Coessentiality of Money and Credit

Luis Araujo* and Tai-Wei Hu†

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Abstract

We use a random matching model with limited record-keeping to study the es-
sentiality of money and credit. The model is based on Lagos-Wright (2005), but with
two rounds of pairwise meetings in each period. The mechanism designer chooses
which meetings to use the record-keeping technology, and designs the trading mech-
anism for each meeting to optimize social welfare. We characterize implementable
outcomes subject to both individual rationality and pairwise core requirement; this
result extends the results in Hu-Kennan-Wallace (2009) by allowing limited record-
keeping. Under limited record-keeping, we obtain coexistence of money and credit
as essential means-of-payments in the sense that both are used in any optimal
mechanism, and show that the optimal inflation rate is (strictly) positive, with
the seigniorage revenue used to purchase privately issued debt. Our results show
that expansionary monetary policy that subsidizes credit trades is beneficial for a
large set of parameters.

1 Introduction

There is a large literature on the role of money and credit as means of exchange. How-
ever, there is much less work on the crucial frictions under which it is socially beneficial
to use both money and credit, let alone the implications of this coessentiality to the mon-
etary policy. In one direction, neo-monetarist models based on Lagos and Wright (2005)
(LW), while quite equipped to examine the liquidity effects of monetary policy, encounter
difficulties addressing coessentiality of money and credit. In fact, Hu, Kennan, and Wal-
lace (2009) show that, in the LW environment, money alone provides sufficient liquidity
to achieve good allocations and credit has no beneficial role. In another direction, neo-
keynesian models abstract away the role of money as a medium of exchange, and focus

*Michigan State University
†Kellogg School of Management, Northwestern University
on the impacts of monetary policy on credit market, through manipulation of nominal interest rates.

In this paper, we aim at providing a tractable framework based on LW in which money and credit are coessential. Although coexistence of money and credit emerges in some models based on LW, they assume particular trading mechanisms, which render implications to optimal monetary policy that are not robust when a more general class of mechanisms is considered. Instead, we take a mechanism-design approach, and we say that money and credit are coessential if they are both used in optimal mechanisms.

Our environment differs from LW in two ways. First, we have three stages in each period: in the first two stages agents meet in pairs, and in the last stage they meet in a centralized location. Second, we introduce a record-keeping technology which keeps track of some actions made by buyers and which can be accessed in some pairwise meetings. We consider three cases: first, the technology is available in all meetings (full credit); second, only meetings in one stage can access the technology (limited credit); third, no meeting can access the technology (no credit). We also look at a benchmark scenario in which the supply of money is constant and there is no intervention and a scenario in which monetary policy is active.

We first show that money is not essential when there is full credit. This result, although similar in spirit, does not follow directly from the results in Kocherlakota (1998), since we use a weaker notion of record keeping. Precisely, our record-keeping technology only keeps track of the identities of the agents involved in the transaction and the amount the buyer promised to pay the seller. This promise can be thought of as an IOU issued by the buyer to the seller. No other information is recorded, such as monetary trades or refusal to trade. Moreover, records can only be accessed in credit meetings. Given this technology, we show that money is essential without full credit by giving an anti-folk theorem: money must be used to induce positive production in meetings without the record-keeping technology.\(^1\)

We obtain a full characterization of implementable allocations under no intervention. It turns out that the limited credit case and the no credit case are identical, and produce a smaller set of implementable allocations than that obtained under full credit. This result differs from the findings in Hu, Kennan and Wallace (2009) because our model has two stages of pairwise meetings instead of one. Intuitively, when there are two stages and no credit, buyers can choose to hold only enough money to participate in the stage-two trade, and this temptation gives an extra constraint that is absent under full credit. Interestingly, the same constraint applies to the case with limited credit. However, the above result also implies that, absent interventions, money and credit are not coessential.

We then examine the coessentiality of money and credit under active monetary policies.

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\(^{1}\)Money is not necessarily essential in settings with limited record-keeping. Precisely, we follow Kocherlakota (1998) in the sense that the record of a buyer who participated in a credit meeting can only be observed by his partners. If his record is publicly observable, as in Kocherlakota and Wallace (1998) and Cavalcanti and Wallace (1999), it is possible to construct equilibria in which a deviation in a non-credit meeting eventually leads to an action by some agents which reveals the initial deviation to the entire population (see Araujo and Camargo (2013)).
We restrict attention to interventions which respect voluntary participation and incentive compatibility. This excludes, for instance, compulsory lump-sum taxes (as one version of the Friedman rule is usually modeled), and also requires interventions to condition only on information obtained through the record-keeping technology.\footnote{This constraint is also imposed by Gomis-Porqueras and Sanches (2013) and captures the idea that there are limits to what the policy maker can implement in terms of money distribution. For instance, in a lump sum injection, if there are no technology which keeps track of who gets the money, how can one prevent an agent from coming twice to receive the money transfer?}

We consider a class of monetary policies, labeled OMO, which mimic the way in which open market operations work. Precisely, the policy sets a maximal amount of IOU’s issued by each buyer that the mechanism commits to purchase from the sellers. As a result, the buyer only has to repay what is not purchased by the mechanism. The mechanism issues money to finance this purchase, and, in equilibrium, has to satisfy a feasibility constraint that binds the inflation rate to the set amount of debt purchases. Since the class of policies we consider involve the purchase of private debt, it resembles the quantitative easing recently implemented by the FED.

We say that OMO is essential if the optimal mechanism involves a strictly positive inflation rate. We obtain two results regarding optimal monetary policies in the case of limited credit. First, OMO is generically essential when the first best allocation is implementable, and it is still essential for a range of lower discount factors for which the first best allocation is not implementable. Second, for lower discount factors, we show that OMO is essential if the matching probability for stage-1 DM is sufficiently high. Because money and credit are not coessential without interventions, and credit is necessary to implement OMO, OMO is essential if and only if money and credit are coessential. As a result, money and credit are coessential for a large set of parameter values.

The paper proceeds as follows. In the next section, we present the environment, and in the following section we define trading mechanisms, strategies and equilibrium. In section 4, we characterize the implementable allocations under no intervention. In section 5 we introduce active monetary policies and characterize the set of implementable allocations under such policies. Section 6 describes optimal allocations under active monetary policies. Section 7 compares our results to the existing literature and section 8 concludes. All the proofs are in the Appendix.

\section{Environment}

Time is discrete and the horizon is infinite. The economy is populated by buyers and sellers. The set of buyers is denoted $\mathbb{B}$ and the set of sellers is partitioned into two subsets, $S_1$ and $S_2$ both with measure one. Each period is divided into three stages. Buyers randomly meet sellers in $S_i$ in sub-period $i \in \{1, 2\}$, and the probability of a successful meeting is $\sigma_i$. There are three goods, one for each stage. At stage 1, a seller from $S_1$ can produce $x$ units of stage-1 good for a buyer at cost $c(x)$ and the buyer’s utility is $u(x)$. At stage 2, a seller from $S_2$ can produce $y$ units of stage-2 good for a buyer at...
cost \( c(y) \) and the buyer’s utility is \( v(y) \). Let \( x^* \) be the solution to \( u'(x) = c'(x) \) and let \( y^* \) be the solution to \( v'(y) = c'(y) \). In the last stage, agents meet in a centralized market. In this market, they can all consume and produce, and the utility is linear, represented by \( z \). Agents maximize their life-time expected utility with discount factor \( \delta \). We let \( r = \frac{1-\delta}{\delta} \).

We call the first two stages DM rounds and the last stage the CM round.

There is an intrinsically useless and storable object, called money. Money is perfectly divisible and its supply is constant and equal to \( M \). There is also a record-keeping technology, which keeps track of buyers’ trading histories. The technology may not be accessible in all meetings. We call a meeting a credit meeting if the technology is accessible, and call a meeting a noncredit meeting otherwise. This technology works as follows. For each buyer \( b \in \mathbb{B} \), a current history at period \( t \) is a triple, \( h^t = (h_1, h_2, h_3) \in H \), such that for \( i = 1, 2 \), \( h_i^t \) records the buyer’s round-\( i \) DM promise, \((b, s, z)\), where \( b \) is the identity of the buyer, \( s \) is the identity of the seller, and \( z \) is the promise in terms of CM good made by the buyer, and \( h_3 \) records his repayment of either trade, denoted by \((p_1, p_2) \in \{0, 1\}^2 \) (where 0 denotes no repayment and 1 denotes repayment). If the buyer does not meet a seller in round-\( i \), or if the buyer meets a seller but there is no trade, \( h_i^t \) is empty. The history \( h_i^t \) is also empty if the technology for noncredit meetings.

The technology does not keep all the past histories \((h^0, h^1, ..., h^{t-1})\). Rather, it suppresses the past histories into a single credit record \( r \in R \), where \( R \) is a finite set. How the credit records are updated will depend on the trading mechanism given later. At the end of period \( t \), the technology then update the buyer’s credit record using the current history \( h^t \) and his credit record from last period. The credit record and the current history of a buyer can be accessed freely by the seller in any credit meeting, but it cannot be accessed in non-credit meetings. Finally, credit may be limited, i.e., the number of total DM rounds with record-keeping is given by \( \ell \in \{0, 1, 2\} \).

**Remark** The record-keeping technology we consider here is very similar to typical operations of credit cards. First, it only records the identities of the agents involved in the transaction and the amount the buyer promised to pay the seller. In particular, it does not record transfers of real balances or the exact quantity of good produced by the seller. Second, it can only be accessed in credit meetings, i.e., if the buyer uses money only in a transaction, the seller has no access to the information contained in his record. Finally, if \( \sigma_i < 1 \), it cannot distinguish between not meeting a seller and not trading in a credit meeting, as both events lead to an empty record. In this sense, our record-keeping technology is much weaker than the notion of memory put forth by Kocherlakota (1998), which includes all actions of all direct and indirect partners of an agent. However, as in Kocherlakota (1998), we assume that the record of a buyer can only be observed by his partners, i.e., it is not publicly observable, as in Kocherlakota and Wallace (1998) and Cavalcanti and Wallace (1999). As it will become clear later, this difference matters for the essentiality of money in non-credit meetings.

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3 In general, \( \ell \) may take any real number between 0 and 2. However, we shall show in the Appendix that restricting \( \ell \) to \( \{0, 1, 2\} \) is without loss of generality.

4 Clearly, if \( \sigma_i = 1 \), an empty record in a credit meeting is evidence of no trade.
3 Implementation

3.1 Trading mechanisms

We study outcomes that can be implemented by proposals from a mechanism designer. A proposal consists of the following objects:

(P1) A subset $C \subseteq \{1, 2\}$ of DM rounds which have access to credit records.

(P2) A finite set of records $R$ and a function $\omega_t : H \times R \rightarrow R$ which updates the record of the buyer at the end of each period.

(P3) A function $o_1^t$ given as follows: if $1 \in C$, then

$$o_1^t(m, r) = (x, z_{1,p}, z_{1,m}),$$

where $m$ is the buyer’s announcement of real balance holdings, $r$ is his record, $(x, z_{1,p}, z_{1,m})$ is the trade—$x$ is the quantity to be produced by the seller, $z_{1,p}$ is the promise to pay of the buyer, and $z_{1,m}$ is the transfer of real balances from the buyer to the seller; if $1 \notin C$, then

$$o_1^t(m) = (x, z_{1,m}),$$

where $m$ is the buyer’s announcement of real balance holdings and $(x, z_{1,m})$ is the trade—$x$ is the quantity to be produced by the seller and $z_{1,m}$ is the transfer of real balances from the buyer to the seller.

(P4) A function $o_2^t$ given as follows: if $2 \in C$, then

$$o_2^t(m, r, h_1) = (y, z_{2,p}, z_{2,m}),$$

where $m$ is the buyer’s real balance holding, $r$ is his record and $h_1$ is his round 1 trading history, $(y, z_{2,p}, z_{2,m})$ is the trade; if $2 \notin C$, then

$$o_2^t(m) = (y, z_{2,m}).$$

(P5) The price for money, $\phi_t$, in the CM, and an initial distribution of money holdings, $\mu$.

