strong incentive to supply liquidity relative to Frank, so she is again willing to buy the asset from Frank. Thus, the consumption smoothing motive encourages Frank to sell and Ellen to buy in period 1 in all three states, $b$, $L$, and $H$.

Similarly, in period 2, Ellen has a large liquidity demand in her information set $\{H\}$, while Frank has a strong incentive to supply liquidity in $\{L, H\}$. The gains from smoothing his consumption in state $H$ between periods 2 and 3 are sufficiently large that Frank is willing to run the risk that the true state is $L$. Thus, the consumption smoothing motive encourages Ellen to sell and Frank to buy in period 2 in states $L$ and $H$.

Trading the bubble asset therefore allows agents to smooth their consumption over time. To put this more precisely, trading the bubble asset allows each agent’s consumption paths to more closely follow the paths of the overall supply of the consumption good in the aggregate economy. Table 7 shows the economy’s aggregate supply of the consumption good across states and over time, i.e., the total amounts of the consumption good from endowments and dividends.

For the purposes of understanding consumption smoothing and welfare, we distinguish between two possible outcomes: a trading equilibrium, and a situation in which no trade is allowed, called
Table 7: Aggregate Economy (Endowments Plus Dividends)

<table>
<thead>
<tr>
<th>State</th>
<th>b</th>
<th>L</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Period 2</td>
<td>25</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>Period 3</td>
<td>13</td>
<td>13</td>
<td>48</td>
</tr>
</tbody>
</table>

**autarky.** Autarky exogenously prohibits all trades between agents for all states and periods. That is, agents are required to hold their endowed shares, \( s_0^j(\omega) \), to maturity, so \( C_t^b(\omega) = e_t^b(\omega) \) for \( t = 1, 2 \) and \( C_t^H(\omega) = e_t^H(\omega) + d(\omega)s_0^H(\omega) \). This autarky case provides us with a benchmark that is useful in measuring the welfare impacts, both of bubbles and of anti-bubble policy. \(^{18}\)

Table 8 shows Ellen’s and Frank’s gross consumption growth ratios, \( C_{t+1}(\omega)/C_t(\omega) \), under both autarky and trade, as well as the growth ratios for the aggregate economy. This table has two panels: Panel A considers a trading equilibrium in which a bubble exists, i.e., a bubble equilibrium; Panel B considers a trading equilibrium in which a bubble does not exist, i.e., a no-bubble equilibrium – as, for example, when the central bank follows a policy of deflating overpriced assets. \(^{19}\) There are two consumption growth intervals in the table: \( 1 \rightarrow 2 \) refers to the growth between periods 1 and 2, and \( 2 \rightarrow 3 \) refers to the growth between periods 2 and 3. The growth ratio of 2.8, for example, in the second row of Panel A, in Ellen’s column labeled Trade, indicates that Ellen’s state-\( H \) consumption would grow by a factor of 2.8 = 28/10 from period 2 to period 3 in the bubble equilibrium.

As is clear from the table, agents’ consumption growth ratios in the bubble equilibrium are generally closer to the aggregate growth ratios than are the autarky growth ratios. \(^{20}\) For instance, the aggregate grows by a factor of 3 between periods 1 and 2 in state \( H \). Ellen’s consumption grows by a factor of 2 in the bubble equilibrium, but only 1 in autarky. Similarly, Frank’s consumption grows by a factor of 4 in the bubble equilibrium, but 6 in autarky. Thus, since agents’ intertemporal

\(^{18}\)Note that the asset-deflation policy we discuss in this paper – simply announcing whether states \( b \) or \( L \) occur – is not the only, or even the most realistic, anti-bubble policy a central bank might pursue. In reality, instead of announcing the true state of the world, central banks often use open market operations to counteract bubbles – selling bonds to raise interest rates. Presumably such bond sales compete with the overpriced asset for investors’ savings, and so, push down the price of the overpriced asset (see Allen, Barlevy and Gale, 2018). The welfare effect of such a policy is a very interesting topic for future research. However, here we consider a much simpler anti-bubble policy, since our major goal is to illustrate the role of intertemporal consumption smoothing in markets with overpriced assets.

\(^{19}\)Consumption growth ratios for states \( b \) and \( L \) are not presented in Panel B because an asset-deflation policy would eliminate trade in these states, so these growth ratios would simply equal the autarky growth ratios.

\(^{20}\)Note that consumption smoothing is an issue primarily for the unshaded growth ratios in Table 8. The shaded growth ratios are essentially arbitrary, since the (shaded) endowments they depend on in Tables 1 and 2 are also arbitrary and don’t affect the equilibrium (see footnote 14 above).
consumption becomes smoother, a bubble has the potential to improve agents’ welfare relative to autarky.