The trading mechanism in meetings in the DM is as follows. The buyer first announces his real balances, and then both the buyer and the seller respond with yes or no to the corresponding proposed trade. If both respond with yes then the trade is carried out; otherwise, there is no trade. As for now, the individual responses only ensure the proposed trade to be individually rational, but it may still leave room for Pareto improvement within the pair. We will return to this pairwise core requirement later. In turn, the trading mechanism in the CM stage is as follows. Agents trade competitively against $\phi$ to rebalance their money holdings, and each buyer chooses whether to repay his promises to the mechanism.
3.2 Strategies and equilibrium

We denote by $s_b$ the strategy of a buyer $b \in B$. For any given trading history, the strategy $s_b$ has three components: (i) the first component maps the buyer’s money holding and his record, to the buyer’s announcement, $m \geq 0$, and to his response $\{yes, no\}$; (ii) conditional on his history in the first DM round, the second component maps the buyer’s money holding and his record to the buyer’s announcement, $m \geq 0$, and to his response $\{yes, no\}$; (iii) the third component maps the buyer’s trading history in the first and second DM rounds to his final money holdings after the CM and to his repayment decisions.

We denote by $s_i$ the strategy of a seller $s \in S_i$, where $i \in \{1, 2\}$. For any given trading history, the strategy $s_1$ is as follows. In a credit meeting, the function maps the buyer’s announced money holding and his record to the seller’s response $\{yes, no\}$; in a money meeting, the function maps the buyer’s announced money holding to the seller’s response $\{yes, no\}$. In turn, for any given trading history, the strategy $s_2$ is as follows. In a credit meeting, the function maps the buyer’s announced money holding, his record, and his recorded history in the first DM round, to the seller’s response $\{yes, no\}$; in a money meeting, the function maps the buyer’s announced money holding to the seller’s response $\{yes, no\}$. We assume that sellers do not carry money across periods. We restrict attention to equilibria in stationary strategies.

**Definition 3.1.** An equilibrium is a list,

$$E = \{(s_b : b \in B), (s_{s_i} : s_i \in S_i)_{i=1,2}, [C, (R, \omega), (o_1, o_2), (\phi, \mu)]\},$$

composed of one strategy for each agent and the proposals,

$$P = [C, (R, \omega), (o_1, o_2), (\phi, \mu)],$$

such that: (i) each strategy is sequentially rational given other players’ strategies and the price of money; (ii) the centralized market for money clears at every date; (iii) the number of the total DM rounds with record-keeping per period is limited by $l$.

Throughout the paper we restrict attention to symmetric equilibria with the following characteristics: (1) the buyer always announces the truth about his money holdings, (2) both the buyer and the seller respond with yes in all DM meetings; (3) the initial distribution of money across buyers is degenerate - all buyers hold $M$ units of money; (4) all buyers repay their promises at every date. We call such equilibria simple equilibria. The outcome associated with a simple equilibrium $\sigma$ is characterized by a list

$$O(E) = [(x, z_{1,p}, z_{1,m}), (y^0, z_{2,p}^0, z_{2,m}^0), (y^1, z_{2,p}^1, z_{2,m}^1), z],$$

where $z$ denotes the amount of real balances the buyers hold across periods, $(x, z_{1,p}, z_{1,m})$ denotes round 1 DM trade; $(y^0, z_{2,p}^0, z_{2,m}^0)$ denotes the round 2 DM trade, conditional on the buyer not meeting a seller in the round 1 DM; $(y^1, z_{2,p}^1, z_{2,m}^1)$ denotes the round 2 DM trade, conditional on the buyer meeting a seller in the round 1 DM. The allocation associated with an outcome $O(E)$ is characterized by a list

$$L(E) = [(x, y^0, y^1), (z_1, z_{2,ty}^0, z_{2,ty}^1)],$$
where $x$ denotes a buyer’s round 1 DM consumption, conditional on meeting a seller; $y^0$ denotes a buyer’s round 2 DM consumption, conditional on meeting a seller at round 2 but not meeting a seller at round 1; $y^1$ denotes a buyer’s round 2 DM consumption, conditional on meeting a seller at round 2 and round 1; $z_1 = z_{1,p} + z_{1,m}$ denotes the CM consumption of a round 1 seller, conditional on meeting a buyer; $z_2^0 = z_{2,p}^0 + z_{2,m}^0$ denotes the CM consumption of a round 2 seller, conditional on meeting a buyer who has not matched at round 1; $z_2^1 = z_{2,p}^1 + z_{2,m}^1$ denotes the CM consumption of a round 2 seller, conditional on meeting a buyer who has matched at round 1. Henceforth, we let $ar{z} = \max\{z_1 + z_2^1, z_2^0\}$.\footnote{The distinction between $(y^0, z_{2,p}^0, z_{2,m}^0)$ and $(y^1, z_{2,p}^1, z_{2,m}^1)$ is only meaningful if there is match uncertainty in the first DM round, i.e., if $\sigma_1 < 1$. If $\sigma_1 = 1$, the outcome associated with a simple equilibrium $\sigma$ is characterized by a list $O(\mathcal{E}) = [(x, z_{1,p}, z_{1,m}), (y^1, z_{2,p}^1, z_{2,m}^1), z]$. In turn, the allocation associated with an outcome $O(\mathcal{E})$ is characterized by a list $\mathcal{L}(\mathcal{E}) = [(x, y^1), (z_1, z_2^1)]$.}

We say that an outcome $O$ is implementable if it is the outcome of a simple equilibrium $\mathcal{E}$ for some proposal $\mathcal{P}$. We say that an allocation $\mathcal{L}$ is implementable if it corresponds to an implementable outcome $O$. Finally, if $\sigma_1 < 1$, we say that an allocation is symmetric if $y^0 = y^1$ and $z_2^0 = z_2^1$. In this case, an allocation is characterized by

$$\mathcal{L}(\mathcal{E}) = [(x, y), (z_1, z_2)].$$

An anti-folk result In our main result we provide conditions under which credit and money are co-essential. We first show that, in the absence of money, there is no production in non-credit meetings. This is the purpose of Lemma 1.

**Lemma 3.1.** Assume that $M = 0$ and $\ell \in \{0, 1\}$. In every implementable allocation, positive production can only occur in credit meetings.

The key for the result in Lemma 1 is the assumption, akin to Kocherlakota (1998), that the record of a buyer who participated in a credit meeting can only be observed by his partners. If his record was publicly observable, as in Kocherlakota and Wallace (1998) and Cavalcanti and Wallace (1999), one could construct equilibria in which a deviation by seller $s$ in a non-credit meeting with buyer $b$ eventually leads to an action by some agents which reveals the initial deviation to the entire population (see Araujo and Camargo (2013)).

## 4 Constant Money Supply

Here we present two characterization results: one for $\ell = 2$ and another for $\ell < 2$.

**Theorem 4.1 (Implementability under Full Credit).** Suppose that $\ell = 2$, $\sigma_1 < 1$ and $\sigma_2 < 1$. An allocation $\mathcal{L} = [(x, y^0, y^1), (z_1, z_2^0, z_2^1)]$ is implementable if and only if

\begin{align}
-r\bar{z} + \sigma_1[u(x) - z_1] + \sigma_1\sigma_2[v(y^1) - z_2^1] + (1 - \sigma_1)\sigma_2[v(y^0) - z_2^0] & \geq 0; \\
[u(x) - z_1] + \sigma_2[v(y^1) - z_2^1] & \geq \sigma_2[v(y^0) - z_2^0]; \\
z_1 & \geq c(x), \quad v(y^1) \geq z_2^1 \geq c(y^1), \quad v(y^0) \geq z_2^0 \geq c(y^0).
\end{align}

\footnote{The distinction between $(y^0, z_{2,p}^0, z_{2,m}^0)$ and $(y^1, z_{2,p}^1, z_{2,m}^1)$ is only meaningful if there is match uncertainty in the first DM round, i.e., if $\sigma_1 < 1$. If $\sigma_1 = 1$, the outcome associated with a simple equilibrium $\sigma$ is characterized by a list $O(\mathcal{E}) = [(x, z_{1,p}, z_{1,m}), (y^1, z_{2,p}^1, z_{2,m}^1), z]$. In turn, the allocation associated with an outcome $O(\mathcal{E})$ is characterized by a list $\mathcal{L}(\mathcal{E}) = [(x, y^1), (z_1, z_2^1)]$.}
The intuition underlying Theorem 4.1 runs as follows. Condition (2) ensures that, at the end of every period, the buyer is willing to honor his current debt in order to keep a good record and be allowed to participate in both DM rounds in the next period. Condition (3) ensures that the buyer wants to participate in the first DM round. It is necessary because, when $\sigma_1 < 1$, the record-keeping technology cannot distinguish between not meeting a seller and meeting a seller and saying no. Finally, condition (4) ensures the participation of buyers and sellers in the DM rounds. Note that, since payoffs in the second DM round can condition on participation in the first DM round, the buyer can have a negative payoff in the first DM round. However, if the allocation is symmetric, which will be the case of optimal allocations as we will discuss in section 5, (3) becomes $u(x) \geq z_1$.

Theorem 4.1 assumes that $\sigma_1 < 1$ and $\sigma_2 < 1$. It is straightforward to adapt the proof to the case where either or both inequalities fail to hold. The only restrictions which can be relaxed are those in (4), associated with the incentives of buyers to participate in DM rounds. In fact, if $\sigma_1 = 1$ or $\sigma_2 = 1$, an empty history is evidence of no trade. Thus, if a buyer refuses to participate in trade, he can be punished with a bad record. This implies that, if $\sigma_1 < 1$, we can have $v(y^1) < z_2^1$ and $v(y^0) < z_2^0$, while if $\sigma_1 = 1$, we can either have $u(x) < z_1$ or $v(y^1) < z_2^1$. However, in what follows, we restrict attention to allocations satisfying $u(x) \geq z_1$, $v(y^1) \geq z_2^1$ and $v(y^0) \geq z_2^0$. There are two reasons for this restriction. First, it simplifies the discussion and, as it will be seen in section 5, it has no impact when it comes to the characterization of optimal allocations. Second, punishing a buyer with a bad record if he refuses to trade in a credit meeting, although feasible under our record-keeping technology, is not very appealing. In particular, it is not consistent with our intent of defining credit in a way that captures the mechanics of credit cards in actual economies.

Two final remarks are in order as it refers to Theorem 4.1. First, it should be clear that the use of money does not help relaxing any of the conditions required in the proof of the theorem. This suggests that, similarly to Kocherlakota (1998), imperfect credit is a necessary condition for money to be essential. Second, the condition (2) also gives an endogenous debt limit,

$$\bar{z} = \frac{1}{r} \{\sigma_1[u(x) - z_1] + \sigma_1\sigma_2[v(y^1) - z_2^1] + (1 - \sigma_1)\sigma_2[v(y^0) - z_2^0]\}.$$ 

Given this debt limit, the pairwise core requirement implies that there should be no unrealized gain from trade at either stage given the buyer’s payment capacity, $\bar{z}$, and his expected gain from trade in the next stage. We will return to this point later.

**Theorem 4.2 (Implementability under Limited or No Credit).** Suppose that $\ell < 2$. If $\sigma_1 < 1$, an allocation $L = [(x, y^0, y^1), (z, z_2^0, z_2^1)]$ satisfying $u(x) \geq z_1$, $v(y^1) \geq z_2^1$, and $v(y^0) \geq z_2^0$, is implementable if and only if (2), (3),

$$z_1 \geq c(x), \quad z_2^1 \geq c(y^1), \quad z_2^0 \geq c(y^0), \quad (5)$$

and

$$v(y^0) - z_2^0 \geq v(y^1) - z_2^1; \quad (6)$$

$$-r z_1 + \sigma_1[u(x) - z_1] + (1 - \sigma_1)\sigma_2\{v(y^0) - z_2^0\} - [v(y^1) - z_2^1] \geq 0. \quad (7)$$
Alternatively, if \( \sigma_1 = 1 \), an allocation \( \mathcal{L} = [(x, y^1), (z, z_2^1)] \) satisfying \( u(x) \geq z_1 \) and \( v(y^1) \geq z_2^1 \) is implementable if and only if (2)

\[
z_1 \geq c(x), \quad z_2^1 \geq c(y^1),
\]

and

\[
-rz_1 + \sigma_1[u(x) - z_1] \geq 0.
\]

In the case of perfect credit, if \( \sigma_1 < 1 \), the record-keeping technology cannot distinguish between a buyer who does not meet a seller and a buyer who meets a seller and says no, i.e., buyers can always pretend not to have met a seller in the first DM round. In the case of limited credit, buyers can also pretend to have participated in a trade meeting in the first DM round. Conditions (6) and (7) deal with this problem. Indeed, (6) ensures that buyers who did not meet a seller in the first DM round have no incentive to pretend they did so by hiding their real balances. In turn, (7) ensures that buyers have no incentive to bring less real balances which prevent their participation in the first DM round but allow for participation in the second DM round.

Theorem 4.2 shows that, if \( \ell < 2 \), money alone achieves the same set of allocations which can be achieved with money and credit. Thus, credit is not essential if it is limited. Together with Theorem 4.1, this implies that, irrespective of the value of \( \ell \), money and credit are not co-essential. If credit is perfect, money is irrelevant, and if credit is limited, money is sufficient. We conjecture that this result is partly driven by our assumption that buyers face only two rounds of DM trade. For instance, assume there are three rounds of DM trade and \( C = \{1, 2\} \). In this case, a buyer who does not participate in DM trade in round 1 cannot pretend to have done so in round 2. The same is not true if \( C = \emptyset \) and only money can be used, as a buyer can always hide his real balances in round 2 of DM trade so as to pretend that he participated in a trade meeting in round 1. Thus, the restriction to two rounds of DM trade is not without loss of generality as it refers to the coessentiality between money and credit. We do not pursue this topic further here because we are interested on how monetary policy, particularly open-market operations, can increase the set of implementable outcomes when credit is limited and, at the same time, make both money and credit essential. To address this issue, two rounds of DM trade suffice.

Lastly, we assumed that the supply of money is constant. It should be intuitive that nothing is gained if new money is uniformly injected in the CM round in every period. Indeed, for the usual reasons, such an injection would only reduce the incentives of buyers to hold money across periods. In turn, we have not considered the possibility of using lump-sum taxes as an instrument because we are only interested in interventions which result in a net transfer of assets to private agents so as to respect incentive compatibility. This will the case of open market operations, to be considered in the next section.
5 Monetary Interventions

Here we introduce monetary interventions in our model and show that it increases the set of implementable outcomes for \( \ell = 1 \), and, in fact, next section shows that it makes both money and credit essential in many cases. For the most of the section we confine our consideration to schemes without taxation; in Section 5.4 we discuss taxes. Without the coercion power to tax, any monetary intervention increases the money supply and the real effects of such intervention is determined by how the seignorage revenue is used. In the literature the seignorage revenue is returned to the agents in a lump-sum fashion (as in Lagos-Wright (2005)), or conditional on the agent’s money holding (as in Wallace (2013)). However, as been argued else where (Hu-Kennan-Wallasce (2009) for the lump-sum case), such interventions are not useful here.

In contrast, we consider interventions that use the seignorage revenue to subsidize the credit market, which we label expansionary monetary (EM) policies. Consider a mechanism where at least one of the DM rounds has credit meetings. Then buyers may issue some IOU’s that are recorded for meetings where the technology is available. The EM sets a maximum amount (in terms of the CM good), \( k_i \), of IOU’s for which the mechanism will repay for each buyer (to seller from \( S_i \)) using newly printed money. Therefore, for any recorded promise at period \( t \), \( (b, s_i, z_{i,p}) \), the mechanism will pay \( \min\{k_i, z_{i,p}\} \) for the buyer. Let \( \pi \) be the net money growth rate. Thus, for each \( t \), \( M_{t+1} = (1 + \pi)M_t \) and we focus only on proposals with constant real balances, that is, \( \phi_{t+1} = \phi_t/(1 + \pi) \). Then, if, for each buyer \( b \), \( z_{i,p}^b \) is the amount of debt that \( b \) has for his stage-\( i \) trade, feasibility requires a corresponding inflation rate \( \pi \) such that

\[
\int_{b \in \mathbb{B}} \min\{z_{1,p}^b, k_1\} db + \int_{b \in \mathbb{B}} \min\{z_{2,p}^b, k_2\} db = \pi \phi_t M_{t-1}. \tag{10}
\]

Here we assume that the mechanism commits to purchase the set amount of IOU’s from sellers. Note that for a given policy and a given path of prices of money, there is an upper bound on the inflation rate to finance such purchases. For implementation, we require that the equilibrium path of prices of money and the equilibrium issuance of private IOU’s are consistent with the inflation rate and the amount of debt purchases set by the EM.

Formally, under EM, a proposal includes \( \mathcal{P} \) given by the list (1) and three parameters, \(((k_1, k_2), \pi)\). We assume that \( k_i > 0 \) only if \( i \in C \) in \( \mathcal{P} \), that is, round-\( i \) DM has credit meetings. An outcome

\[
\mathcal{O} = [(x, z_{1,m}, z_{1,m}), (y_0, z_{2,p}, z_{2,m}), (y_1, z_{2,p}, z_{2,m})]
\]

is implementable under the proposal \( \langle \mathcal{P}, ((k_1, k_2), \pi) \rangle \) if it is the equilibrium outcome of a simple equilibrium consistent with the given proposal \( \mathcal{P} \) and the parameters \(((k_1, k_2), \pi)\) with respect to (10). An outcome is implementable with EM if it is implementable under some proposal \( \langle \mathcal{P}, ((k_1, k_2), \pi) \rangle \). Note that if an outcome is implementable, it is implementable with EM.
5.1 Benchmark case: $\sigma_1 = \sigma_2 = 1$

In this section we assume that $\sigma_1 = \sigma_2 = 1$. This assumption helps to simplify notations significantly but the main message generalizes (see Section 5.3 below). In this case an allocation $\mathcal{L}$ consists of $[(x, y), (z_1, z_2)]$ as there is no matching uncertainty. The following theorem shows that, with expansionary monetary policies, any allocation that is implementable with $\ell = 2$ is also implementable with $\ell = 1$.

**Theorem 5.1 (Expansionary Monetary Policy).** Suppose that $\sigma_1 = \sigma_2 = 1$ and that $\ell = 1$. An allocation, $\mathcal{L} = [(x, z_1), (y, z_2)]$ satisfying $u(x) \geq z_1$ and $v(y) \geq z_2$, is implementable with EM if and only if

\[
\begin{align*}
\{-rz_1 + [u(x) - z_1]\} + \{-rz_2 + [v(y) - z_2]\} &\geq 0, \\
z_1 &\geq c(x), \quad z_2 \geq c(y).
\end{align*}
\]

**Remarks.**

(a) When $\sigma_1 = \sigma_2 = 1$, the EM is active only if $C = \{1\}$. Assume, consistent with the proof of Theorem 5.1, that debt is only used in the first DM round. Then, in the centralized market, the designer prints an amount $\pi M_t$ of new money, where $\pi \phi_t M_t = k$. The new money is uniformly distributed across all sellers who participated in trade in the stage-1 DM, in such a way that each seller is able to buy $z_1$ units of the CM good.

(b) Note that an active EM, defined as $k > 0$, is necessary to implement the candidate allocation whenever money alone cannot implement it, that is, whenever money and credit are coessential. Moreover, when money and credit are essential, the record-keeping technology is used in the stage-1 DM and hence buyers will carry both money and credit debt into the stage-2 DM. This result rationalizes the setup in Telyukova and Wright (2008), but a key difference here is that coessentiality requires interventions. However, when this is the case, the EM defined by (29) is such that the buyer is only partially subsidized, that is, $k \in (0, z_1)$.

(c) Another feature in the mechanism constructed in the proof of Theorem 5.1 is that money trades and credit trades are totally independent in the sense that money holding does not matter in the credit trades. In later section we will see that money-holdings can be an important information in the presence of matching uncertainty (i.e., $\sigma_1 < 1$).

(d) In contrast to the usual models of monetary policy, here money is given only to sellers with IOU’s from buyers. This assumption can be justified by the fact that the identities of buyers and sellers who participated in credit meetings in the first DM round can be accessed through the record-keeping technology. Indeed, the usual assumption that money can be distributed to people according to various schemes (lump-sum in Lagos-Wright (2005), interest-payment in Andolfatto (2010) and Wallace (2013)) requires some care in terms of record-keeping of money-holdings that is largely ignored (except for Sanches and Gomis-Porqueras (2013)).

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6One crucial question in those papers is the following: How can one prevent an agent from coming twice to receive the money transfer without keeping records about people who have already done so? See also the discussion in Sanches and Gomis-Porqueras (2013), page 703.
5.2 Pairwise-core implementability

Here we revise the trading mechanism to allow for renegotiation. The proposals are defined as before. In meetings in the DM, the trading procedure is given by the following. The buyer first announces his real balances, and then both the buyer and the seller respond with yes or no to the corresponding proposed trade. If both respond with yes then they move to the next stage; otherwise, the meeting is autarkic. If they move to the next stage, the buyer makes a take-it-or-leave-it offer, which is implemented if seller responds with a yes while the originally proposed trade by the mechanism is carried out otherwise. The trading procedure in the CM stage is the same as before. We call such trading procedures the PC procedure. The definition of simple equilibria remains the same; in particular, this implies that the proposed trades are such that renegotiations never take place in equilibrium, following any history. An allocation \( L \) is said to be PC-implementable if there is a proposal \( (P, ((k_1, k_2), \pi)) \) for which \( L \) corresponds to the equilibrium outcome of a simple equilibrium associated with that proposal.

**Theorem 5.2 (Expansionary Monetary Policy under PC).** Suppose that \( \sigma_1 = \sigma_2 = 1 \) and that \( \ell = 1 \). An allocation, \( L = [(x^{pc}, z_1^{pc}), (y^{pc}, z_2^{pc})] \), satisfying \( u(x^{pc}) \geq z_1^{pc} \) and \( v(y^{pc}) \geq z_2^{pc} \) is PC-implementable with OMO if and only if (11) and (12) hold, \( x^{pc} \leq x^* \), \( y^{pc} \leq y^* \), and, by setting \( \bar{x} = \min\{x^*, c^{-1}(z_1^{pc} + z_2^{pc})\} \),

\[
u(x^{pc}) - c(x^{pc}) + v(y^{pc}) - z_2^{pc} \geq u(\bar{x}) - c(\bar{x}).\]

5.3 The general case

Now we consider the case where \( \sigma_1 < 1 \) and \( \sigma_2 < 2 \). In this case, the EM also provides insurance to hedge against the matching uncertainty, and, as a result, it can be used to implement more allocations than full credit. Here we concentrate on symmetric allocations only.

**Theorem 5.3.** Suppose that \( \ell = 1 \). A symmetric allocation, \( L = [(x, z_1), (y, z_2)] \), that satisfies \( u(x^p) \geq z_1^p \) and \( v(y^p) \geq z_2^p \) is implementable with EM and \( C = \{1\} \) if and only if

\[
\begin{align*}
\sigma_1 \{-rz_1 + [u(x) - z_1]\} + \{-rz_2 + \sigma_2[v(y) - z_2]\} &\geq 0, \quad (13) \\
z_1 &\geq c(x), \quad z_2 \geq c(y). \quad (14)
\end{align*}
\]

Moreover, \( L \) is PC-implementable if \( x^p \leq x^* \), \( y^p \leq y^* \), and (1) \( x = x^* \) or (2) the constraint (13) holds with equality and

\[
u(x) - c(x) + \sigma_2[v(y) - z_2] \geq u(x^*) - c(x^*).\]

(15)

Theorem 5.3 is particularly useful when \( \sigma_1 < 1 \); indeed, when \( \sigma_1 = 1 \), the condition coincides with those for full credit, and, when \( \sigma_1 = \sigma_2 = 1 \), coincides with those in Theorem 5.1. Now we consider another scheme that is particularly useful for \( \sigma_2 < 1 \).
Theorem 5.4. Suppose that \( \ell = 1 \). A symmetric allocation, \( \mathcal{L} = [(x, z_1), (y, z_2)] \), that satisfies \( u(x^p) \geq z_1^p \) and \( v(y^p) \geq z_2^p \) is implementable with EM and \( C = \{2\} \) if and only if

\[
\begin{align}
-r z_1 + \sigma_1 [u(x) - z_1] + \frac{(1 + r) \sigma_2}{r + \sigma_2} \{ -r z_2 + \sigma_2 [v(y) - z_2] \} & \geq 0, \\
-r z_1 + \sigma_1 [u(x) - z_1] & \geq 0, \\
z_1 & \geq c(x), \quad z_2 \geq c(y).
\end{align}
\]

5.4 Interventions with taxes

Now we turn to interventions that use taxes. If the lump-sum taxes are available, then we can achieve the first-best allocation. However, such taxes require both detailed record-keeping and coercion power that are not present in our model, and hence are not available here. Instead, we assume that the mechanism may tax the agent only if the agent is in a credit meeting and decides to engage in a credit trade. The only punishment for not paying the taxes is to give the individual a bad credit record. We consider interventions that use the tax revenue to buy back money and hence provides interest on money holdings. We label such interventions contractionary monetary policy (CMP).