In particular, the bubble equilibrium allows agents to smooth their consumption in state \( L \) between periods 1 and 2. To illustrate this point, consider an asset-deflation policy, for example, where the central bank follows a policy rule of revealing to all agents whether the bubble or semi-bubble state, \( b \) or \( L \), occurred. This policy will disrupt the bubble equilibrium, so the price of the asset will collapse to zero in states \( b \) and \( L \). Since there is no useful (nonzero-priced) asset to trade in these states, agents revert to autarky and so cannot smooth their consumption. Because of this interference with consumption smoothing, asset-deflation policy can be welfare reducing in state \( L \).

In fact, in the current example, Frank is indeed hurt in state \( L \) by an asset-deflation policy, even though he is the ultimate greater fool who would be exploited by Ellen in state \( L \) of the bubble equilibrium. Specifically, Frank’s lifetime state-\( L \) utility under the policy is \( \ln(4) + \ln(24) + \ln(12) = \ln(1152) \), while his utility in the bubble equilibrium is \( \ln(5) + \ln(20) + \ln(12) = \ln(1200) \). While this might not always happen, it is reminiscent of results in Holt (2018), who first showed that this sort of asset-deflation policy tends to reduce welfare in greater-fool bubble models, since it interferes with trade – though in his case it interferes with risk-sharing, not intertemporal consumption smoothing.\textsuperscript{21} It is also consistent with the results in infinite-horizon bubble models like Samuelson (1958), where the bubble asset tends to improve welfare by allowing agents to smooth consumption

\textsuperscript{21}Note that our welfare effect may also have a risk-sharing component like Holt (2018), since the asset, in the bubble equilibrium, may allow Ellen and Frank to shift wealth across states \( L \) and \( H \).
from youth to old age.\footnote{Ellen’s utility in state $L$ is also hurt by asset-deflation policy, since it interferes with her exploitation of Frank. This should not be surprising. By purchasing Frank’s share in $\{b, L\}$ in period 1, Ellen is effectively choosing to participate in the bubble equilibrium, in spite of the risk that the true state might be $b$. Thus, by a revealed preference argument, Ellen must be better off in state $L$ in the presence of a bubble; otherwise, she would never have chosen to participate in the bubble equilibrium in her information set $\{b, L\}$.}

Note, however, that even though the bubble equilibrium facilitates intertemporal consumption smoothing in state $L$, it also creates a potential lemons problem which may distort consumption smoothing in state $H$. An asset-deflation policy, which eliminates this lemons problem, may therefore improve consumption smoothing in state $H$. As shown by comparing Panels A and B of Table 8, agents’ state-$H$ consumption paths, in our example, are indeed smoother under the asset-deflation policy than they are in the bubble equilibrium. This occurs because buyers know in state $H$ that the asset has intrinsic value, so they are willing to pay more for the asset. By paying a higher price, the high-liquidity buyers help the low-liquidity sellers to better smooth their consumption intertemporally. Because of this smoother consumption, asset-deflation policy may potentially improve welfare in state $H$, but it could also shift wealth between buyers and sellers, which complicates the welfare analysis.\footnote{In fact, in the current example, while the asset-deflation policy leads to smoother consumption paths in state $H$, it also turns out to transfer wealth from Frank to Ellen, so Frank is actually hurt by the policy in state $H$, while Ellen benefits. See the working paper version of this paper for details.} Thus, the total welfare impact of asset-deflation policy is ambiguous, though again, a full welfare analysis is beyond the scope of this paper.

5 Conclusion

As in other Allen et al. (1993)-style strong bubble models, this paper maintains the assumptions of a finite horizon, rational agents, asymmetric information, constrained short-selling, equal investment opportunities, and Walrasian markets. We show that, if one introduces intertemporal consumption smoothing as a motive for trade, then a strong bubble can exist in a very simple model, with only three states of the world, instead of five. The “rational bubble” model in Samuelson (1958) is also generated from intertemporal consumption smoothing, but it relies on an infinite horizon, while our bubble bursts after a bounded number of periods. Our intertemporal consumption structure also links Allen et al. (1993)-style bubble models with the Doblas-Madrid (2012, 2016) version of the Abreu and Brunnermeier (2003) bubble-riding model. Finally, our simple three-state strong bubble model may provide a useful framework in which to reexamine the welfare effects of asset-deflation
policies studied in Conlon (2015) and Holt (2018), and may also be simple enough to make other applied work on greater-fool bubble models extremely straightforward.

References