Consider a mechanism where at least one of the DM rounds has credit meetings. Then buyers may issue some IOU’s that are recorded for meetings where the technology is available. The CMP sets a proportional tax (in terms of the CM good), \( \rho_i \), on the IOU’s each buyer (in addition to their payments to sellers from \( S_i \)) issues and then buy back money with those tax revenues. Therefore, for any recorded promise \( z_{i,p}^b \) at period \( t \) made by buyer \( b \), the buyer has to pay extra \( \rho_i z_{i,p}^b \) to keep his good record. Let \( \tau \) be the net money contraction rate. Thus, for each \( t, M_{t+1} = (1 - \tau) M_t \) and we focus only on proposals with constant real balances, that is, \( \phi_{t+1} = \phi_t / (1 - \tau) \). Then, if, for each buyer \( b \), \( z_{i,p}^b \) is the amount of debt that \( b \) has for his stage-\( i \) trade, feasibility requires a corresponding deflation rate \( \tau \) such that

\[
\int_{b \in B} \rho_1 z_{1,p}^b db + \int_{b \in B} \rho_2 z_{2,p}^b db = \tau \phi_t M_{t-1}.
\]

Here we assume that the mechanism can only tax buyers with a nonempty credit history.

Formally, under CMP, a proposal includes \( \mathcal{P} \) given by (1) and three parameters \( ((\rho_1, \rho_2), \tau) \). We assume that \( \rho_i > 0 \) only if \( i \in C \) in \( \mathcal{P} \), that is, round-\( i \) DM has the record-keeping technology. An outcome

\[
\mathcal{O} = [(x, z_{1,p}, z_{1,m}), (y^0, z_{2,p}^0, z_{2,m}^0), (y^1, z_{2,p}^1, z_{2,m}^1)]
\]

is implementable under the proposal \( \langle \mathcal{P}, ((\rho_1, \rho_2), \tau) \rangle \) if it is the equilibrium outcome of a simple equilibrium consistent with the given proposal and the parameters \( ((\rho_1, \rho_2), \tau) \) with respect to (19). An outcome is implementable with CMP if it is implementable under some proposal \( \langle \mathcal{P}, ((\rho_1, \rho_2), \tau) \rangle \). Note that if an outcome is implementable, it is implementable with CMP.
The following theorem shows that, at least for symmetric allocations, contractionary monetary policies do not expand the set of implementable outcomes than expansionary ones, and, in general, it shrinks.

**Theorem 5.5 (Contractionary Monetary Policy).** Suppose that \( \ell = 1 \). A symmetric allocation, \( L = [(x, z_1), (y, z_2)] \), that satisfies \( u(x) \geq z_1 \) and \( v(y) \geq z_2 \) is implementable with CMP if and only if either it is implementable with constant money supply, or it satisfies (18), and
\[
\frac{1}{(1 + r)\sigma_2}\{-rz_1 + \sigma_1[u(x) - z_1]\} + \frac{1}{\sigma_2 + r}\{-rz_2 + \sigma_2[v(y) - z_2]\} \geq 0, \tag{20}
\]
\[
-rz_1 + \sigma_1[u(x) - z_1] < 0. \tag{21}
\]
Moreover, \( L \) is implementable with CMP under PC only if (20) holds with strict inequality whenever \( \sigma_1 < 1 \).

### 6 Optimal Allocation and Monetary Policies

The mechanism designer’s problem is to choose an optimal mechanism that maximizes the social welfare, subject to the implementability constraints. For a given allocation, \( L = [(x, y^0, y^1), (z_1, z_2^0, z_2^1)] \), its welfare is given by
\[
W(L) = \frac{1}{r} \{\sigma_1[u(x) - c(x)] + \sigma_2\sigma_1[v(y^1) - c(y^1)] + \sigma_2(1 - \sigma_1)[v(y^0) - c(y^0)]\}. \tag{22}
\]
Here we focus on the case where \( \ell = 1 \).

#### 6.1 Optimal allocation: \( \sigma_1 = \sigma_2 = 1 \)

As mentioned in the introduction, we are interested in whether money and credit are coessential and, when they are, what the optimal monetary policy is like. By essentiality we mean that it is a feature of the optimal mechanism, and its formal definition is given in the following.

**Definition 6.1.** An allocation \( [(x^p, z_1^p), (y^p, z_2^p)] \) is said to be **constrained efficient** if it maximizes the social welfare (22) among all implementable allocations. We say that *both money and credit are essential* if, in order to implement the constrained efficient allocation, we need a proposal with \( C \neq \emptyset \). The expansionary monetary policy is said to be **essential** if, in order to implement the constrained efficient allocation, the EM is necessary, that is, with \( k_i > 0 \) for some \( i \).

By Theorem 5.1 we know that the set of implementable allocations is characterized by (11). However, it turns out that to solve for a constrained efficient allocation, it is with
out loss of generality to give all the surplus to the buyer in equilibrium. The following lemma characterizes the optimal allocation by giving all the surpluses to the buyers.

**Lemma 6.1.** Consider the following optimization problem.

\[
\max_{(x,y)} \left[ u(x) - c(x) + v(y) - c(y) \right]
\]

subject to

\[
u(x) - (1 + r)c(x) + v(y) - (1 + r)c(y) \geq 0.
\]

Its solution exists and is unique, denoted by \((x^p, y^p)\), and can be characterized as follows. Let \(\bar{x} > 0\) and \(\bar{y} > 0\) be such that

\[
u(\bar{x}) - (1 + r)c(\bar{x}) = 0 = v(\bar{y}) - (1 + r)c(\bar{y}).
\]

1. The solution, \((x^p, y^p)\), is equal to \((x^*, y^*)\) if and only if \((x^*, y^*)\) satisfies (24).

2. Suppose that \((x^*, y^*)\) does not satisfy (24).

   - \((x^p, y^p)\) is the unique pair such that
     \[- u'(x^p)/c'(x^p) = v'(y^p)/c'(y^p);\]
     \[- u(x^p) - (1 + r)c(x^p) + v(y^p) - (1 + r)c(y^p) = 0.\]
   - \(u(x^p) - (1 + r)c(x^p) \geq 0\) if and only if \(u'(\bar{x})/c'(\bar{x}) \leq v'(\bar{y})/c'(\bar{y}).\)

Lemma 9.1 gives a simple characterization of the optimal allocation. By Theorem 4.2, if the optimal allocation, \([(x^p, y^p), (c(x^p), c(y^p))]\) is such that

\[
u(x^p) - (1 + r)c(x^p) < 0,
\]

then it is not implementable by money alone. As a result, the EM is essential and both money and credit are essential. The following theorem, based on the results in Lemma 9.1, gives a full characterization of essentiality of EM in terms of the fundamentals.

**Theorem 6.1.** [Essentiality of expansionary monetary policy] Suppose that \(\sigma_1 = \sigma_2 = 1\) and that \(\ell = 1\).

1. Suppose that \((x^*, y^*)\) satisfies (24). EM is essential if and only if

   \[
u(x^*) - (1 + r)c(x^*) < 0.
   \]

2. Suppose that \((x^*, y^*)\) does not satisfy (24). EM is essential if and only if

   \[
u'(\bar{x})/c'(\bar{x}) > v'(\bar{y})/c'(\bar{y}),
   \]

where \((\bar{x}, \bar{y})\) is defined by (25).

\(^7\)However, the optimal mechanism may not give all the surplus to the buyer is the buyer does not have equilibrium amount of real balances.
Here we also consider PC-implementability. We modify the definitions of essentiality to accommodate PC as follows.

**Definition 6.2.** An allocation \([\langle x^p, z^p_1 \rangle, \langle y^p, z^p_2 \rangle]\) is said to be constrained efficient under PC if it maximizes the social welfare (22) among all PC-implementable allocations. We say that both money and credit are PC-essential if, in order to implement the constrained efficient allocation under PC, we need a proposal with \(C \neq \emptyset\). The EM is said to be PC-essential if, in order to implement the constrained efficient allocation under PC, the EM is necessary, that is, with \(k_i > 0\) for some \(i\).

The pairwise-core requirement is not a binding constraint if the first best allocation is implementable. Indeed, in that case, the agents cannot improve the gain from trade relative to the first-best. Because the first-best is implementable when \(r\) is sufficiently small, by continuity it also implies that the pairwise-core requirement is not binding when \(r\) is not too large. However, for large \(r\)'s, or, equivalently, for low discount factors, the PC requirement can be binding. However, the following theorem gives a sufficient condition under which this does not happen even for low discount factors.

**Theorem 6.2.** [Essentiality of expansionary monetary policy under PC] Suppose that \(\sigma_1 = \sigma_2 = 1\) and that \(\ell = 1\).

1. Suppose that \((x^*, y^*)\) satisfies (24). EM is essential under PC if
   \[u(x^*) - (1 + r)c(x^*) < 0.\]  
2. Suppose that \((x^*, y^*)\) does not satisfy (24).
   - (a) There exists a lower bound \(r_0\) such that for all \(r \geq r_0\), EM is essential under PC if and only if (26) holds.
   - (b) Suppose that \(v(x) \geq u(x)\) for all \(x\) and \(c_1(x) \geq c_2(x)\) for all \(x\). Then, EM is essential under PC if (26) and (27) hold.

The following corollary follows immediately from Theorem 4.2.

**Corollary 6.1.** Suppose that \(\sigma_1 = \sigma_2 = 1\) and that \(\ell = 1\). Then, both money and credit are essential (under PC) if and only if EM is essential (under PC).

Theorem 6.1 and Theorem 6.2 demonstrate that for a large set of parameters the EM is essential and the optimal monetary policy requires a strictly positive inflation rate. Moreover, Corollary 6.1 shows that these are the exact situations where money and credit are both essential.

We end the section with an example where the sufficient conditions in Theorem 6.2 are satisfied.
Example 6.1. Suppose that $u(x) = v(x)$ and that $c_1(x) = x$ and $c_2(x) = \max\{x - a, 0\}$ for all $x \in \mathbb{R}_+$. Assume that $x^* > a$. Then, EM is essential (under PC) if and only if $r > r_0$, where $r_0$ solves

$$u(x^*) - (1 + r)c(x^*) = 0.$$ 

6.2 Optimal allocations: the general case

To characterize the constrained efficient outcomes in the general case is much more difficult. The difficulties come from the asymmetric allocations in round-2 DM meetings. However, we are still able to fully characterize essentiality of EM when the first-best is implementable. In fact, we show that EM is always essential except for very high discount factors in this case. When the first-best is not implementable things are more complicated, and we are only able to give sufficient conditions for essentiality of EM.

6.2.1 First-best allocations

Here we consider the general case where the only assumption is that $\sigma_1 > 0$ and $\sigma_2 > 0$.

To verify whether the first best allocation is implementable only requires checking the inequalities given by the incentive compatibility constraints. These give rise to the following thresholds:

$$r_0 = \frac{\sigma_1[u(x^*) - c(x^*)]}{c(x^*)};$$
$$r_1 = \frac{\sigma_1[u(x^*) - c(x^*)] + \sigma_2[v(y^*) - c(y^*)]}{c(x^*) + c(y^*)};$$
$$r_2 = \frac{\sigma_1[u(x^*) - c(x^*)] + \sigma_2[v(y^*) - c(y^*)]}{\sigma_1 c(x^*) + c(y^*)}.$$

It is easy to verify that $r_0 \leq r_1$ if and only if

$$\frac{\sigma_1[u(x^*) - c(x^*)]}{c(x^*)} \leq \frac{\sigma_2[v(y^*) - c(y^*)]}{c(y^*)}. $$

Moreover, $r_1 \leq r_2$, and $r_1 < r_2$ if and only if $\sigma_1 < 1$. Given these thresholds, we can characterize the essentiality of EM as follows.

Theorem 6.3. Suppose that $\ell = 1$.

1. Suppose that $\sigma_1 = 1$ and that $\sigma_2 < 1$. If

$$\frac{\sigma_1[u(x^*) - c(x^*)]}{c(x^*)} \neq \frac{\sigma_2[v(y^*) - c(y^*)]}{c(y^*)},$$

then there exists $\bar{r} > \bar{r} \equiv \min\{r_0, r_1\}$ such that for all $r \in (\bar{r}, \bar{r}]$, the first-best is implementable and EM is essential (under PC).

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2. Suppose that $\sigma_1 < 1$. There exists $\bar{r} \geq r_2 > \bar{r}$ such that for all $r \in (\bar{r}, \bar{r}]$, the first-best is implementable and EM is essential (under PC).

Theorem 6.3 (1) shows that, when $\sigma_1 = 1$, EM is essential generically whenever the first-best is implementable as long as the first-best is implementable, except for very high discount factors. Theorem 6.3 (2) shows that the genericity condition can be removed for $\sigma_1 < 1$. The following corollary expands the set of discount factors for which the EM is essential downward.

**Corollary 6.2.** Suppose that $\ell = 1$ and that $\sigma_1 < 1$. Let $r^*$ be the highest $r$ such that the first-best is implementable for all $r \in (0, r^*)$. Then, there exists $\hat{r} > r^*$ such that for all $r \in (\hat{r}, \bar{r})$, EM is essential (under PC).

### 6.2.2 Low discount factors

In this section we ignore the PC requirement. However, the sufficient condition for PC implementation given in Theorem 5.3 is still valid.

**Theorem 6.4.** Suppose that $\ell = 1$ and that $\sigma_2 < 1$. Suppose that the first-best is not implementable with EM, i.e., $r > r^*$.

1. Suppose that $\sigma_1 = 1$. Then, EM is essential if

$$\frac{u'(\bar{x}) - c'(\bar{x})}{u'(\bar{x}) - (1 + r)c'(\bar{x})} \neq \frac{\sigma_2[u'(\bar{y}) - c'(\bar{y})]}{\sigma_2 u'(\bar{y}) - (\sigma_2 + r)c'(\bar{y})}. \quad (28)$$

2. Suppose that (28) holds. Then, there exists $\bar{\sigma}_1 < 1$ (which may depend on $r$) such that if $\sigma_1 > \bar{\sigma}_1$, then EM is essential.

### 7 Literature review

A relatively large number of papers examine the coexistence between credit and money in environments based on Lagos and Wright (2005) (LW). In these environments, trade alternates between a goods market in which standard frictions (e.g., anonymity, absence of record keeping, lack of commitment) render money essential to conduct some transactions; and a Walrasian market whose main function is inducing a degenerate distribution of money at the beginning of the following goods market. In what follows we restrict attention to contributions to this literature which assume that policy interventions are consistent with the view that all trade, including trade between agents and the policy maker, must be voluntary and satisfy incentive compatibility. This excludes, for instance, deflation schemes which are financed by compulsory lump sum taxes which contract the money supply.
Berentsen, Camera and Waller (2007) (BCW) are among the first to introduce credit in the LW set up. They do so by adding banks, i.e., agents which transfer money from sellers to buyers at the beginning of the goods market.\(^8\) They obtain that money and credit (modeled as financial intermediation by banks) are coessential. Intuitively, by offering a positive interest rate to agents with idle cash balances, banks reduce the opportunity cost of holding those balances. They also obtain that coessentiality requires a positive rate of inflation, as positive inflation reinforces the incentives of borrowers to repay their debts to banks if failure to do so is permanent exclusion from the credit market. A key element of BCW is the assumption that banks keep financial records which allows them to pay interest to agents who deposited money in their vaults. Andolfatto (2010) builds on this idea and shows that, if the money issuer (government) itself is able to intervene in the economy and pay interest directly to all money holders, there is no need for banks or any other form of credit.\(^9\) Precisely, the government can inject money and equalize the real rate of return on money to the rate of time preference. This way, it implements the Friedman rule with no violation of incentive compatibility.\(^10\)

There are key differences between our environment and that of BCW. First, BCW (and Andolfatto (2010), for that matter) assume centralized trade in the goods market, while we assume pairwise meetings. Hu, Kennan and Wallace (2009) (HKW) show that, with pairwise meetings and no restrictions on the class of trading mechanisms, the first best can be implemented with a constant money supply if agents are patient enough. Thus, the coessentiality result of BCW does not extend to pairwise meetings, as in the original LW set up. Second, in BCW credit is modeled as an interaction between agents and banks, so money must be used in all transactions among agents in the goods market. In our model there are no banks and credit relations are established among agents in the goods market. There are also key differences between our results and those of BCW. First, banks in BCW improve welfare but, in contrast to our paper, the first best can never be achieved. Second, although we share with BCW the result that money and credit are coessential if and only if there is inflation, the mechanism is quite different. In BCW inflation is good because it makes money an inferior outside option, thus reducing the incentives to default on debt. In our case, inflation is a feasible way to buy the private debt incurred by consumers in credit meetings.

A number of papers attempt to examine the coessentiality between money and credit in versions of the LW environment with pairwise meetings in the goods market. A recent example is Gomis-Porqueras and Sanches (2013). Following Sanches and Williamson

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\(^8\) He, Huang, and Wright (2005) is an earlier contribution which also introduce banks in a LW type of environment. In their model, banks issue notes which circulate in the goods market and are safer than money as a medium of exchange. Their focus though is not on the coessentiality of money and credit.

\(^9\) The assumption that the money issuer is able to distribute money according to various schemes is common in monetary models. For example, Wallace (2013) uses distribution schemes which are proportional to money holdings to show that money creation can be welfare improving in a large class of environments. As said in the introduction, we think that one should be precise as to the feasibility of such distribution schemes.

\(^10\) Since Andolfatto (2010) follows BCW and considers a version of LW with a centralized goods market, one can interpret his result as providing a rationale for the existence of banks in environments in which trade is centralized and a direct policy of paying interest to money holders is not feasible.
(2010), Gomis-Porqueras and Sanches (2013) add credit to the LW set up by assuming that a subset of sellers has access to a record-keeping technology which allows the public observation of the buyer’s identity and of his promise of future payment to the seller. Under the assumption that buyers make take it or leave it offers to sellers, they obtain that money and credit are coessential and that coessentiality requires a positive rate of inflation. This result is similar to the one in BCW and the intuition is also similar: the lower rate of return on money implied by a positive inflation reduces the incentive to default if default is punished with exclusion from the credit system. The coessentiality result of Gomis-Porqueras and Sanches (2013) arises under a particular trading protocol. As said above, if one allows for general trading mechanisms, HKW show that credit is irrelevant if agents are patient enough. Our results on the coessentiality of money and credit emerges under an optimal trading mechanism, which bears some resemblance to the one in HKW.

The coessentiality of money and credit is difficult to achieve even if one assumes that money does not work very efficiently by considering a class of trading mechanisms which does not include the optimal mechanism considered by HKW. For instance, Gu, Mattesini and Wright (2013) consider a fairly general version of LW with pairwise meetings and show that money and credit are not coessential. In particular, they model credit as an upper bound on how much a buyer can borrow in any given pairwise meeting and obtain that, if credit is limited, money alone can achieve any allocation that money and credit combined achieve. Intuitively, the price of money in the centralized market adjusts to fill in the gap between the debt limit and the amount of liquidity buyers want to hold.

Telyukova and Wright (2008) also examine the coexistence between money and credit. They add a third market to the LW set up, in which credit is possible due to the existence of an exogenous enforcement technology. Since it is assumed that this technology does not work in the goods market, money remains essential. They obtain that consumers want to use credit even though they have cash on hand and credit is costly in terms of interest payments (the so-called credit card debt puzzle). Intuitively, consumers do not want to spend their cash because there is a chance they may need it later in the decentralized market, where credit is not available. Telyukova and Wright (2008) are not focused on the coessentiality of money and credit as they do not consider general trading mechanisms or optimal monetary policies. We share with their paper the assumption that there are two rounds of trade in between the Walrasian market. However, while they exogenously assume that the first round involves credit and the second round involves money, in our paper the decomposition of the two rounds between credit and money is part of the optimal mechanism. Deviatov and Wallace (2009) study optimal monetary policy in an environment where

\[\text{(Sanches and Williamson (2010) also assume take it or leave it offers by buyers and obtain that the coessentiality of money and credit is linked to inflation rates away from the Friedman rule. The overall intuition is similar to BCW and Gomis-Porqueras and Sanches (2013). However, what makes money a bad option and the Friedman rule a non optimal policy is the possibility of money theft.}\]

\[\text{(Zhang and Lotz (2013) also examine the coexistence between money and credit in a version of LW in which credit is introduced by retailers who endogenously invest in a record-keeping technology. As Telyukova and Wright (2008), they are not focused on the coessentiality of money and credit.}\]
credit and money coexist and there are two rounds of interactions between agents in every period, a decentralized and a centralized round. In contrast to the LW set up though, the centralized round is only used to implement monetary policy. They follow Cavalcanti and Wallace (1999) and introduce credit by assuming that the actions of a subset of agents, labeled monitored agents, are common knowledge while the actions of the complementary set are private. They construct an example in which the optimal monetary policy involves loans to monitored agents so as to fund their purchases in the goods market. These loans bear some resemblance to the open market operations we obtain in our model, which is quite remarkable given the innumerable differences between our environments.

8 Conclusion

9 Appendix: Proofs

Proof of Lemma 3.1

The proof is by contradiction. Without loss of generality, let the first DM round consist of non-credit meetings. Assume there exists an implementable allocation with positive production in meetings in the first DM round. Formally, \( o_1(0) = (x, 0) \), with \( x > 0 \). Consider a deviation by a seller in any such meeting, i.e., assume that seller \( s \) announces \( no \). Then, it must be that her continuation payoff after saying \( no \) is smaller than her continuation payoff after saying \( yes \). Now, since sellers have no record, the impact from saying \( no \) must come from some future action that her partner, say buyer \( b \), takes. There are two potential channels through which an action by buyer \( b \) could lead to an eventual impact on seller \( s \). The first channel consists on a contagion process, through which buyer \( b \) and all his direct and indirect future partners choose actions in the DM rounds which communicate to their ongoing partners that a deviation from the equilibrium path took place. If seller \( s \) is patient and anticipates that this contagion could eventually reach her, it could prevent the initial deviation. The problem with this channel is that, since there is a continuum of buyers and sellers, there is a zero probability that the contagion eventually reaches the initial deviator. The second channel consists on actions taken in the CM round by buyer \( b \) and all his direct and indirect future partners which would induce a noticeable change in the price of money and, as a result, trigger a collective change of action which would affect seller \( s \). The problem with this channel is that, since there is a continuum of agents and at most a countable number of agents which would choose such actions, there would be no impact on the price of money.

\[ \text{Since the centralized round is not intended to produce a degenerate distribution of money, for tractability, they restrict money holdings at the beginning of the first round to lie in the set \{0,1\} and money holdings at the end of the first round to lie in the set \{0,1,2\}.} \]

\[ \text{In fact, the same result applies if the population is finite but sufficiently large. The argument of the proof is essentially the same as in Kandori (1992, page 78).} \]

\[ \text{If the population is finite and there is a deterministic map from actions to prices in the CM, buyer \( b \) can communicate the deviation of seller \( s \) by choosing an action which produces a noticeable change in the} \]
Proof of Theorem 4.1

The necessity of (2) comes from the fact that, if it does not hold, the buyer is better off not participating in any trade. The necessity of (3) follows from the buyer’s decision to respond with yes at round 1. The left-hand side gives his payoff by responding yes and the right-hand side gives his payoff by responding with no. Indeed, if he says no, both his record and his money holdings coincide with those of a buyer who did not meet a seller, and his expected payoff in the second round must be \( \sigma_2 [v(y^0) - z^0_2] \). The necessity of (4) is to ensure that sellers are willing to participate in trade.

Now we turn to sufficiency. Consider the following proposal: \( C = \{1,2\} \), \( R = \{g,b\} \), \( \omega(\emptyset) = g \), \( \omega(h,g) = g \) if \( h_3 \) indicates repayment and \( \omega(h_3,g) = b \) if \( h_3 \) indicates no repayment on either debt, \( \omega(h,b) = b \) for all \( h \); \( o_1(m,g) = (x,z_1,0) \) and \( o_1(m,b) = (0,0,0) \) for all \( m \); \( o_2(m,g,h_1) = (y^0,z^0_2,0) \) if \( h_1 = \emptyset \) and \( o_2(m,g,h_1) = (y^1,z^1_2,0) \) if \( h_1 \neq \emptyset \) for all \( m \), \( o_2(m,b,h_1) = (0,0,0) \) for all \( m \) and for all \( h_1; \phi = 0 \). Given the proposal, the strategies that both buyers and sellers always respond with yes and that buyers always repay their debt constitute an equilibrium. Indeed, (3) and (4) guarantee the yes responses, and (2) guarantees repayment being optimal. As the price for money is zero, we assume that buyers hold \( M \) units of money forever. □

Proof of Theorem 4.2

Let \( \sigma_1 < 1 \). We start with necessity. First, it is clear that the use of money does not help relaxing (2), (3), and (4). Thus, we only need to check (6) and (7). If condition (6) does not hold, participation in the first DM round leads to a relatively higher payoff in the second DM round. If the first DM round is not a credit meeting, a buyer who brought real balances consistent with behavior on the equilibrium path can always pretend to have participated in a meeting in the first DM round by hiding real balances. Thus, the first DM round must be a credit meeting if (6) is to be relaxed. Since \( \ell < 2 \), this means that the second DM round must be a non-credit meeting, in which case the history \( h_1 \) of the first DM round cannot be observed. This implies that information about the first DM round can only be conveyed by the real balances of the buyer. Now, buyers who have met sellers in the first DM round always end up with a weakly lower real balances at the beginning of the second DM round. Thus, a buyer who did not meet a seller can always mimic the real balances of a buyer who has met a seller, which implies that (6) must hold for buyers to behave truthfully.

We now consider condition (7). Let \( \tilde{z} \geq \bar{z} \) be the equilibrium real balances that buyers have to hold across periods. If \( \ell = 0 \), the buyer may hold only \( \tilde{z} - z_1 \) real balances into the DM trades and still participate in the second round of DM trade. He has no incentives price of money in the CM. However, if the population is large and prices are noisy, it can be shown that the expected amount of time it takes for a defection in the decentralized market to affect prices in the centralized market in a substantial way increases to infinity as the population size grows. Thus, when the population is large, the second channel is also not effective in disciplining the behavior of seller \( s \). The argument of the proof is essentially the same as in Araujo et al. (2012, page 618).
to do so if and only if

\[-rz_1 + \sigma_1[u(x) - z_1] + (1 - \sigma_1)\sigma_2 \left\{ [v(y^0) - z_2^0] - [v(y^1) - z_2^1] \right\} \geq 0,
\]

which is identical to (7). If \( C = \{2\} \), then it must be the case that \( o_1(\bar{z}) = (x, 0, z_1) \); hence, for buyers to hold \( \bar{z} \) real balances at the beginning of a period, as opposed to \( \bar{z} - z_1 \), it must be the case that

\[-rz_1 + \sigma_1[u(x) - z_1] \geq 0,
\]

which, together with (6), implies (7). Finally, let \( C = \{1\} \). The minimum promise to pay in the first DM round is given by \( \bar{z}_{1,p} = \max\{z_1 - (\bar{z} - z_{1,m})^+, 0\} \), to ensure that he could pay \( z_{1,m}^1 \) real balances in the second DM round. Note that if \( \bar{z}_{1,p} = 0 \), then only money is used and the mechanism is identical to those with \( \ell = 0 \), and so we assume that \( \bar{z}_{1,p} > 0 \). The buyer can choose between repaying his debt and participating in both DM rounds and not repaying his debt and only bringing \( \bar{z} - z_{1,m} \) real balances to participate in the second DM round. He does not make such a deviation if and only if

\[-rz_1 + \sigma_1[u(x) - z_1] + (1 - \sigma_1)\sigma_2 \left\{ [v(y^0) - z_2^0] - [v(y^1) - z_2^1] \right\} \geq 0,
\]

which is identical to (7).

We need to check sufficiency. Since we can always ignore records, it suffices to show the case with \( \ell = 0 \). Consider the following proposal: \( M\phi = \bar{z}, C = \emptyset; o_1(m) = (x, z_1) \) if \( m \geq \bar{z} \) and \( o_1(m) = (0, 0) \) otherwise; \( o_2(m) = (y^0, z_2^0) \) if \( m \geq \bar{z}, o_2(m) = (y^1, z_2^1) \) if \( \bar{z} > m \geq \bar{z} - z_1 \), and \( o_2(m) = (0, 0) \) otherwise. Condition (2) ensures that the buyer wants to bring \( \bar{z} \) real balances as opposed to zero real balances. Condition (3) ensures that the buyer wants to say yes in the first DM round. Condition (4) ensures that the buyer wants to say yes in the second DM round and that sellers always want to say yes. Condition (6) ensures that buyers who did not meet sellers in the first DM round announce their money holdings truthfully, i.e., do not hide money. Finally, (7) ensures that the buyer does not want to hold \( \bar{z} - z_1 \) real balances as opposed to \( \bar{z} \) real balances at the beginning of the period. Given \( o_1 \) and \( o_2 \), this is the only deviation to a non-zero real balances that we need to check.

Let now \( \sigma_1 = 1 \). We start with necessity. As in the case of \( \sigma_1 = 1 \), irrespective of \( \ell \in \{0, 1\} \), we need to ensure that buyers are better off participating in DM trades, which corresponds to condition (2). We also need to make sure that sellers are better off participating in trades, i.e., (5) is necessary. Finally, we need to ensure that buyers are willing to participate in the first round of DM trade, i.e., (9) must hold. The need for the latter condition is clear if \( \ell = 0 \) or \( C = \{2\} \) since, in this case, the first round of DM trade is a non-credit meeting. It is less clear though if \( C = \{1\} \). In this case, the buyer can be punished with a bad record if he chooses not to participate in the first round of DM trade. However, if (7) does not hold, the buyer would prefer to keep a bad record and only participate in the money meeting in the second round of DM trade. Lastly, we need to check sufficiency. As in the case of \( \sigma_1 < 1 \), we can always ignore records, so it suffices to show the case with \( \ell = 0 \). Consider the following proposal: \( M\phi = \bar{z}, C = \emptyset; o_1(m) = (x, z_1) \) if \( m \geq \bar{z} \) and \( o_1(m) = (0, 0) \) otherwise; \( o_2(m) = (y^1, z_2^1) \) if \( m \geq \bar{z} - z_1 \), and \( o_2(m) = (0, 0) \) otherwise. Condition (2) ensures that the buyer wants to bring \( \bar{z} \) real
balances as opposed to zero real balances. Condition (4) ensures that sellers always want to say \textit{yes}. Finally, (7) ensures that the buyer does not want to hold $\tilde{z} - z_1$ real balances as opposed to $\tilde{z}$ real balances at the beginning of the period. Given $o_1$ and $o_2$, this is the only deviation to a non-zero real balances that we need to check. □

**Proof of Theorem 5.1**

First we prove necessity. Condition (11) is necessary as the buyer would otherwise decline to participate in the whole scheme. Condition (12) is necessary because otherwise some sellers would decline the trade.

Now we prove sufficiency. Suppose that $-rz_1 + [u(x) - z_1] \geq 0$. By Theorem 4.2, we can implement $\mathcal{L}$. So, suppose that $-rz_1 + [u(x) - z_1] < 0$, and consider the following proposal: $\phi_t, M_t = z_2$ for each $t$, $C = \{1\}$, $R = \{g, b\}$, and $\omega(\emptyset) = g$, $\omega(h, g) = g$ if $h_3$ indicates repayment and $\omega(h, g) = b$ if $h_3$ indicates no repayment on either debt, $\omega(h, b) = b$ for all $h$; $o_1(., b) = (x, z_1, 0)$ and $o_1(., b) = (0, 0, 0)$; $o_2(m) = (y, z_2)$ if $m \geq z_2$ and $o_2(m) = (0, 0)$ otherwise.

The OMO is defined as follows. Let

$$k = \frac{1}{1 + r} \{rz_1 - [u(x) - z_1]\} = z_1 - \frac{1}{1 + r} u(x) \in (0, z_1),$$

and let $\pi = k / z_2$.

The equilibrium strategies are as follows. All agents always respond with \textit{yes} to the proposed trades. Buyers always repay their debts when the debts are below $z_1 - k$ (which is a post-OMO amount) and when their records are $g$; otherwise, they renegade on their debts. Buyers always acquire $z_2$ real balances.

First we show that the buyers, conditional on having records $g$, are willing to repay their debt $d \leq z_1 - k$. The buyers are willing to repay $d$ if and only if

$$-d + \frac{\delta}{1 - \delta} [u(x) - (z_1 - k)] \geq 0,$$

that is,

$$d \leq \frac{1}{r} [u(x) - (z_1 - k)] = \frac{1}{r} \left\{ u(x) - z_1 + \frac{1}{1 + r} [(1 + r)z_1 - u(x)] \right\} = \frac{1}{1 + r} u(x) = z_1 - k.$$

We now show that, independent of his past histories and records, the buyer has incentive to carry $z_2$ units of real balances across periods. This is true if

$$-(1 + \pi)z_2 + \delta v(y) \geq 0,$$

which can be rewritten as

$$-rz_2 + [v(y) - z_2] \geq (1 + r)\pi z_2 = (1 + r)k = rz_1 - [u(x) - z_1],$$

(30)
which holds by (11).

Finally we verify that the buyer is willing to participate the whole scheme. This will be the case if and only if

\[-r(z_1 - k) - r(1 + \pi)z_2 + \{[u(x) - (z_1 - k)] + v(y) - (1 + \pi)z_2\} \geq 0,\]

which, using \(k = \pi z_2\), can be rewritten as

\[-r(z_1 + z_2) + [u(x) - z_1] + [v(y) - z_2] \geq 0,\]

which is equivalent to (11). \(\Box\)

**Proof of Theorem 5.2**

(A) Suppose that \(-rz^p_2 + [u(x^p) - z^p_1] < 0\). Consider OMO as defined in (29), and consider the following proposal: the real balances buyers have to hold across periods is \(\phi_t M_t = z^p_t\) for all \(t \in \mathbb{N}, C = \{1\}\), \(R = \{g, b\}\), and \(\omega(\emptyset) = g\), \(\omega(h_3, g) = g\) if \(h_3\) indicates full repayment and \(\omega(h_3, b) = b\) if \(h_3\) indicates no full repayment on either debt, \(\omega(h_3, b) = b\) for all \(h_3\). Let \(\bar{d} = z^p_1 - k\). It remains to define the terms of trade in the DM stages.

(i) First we define \(o_2(m)\)

(a) Suppose that \(m \geq z^p_2\). Then

\[o_2(m) = \arg\max_{(y,z_m):z_m \leq m} -c(y) + z_m \tag{31}\]

subject to

\[v(y) - z_m \geq v(y^p) - z^p_2.\]

(b) Suppose that \(m < z^p_2\). Then

\[o_2(m) = \arg\max_{(y,z_m):z_m \leq m} -c(y) + z_m \tag{32}\]

subject to

\[v(y) - z_m \geq 0.\]

The solutions to (31) and (32) exist and are unique. Moreover, \(o_2(z^p_2) = (y^p, z^p_2)\) iff \(y^p \leq y^*\) (INSERT ARGUMENT HERE). Let \(V_2(m, r, \hat{h}_1)\) be the value function of a buyer who enters the second DM round with \(m\) units of money, a record \(r\) and a first DM round history \(\hat{h}_1\). ADD FOOTNOTE ABOUT \(H HAT\). In turn, let \(W(0, r, \hat{h}_1)\) be the value function of entering the DM round with zero units of money, a record \(r\) and a first DM round history \(\hat{h}_1\). Given \(o_2(m)\), we have

\[V_2(m, r, \hat{h}_1) = v(y^p) + m - z^p_2 + W(0, r, \hat{h}_1) \text{ if } m \geq z^p_2,\]

\[V_2(m, r, \hat{h}_1) = m + W(0, r, \hat{h}_1) \text{ if } m < z^p_2.\]
(ii) We now define $o_1(m, r)$

(a) Suppose that $r = g$. Then

$$o_1(m, g) = \arg \max_{(x, z_p, z_m), z_p \leq d + k, z_m \leq m} -c(x) + z_p + z_m$$

subject to

$$u(x) + V_2[m - z_m, g, (x, z_p, z_m)] \geq u(x^{pc}) + V_2[m, g, (x^{pc}, z_1^{pc}, 0)].$$

(b) Suppose that $r = b$. Then

$$\max_{(x, 0, z_m), z_m \leq m} -c(x) + z_m$$

subject to

$$u(x) + V_2(m - z_m, b, 0) \geq V_2(m, b, 0).$$

Claim. The solutions to (33) and (34) exist, and for all the solutions the constraints on the buyer’s reservation utilities are binding. We CAN CHOOSE $o_1(z_2^{pc}, g) = (x^{pc}, z_1^{pc}, 0)$. Let $V_1(m, r)$ be the value function of a buyer who enters the first DM round with $m$ units of money and a record $r$. Given $o_1(m, r)$, we have

$$V_1(m, g) = u(x^{pc}) - z_1^{pc} + v(y^{pc}) - z_2^{pc} + m + W[0, g, (x^{pc}, z_1^{pc}, 0)] \text{ if } m \geq z_2^{pc},$$

$$V_1(m, g) = u(x^{pc}) - z_1^{pc} + m + W[0, g, (x^{pc}, z_1^{pc}, 0)] \text{ if } m < z_2^{pc},$$

$$V_1(m, b) = v(y^{pc}) - z_2^{pc} + m + W(0, b, 0) \text{ if } m \geq z_2^{pc},$$

$$V_1(m, b) = m + W[0, b, 0] \text{ if } m < z_2^{pc}.$$

Proof. (a) We start with the solutions to (33). We consider two cases: $m \geq z_2^{pc}$ and $m < z_2^{pc}$.

(a.1) $m \geq z_2^{pc}$. First it is clear that, in any solution, $z_p \geq k$, i.e., the buyer is not going to incur a debt lower than what the mechanism designer is willing to pay. If he does so, he can increase the debt without incurring any additional cost. Since the seller will be gaining more, she will be willing to produce an additional amount of the DM good which will increase the utility of the buyer. Since $z_p \geq k$, we can rewrite (33) as

$$o_1(m, g) = \arg \max_{(x, z_p, z_m), k \leq z_p \leq d+k, z_m \leq m} -c(x) + z_p + z_m$$

subject to

$$u(x) - z_m - (z_p - k) + 1_{m-z_m \geq z_2^{pc}}[v(y^{pc}) - z_2^{pc}] \geq u(x^{pc}) - (z_1^{pc} - k) + v(y^{pc}) - z_2^{pc}.$$

The solution to this problem exists. Moreover, at the optimum, $u(x) - z_m - (z_p - k) + 1_{m-z_m \geq z_2^{pc}}[v(y^{pc}) - z_2^{pc}] = u(x^{pc}) - (z_1^{pc} - k) + v(y^{pc}) - z_2^{pc}$ for otherwise, we may decrease $x$ and increase the utility of the seller. We may choose any solution.

Now, consider $m = z_2^{pc}$. Here we show that $o_1(z_2^{pc}, g) = (x^{pc}, z_1^{pc}, 0)$ iff $x^{pc} \leq x^*$ (INSERT ARGUMENT HERE). Suppose, by contradiction, that $(x', z'_p, z'_m)$ satisfies the constraints in (35) and gives a strictly higher utility to the seller. Then

$$-c(x') + z'_p + z'_m > -c(x^{pc}) + z_1^{pc}$$

$$u(x') - z'_m - (z'_p - k) \geq u(x^{pc}) - (z_1^{pc} - k) + v(y^{pc}) - z_2^{pc},$$

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which implies that

\[ u(x') - c(x') > u(x^{pc}) - c(x^{pc}) + v(y^{pc}) - z_{2}^{pc} \geq u(\bar{x}) - c(\bar{x}). \]

However \( c(x') < z_{1}^{pc} - z_{2}^{pc} \), Thus, this inequality cannot be satisfied since \( \bar{x} = \min\{x^*, c^{-1}(z_{1}^{pc} + z_{2}^{pc})\} \).

(a.2) \( m < z_{2}^{pc} \). Using a reasoning similar to the one in (a.1), we can rewrite (33) as

\[
o_1(m, g) = \arg \max_{(x,z_p,z_m),k \leq z_p \leq \bar{d}+k,z_m \leq m} -c(x) + z_p + z_m \tag{36}
\]

subject to

\[ u(x) - z_m - (z_p - k) \geq u(x^{pc}) - (z_{1}^{pc} - k). \]

This problem has a unique solution.

(b) We now consider the solutions to (34). Using a reasoning similar to the one used in (a), the solution is characterized as follows. If \( m \geq z_{2}^{pc} \), then the solution solves (we always choose \( z_m \) to be positive whenever possible)

\[
o_1(m, b) = \arg \max_{(x,0,z_m),z_m \leq m} -c(x) + z_m \tag{37}
\]

subject to

\[ u(x) - z_m + 1_{m-z_m \geq z_{2}^{pc}}[v(y^{pc}) - z_{2}^{pc}] \geq v(y^{pc}) - z_{2}^{pc}. \]

Otherwise, it solves

\[
o_1(m, b) = \arg \max_{(x,0,z_m),z_m \leq m} -c(x) + z_m \tag{38}
\]

subject to

\[ u(x) - z_m \geq 0. \]

In either case, a solution exists and the value function \( V_1(m, b) \) is as given in the claim. □

We now specify the equilibrium strategies. All agents always respond with yes to the proposed trades, on both equilibrium and off-equilibrium paths. The buyers always repay their debts when their records are \( g \) and the debt (after the government purchase) is below \( \bar{d} \), and they never repay otherwise. The buyers always acquire \( z_{2}^{pc} \) real balances in the CM following any history.

First we show that buyers, conditional on having records \( g \), are willing to repay their debt \( d \leq \bar{d} = \delta_1^{pc} - k \). This is the case if and only if

\[ -d + \frac{\delta}{1-\delta}[u(x^{pc}) - (\delta_1^{pc} - k)] \geq 0, \]

that is,

\[ d \leq \frac{1}{\delta^r}[u(x^{pc}) - (\delta_1^{pc} - k)] = \frac{1}{\delta^r} \left\{ u(x^{pc}) - \delta_1^{pc} + \frac{1}{1+r}(1+r)\delta_1^{pc} - u(x^{pc}) \right\} = \frac{1}{1+r}u(x^{pc}) = \delta_1^{pc} - k. \]
We now show that, independent of his past histories and records, the buyer has incentive to carry $z_{2}^{pc}$ units of real balances across periods. This is true if

$$-(1 + \pi)z_{2}^{pc} + \delta v(y^p) \geq 0,$$

which can be rewritten as

$$-rz_{2}^{pc} + [v(y^pc) - z_{2}^{pc}] \geq (1 + r)\pi z_{2}^{pc} = (1 + r)k = rz_{1}^{pc} - [u(x^{pc}) - z_{1}^{pc}],$$

which holds by (11).

Finally we verify that buyers are willing to participate in the whole scheme. This will be the case if and only if

$$-r(z_{1}^{pc} - k) - r(1 + \pi)z_{2}^{pc} + \{[u(x^{pc}) - (z_{1}^{pc} - k)] + v(y^{pc}) - (1 + \pi)z_{2}^{pc}\} \geq 0,$$

which, using $k = \pi z_{2}^{pc}$, can be rewritten as

$$-r(z_{1}^{pc} + z_{2}^{pc}) + [u(x^{pc}) - z_{1}^{pc}] + [v(y^{pc}) - z_{2}^{pc}] \geq 0,$$

which is equivalent to (11).

(B) Now suppose that $-rz_{1}^{pc} + [u(x^{pc}) - z_{1}^{pc}] \geq 0$. (INSERT ARGUMENT HERE). \qed

**Proof of Theorem 5.3**

(1) First we prove sufficiency. Suppose that $-rz_{1} + \sigma_{1}[u(x) - z_{1}] \geq 0$. Since $z_{1} > 0$, it must be that $u(x) \geq z_{1}$. Then, by Theorem 4.2, we can implement $C$. So, suppose that $-rz_{1} + \sigma_{1}[u(x) - z_{1}] < 0$, and consider the following proposal: $\phi_{t}M_{t} = z_{1}$ for each $t$, $C = \{1\}$, $R = \{g, b\}$, and $\omega(\emptyset) = g$, $\omega(h, g) = g$ if $h_{3}$ indicates repayment and $\omega(h, g) = b$ if $h_{3}$ indicates no repayment on either debt, $\omega(h, b) = b$ for all $h$; $O_{1}(m, r) = (x, z_{1}, 0)$ if $m \geq z_{2}$ and $r = g$ and $O_{1}(m, r) = (0, 0, 0)$ otherwise; $O_{2}(m) = (y, z_{2})$ if $m \geq z_{2}$ and $O_{2}(m) = (0, 0)$ otherwise. The OMO is defined as follows: $k = z_{1}$ and $\pi z_{2} = \sigma_{1}k$.

The equilibrium strategies are as follows. All agents always respond with *yes* to the proposed trades. Buyers always repay their debts and acquire $z_{2}$ real balances when their records are $g$. Buyers acquire $z_{2}$ real balances when their records are $b$ if

$$-rz_{2} + \sigma_{2}[v(y) - z_{2}] \geq (1 + r)\sigma_{1}z_{1},$$

and acquire zero real balances when their records are $b$ otherwise.

First we show that the buyer is willing to participate the whole scheme. This will be the case if and only if

$$-r(z_{1} - k) - r(1 + \pi)z_{2} + \{\sigma_{1}[u(x) - (z_{1} - k)] + \sigma_{2}v(y) + (1 - \sigma_{2})z_{2} - (1 + \pi)z_{2}\} \geq 0,$$

which, using $\pi z_{2} = \sigma_{1}k$, can be rewritten as

$$-r(z_{1} + z_{2}) + \sigma_{1}[u(x) - z_{1}] + \sigma_{2}[v(y) - z_{2}] + r(1 - \sigma_{1})k \geq 0.$$
Substituting for the value of \( k \) by \( z_1 \), we obtain

\[
\{-r(z_1 + z_2) + \sigma_1[u(x) - z_1] + \sigma_2[v(y) - z_2]\} + r(1 - \sigma_1)z_1 \geq 0,
\]
which is equivalent to (13).

We now show that the buyer has incentive to carry \( z_2 \) units of real balances across periods, conditional on having a record \( g \). This is true as long as

\[-(1 + \pi)z_2 + \delta \{\sigma_1[u(x) - (z_1 - k)] + \sigma_2v(y) + (1 - \sigma_2)z_2\} \geq 0,
\]
which, using \( \pi z_2 = \sigma_1 k = \sigma_1 z_1 \), can be rewritten as

\[-r z_2 + \sigma_1[u(x) - z_1] + \sigma_2[v(y) - z_2] \geq r \pi z_2 = r \sigma_1 z_1, \tag{41}\]
which holds by (13). Clearly, the buyer will not repay his debt if he has a record \( b \). Finally, we show that the buyer will not hold money if he has a record \( b \). If he holds money with a bad record, his utility within the period is

\[-(1 + \pi)z_2 + \delta \{\sigma_2v(y) + (1 - \sigma_2)z_2\},\]
which, using the definition of \( \pi \), can be rewritten as

\[-r z_2 + \sigma_2[v(y) - z_2] - (1 + r)\sigma_1 z_1.\]

Therefore, the buyer is willing to hold real balances \( z_2 \) when he has a record \( b \) if and only if (40) holds.

(2) Now we show PC-implementability. Here we denote the allocation \( \mathcal{L} \) by \([x^p, z^p_1), (y^p, z^p_2)\]. Here we assume that (40) does not hold. Suppose that \(-r z_1^p + \sigma_1[u(x^p) - z_1^p] < 0\). Then, \( z_1^p > 0 \). The OMO is defined as in (1).

Consider the following proposal: the real balances buyers have to hold across periods is \( \phi_t M_t = z^p_2 \) for all \( t \in \mathbb{N}, C = \{1\}, R = \{g, b\}, \) and \( \omega(\emptyset) = g, \omega(h_3, g) = g \) if \( h_3 \) indicates full repayment and \( \omega(h_3, g) = b \) if \( h_3 \) indicates no full repayment on either debt, \( \omega(h_3, b) = b \) for all \( h_3 \).

Given the equilibrium allocations, we have the following Bellman equations for continuation values. Let \( V_1(m, r), V_2(m, r, h_1), \) and \( W(m, r, h_1, h_2) \) denote the continuation value with \( m \) units of real balances, record \( r \), and trading histories in previous stages when entering stage-1 DM, stage-2 DM (before the matches), and the CM. Then, by standard arguments, \( W \) does not depend on \( h_2 \), and

\[
W[m, g, (x, z_p, z_m)] = m - \max\{z_p - z_1^p, 0\} + W[0, g, \emptyset], \tag{42}
\]
\[
V_1(z^p_2, g) = \sigma_1\{u(x^p) + V_2[z^p_2, g, (x^p, z^p_1)]\} + (1 - \sigma_1)V_2(z^p_2, g, \emptyset), \tag{43}
\]
\[
V_2(z^p_2, g, (x^p, z^p_1)) = \sigma_2\{v(y^p) + W[0, g, (x^p, z^p_1)]\} + (1 - \sigma_2)W[z^p_2, g, (x^p, z^p_1), \emptyset]. \tag{44}
\]
Let
\[ d = -(1 + \pi)z_2^p + \frac{1}{r}\{\sigma_1[u(x^p) - z_1^p] + \sigma_2[v(y^p) - z_2^p]\}. \]

Then, \( \bar{d} \geq 0 \); indeed,
\[ \bar{d} = \frac{1}{r}\left\{-r\sigma_1z_1^p - r(1 + \pi)z_2^p + \sigma_1[u(x^p) - z_1^p] + \sigma_2[v(y^p) - z_2^p]\right\} \]
\[ = \frac{1}{r}\{\sigma_1[\dot{r}z_1^p + u(x^p) - z_1^p] + \dot{r}z_2^p + \sigma_2[v(y^p) - z_2^p]\}\geq 0. \]

Note that \( \bar{d} = 0 \) if (13) holds with equality.

Now we are ready to define the terms of trade in the DM stages.

(i) First we define \( o_2 \).

(a) Suppose that \( m \geq z_2^p \). Then, \( o_2(m) \) solves
\[
\max_{(y,z_m): z_m \leq m} -c(y) + z_m \quad \text{subject to} \quad v(y) - z_m \geq v(y^p) - z_2^p. \tag{46}
\]
(b) Suppose that \( m < z_2^p \). Then, \( o_2(m) \) solves
\[
\max_{(y,z_m): z_m \leq m} -c(y) + z_m \quad \text{subject to} \quad v(y) - z_m \geq 0. \tag{47}
\]

The solutions to (46) and (47) exist and are unique. Moreover, when \( m = z_2^p \), \( o_2(z_2^p) = (y^p, z_2^p) \). Given \( o_2 \), we can then compute \( V_2 \) as follows.

\[
V_2[m, g, h_1] = \sigma_2[v(y^p) - z_2^p] + m + W[0, g, h_1] \text{ if } m \geq z_2^p,
\]
\[
V_2[m, g, h_1] = m + W[0, g, h_1] \text{ if } m < z_2^p,
\]
\[
V_2[m, b, \emptyset] = \sigma_2[v(y^p) - z_2^p] + m + W[0, b, \emptyset] \text{ if } m \geq z_2^p,
\]
\[
V_2[m, b, \emptyset] = m + W[0, b, \emptyset] \text{ if } m < z_2^p.
\]

(ii) Here we define \( o_1 \).

(a) Suppose that \( m \geq z_2^p \) and \( r = g \). Then, \( o_1(m, g) \) solves
\[
\max_{(x, z_p, z_m)} -c(x) + z_p + z_m \quad \text{subject to} \quad u(x) + V_2[m - z_m, g, (x, z_p, z_m)] \geq u(x^p) + V_2[m, g, (x^p, z_1^p)],
\]
\[
z_p \leq \bar{d} + z_1^p, z_m \leq m. \tag{48}
\]
(b) Suppose that \( m < z_2^p \) and \( r = g \). Then, \( o_1(m, g) \) solves

\[
\max_{(x,z_p,z_m)} -c(x) + z_p + z_m \tag{49}
\]

subject to

\[
u(x) + V_2[m - z_m, g, (x, z_p, z_m)] \geq V_2[m, g, \emptyset],
\]

\[
d_p \leq d + z_1^p, z_m \leq m.
\]

(c) Suppose that \( r = b \). Then, \( o_1(m, b) \) solves

\[
\max_{(x,0,z_m),z_m \leq m} -c(x) + z_m \tag{50}
\]

subject to

\[
u(x) + V_2[m - z_m, b, \emptyset] \geq V_2[m, b, \emptyset].
\]

**Claim 0.** The solutions to (48), (49), and (50) exist, and for all the solutions the constraints on the buyer’s reservation utilities are binding. When \((m, r) = (z_2^p, g), o_1(m, r) = (x^p, z_1^p, 0)\). Moreover, the value function \( V_1 \) is given as follows.

\[
V_1[m, g] = \sigma_1[u(x^p) - z_1^p] + \sigma_2[v(y^p) - z_2^p] + m + W[0, g, (x^p, z_1^p)] \quad \text{if} \quad m \geq z_2^p,
\]

\[
V_1[m, g] = m + W[0, g, \emptyset] \quad \text{if} \quad m < z_2^p,
\]

\[
V_1[m, b] = \sigma_2[v(y^p) - z_2^p] + m + W[0, b, \emptyset] \quad \text{if} \quad m \geq z_2^p,
\]

\[
V_1[m, b] = m + W[0, b, \emptyset] \quad \text{if} \quad m < z_2^p.
\]

**Proof.** (a) Here we show that the problem is equivalent to

\[
\max_{(x,z_p,z_m)} -c(x) + z_p + z_m \tag{51}
\]

subject to

\[
u(x) - z_m + z_1^p - z_m \geq u(x^p),
\]

\[
z_1^p \leq z_p \leq d + z_1^p, z_m \leq m - z_2^p.
\]

We first show that in any solution, \( z_p \geq z_1^p \). Suppose that \( z_p < z_1^p \). Then, by increasing \( z_p \) by \( \epsilon \), the buyer only has to pay zero but the seller obtains \( \epsilon \); by increasing \( x \) as well we can increase the buyer welfare. Now we show that \( z_m \leq m - z_2^p \). Suppose not and let the solution be \((x', z'_p, z'_m)\). Then,

\[
-c(x') + z'_p + z'_m \geq -c(x^p) + z_1^p,
\]

\[
u(x') - z'_p + z_1^p \geq u(x^p) + \sigma_2[v(y^p) - z_2^p],
\]

which implies that

\[
u(x') - c(x') \geq u(x^p) - c(x^p) + \sigma_2[v(y^p) - z_2^p] > u(x^*) - c(x^*),
\]

a contradiction.
Now we show that \((x^p, z^p_1, 0)\) solves (51) for \(m = z^p_2\). In case (2), \(\bar{d} = 0\) and hence \((x^p, z^p_1, 0)\) is the only feasible solution. Consider case (1). Suppose, by contradiction, that \((x', z_p', 0) \neq (x^p, z^p_1, 0)\) gives a higher value to (51). Then,

\[
-c(x') + z_p' + z_m' > -c(x^*) + z^1_p,
\]

and hence

\[
u(x') - z_p' + z^1_p \geq u(x^*),
\]

a contradiction.

(b) Using similar reasonings as in (a), the solution solves

\[
\max_{(x, z_p, z_m)} -c(x) + z_p + z_m
\]

subject to

\[
u(x) - (z_p - z^p_1) - z_m \geq 0,
\]

\[z^p_1 \leq z_p \leq \bar{d} + z^p_1, z_m \leq m.
\]

This problem has a unique solution.

(c) Using similar reasonings as in (a), the solution is characterized as follows.

If \(m \geq z^p_2\), then the solution solves (we always choose \(z_m\) to be positive whenever possible)

\[
\max_{(x, 0, z_m)} -c(x) + z_m
\]

subject to

\[
u(x) - z_m + 1_{m - z_m \geq z^p_2} \{\sigma_2[v(y^p) - z^p_2]\} \geq \sigma_2[v(y^p) - z^p_2],
\]

\[z_m \leq m.
\]

Otherwise, it solves

\[
\max_{(x, 0, z_m)} -c(x) + z_m
\]

subject to

\[
u(x) - z_m \geq 0,
\]

\[z_m \leq m.
\]

In either case, a solution exists and the value function \(V_1(m, b)\) is given as in the claim.

Now we specify the equilibrium strategies. All agents always respond with yes to the proposed trades, on both equilibrium and off-equilibrium paths. The buyers always repay their debts and acquire \(z_2\) real balances when their records are \(g\) and the debt (after the government purchase) is below \(\bar{d}\), and they never repay nor hold any money anything otherwise.

To verify that the agents are willing to follow equilibrium behavior, use the same arguments as in (1).

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Proof of Theorem 5.4

We first prove sufficiency. Suppose that \(-rz_2 + \sigma_2[v(y) - z_2] \geq 0\). Then, together with (17), (2) holds. By Theorem 4.2, we can implement \(L\). So, suppose that \(-rz_2 + \sigma_2[v(y) - z_2] < 0\). Then, \(-rz_1 + \sigma_1[u(x) - z_1] > 0\).

Consider the following proposal: \(\phi_t M_t = z_1\) for each \(t\), \(C = \{2\}\), \(R = \{g, b\}\), and \(\omega(\emptyset) = g\), \(\omega(h, g) = g\) if \(h_3\) indicates repayment and \(\omega(h, g) = b\) if \(h_3\) indicates no repayment on either debt, \(\omega(h, b) = b\) for all \(h\); \(o_1(m) = (x, z_1, 0)\) if \(m \geq z_1\) and \(o_1(m) = (0, 0, 0)\) otherwise; \(o_2(m, r) = (y, z_2, 0)\) if \(r = g\) and \(o_2(m, r) = (0, 0, 0)\) otherwise.

The OMO is defined as follows. Let

\[
k = \frac{1}{\sigma_2 + r} \left\{ rz_2 - \sigma_2[v(y) - z_2] \right\} = z_2 - \frac{\sigma_2}{\sigma_2 + r} v(y) \in (0, z_2),
\]

(55)

and let \(\pi = \sigma_2 k / z_1\).

The equilibrium strategies are as follows. All agents always respond with yes to the proposed trades. Buyers always repay their debts when the debts are below \(z_2 - k\) (which is a post-OMO amount) and when their records are \(g\); otherwise, they renege on their debts. Buyers always acquire \(z_1\) real balances.

First we show that the buyers, conditional on having records \(g\), are willing to repay their debt \(d \leq z_2 - k\). The buyers are willing to repay \(d\) if and only if

\[
-d + \frac{\delta}{1 - \delta} \sigma_2[v(y) - (z_2 - k)] \geq 0,
\]

that is,

\[
d \leq \frac{1}{r} \sigma_2 [v(y) - (z_2 - k)] = \frac{1}{r} \sigma_2 \left\{ v(y) - z_2 + z_2 - \frac{\sigma_2}{r + \sigma_2} v(y) \right\} = \frac{\sigma_2}{\sigma_2 + r} v(y) = z_2 - k.
\]

We now show that, independent of his past histories and records, the buyer has incentive to carry \(z_2\) units of real balances across periods. This is true if

\[
-(1 + \pi) z_1 + \delta \sigma_1[u(x) - z_1] \geq 0,
\]

which can be rewritten as

\[
-r z_1 + \sigma_1[u(x) - z_1] \geq (1 + r) \pi z_1 = (1 + r) \sigma_2 k = \frac{(1 + r) \sigma_2}{\sigma_2 + r} \{rz_2 - \sigma_2[v(y) - z_2]\},
\]

(56)

which holds by (16).

Finally we verify that the buyer is willing to participate the whole scheme. This will be the case if and only if

\[
-r(z_2 - k) - r(1 + \pi) z_1 + \{ \sigma_1[u(x) - z_1] + \sigma_2[v(y) - (z_2 - k)]\} + z_1 - (1 + \pi) z_1 \geq 0,
\]

which, using \(\sigma_2 k = \pi z_2\), can be rewritten as

\[
-r(z_1 + z_2) + \sigma_1[u(x) - z_1] + \sigma_2[v(y) - z_2] + r(1 - \sigma_2) k \geq 0,
\]

which is equivalent to (16). □
Proof of Theorem 5.5

Suppose that $\mathcal{L}$ is implementable with CMP and with a mechanism with $C = \{1\}$. Then, for a tax rate $\rho$ to be incentive compatible, one has

$$-(1 + \rho)z_1 + \frac{1}{r}\sigma_1 [u(x) - (1 + \rho)z_1] \geq 0. \tag{57}$$

Buyers are willing to carry $z_2$ units of real balances only if

$$-rz_2 + (1 + r)\tau z_2 + \sigma_2[v(y) - z_2] \geq 0. \tag{58}$$

Combining the above two inequalities, and using (19) to obtain $\sigma_1 \rho z_1 = \tau z_2$, we have

$$\frac{(1 + r)\sigma_1}{\sigma_1 + r} \{-r + \sigma_1[u(x) - z_1]\} + \{-z_2 + \sigma_2[v(y) - z_2]\} \geq 0. \tag{59}$$

Because $-z_1 + \sigma_1[u(x) - z_1] \geq 0$ by (57) and $\frac{(1+r)\sigma_1}{\sigma_1 + r} \leq 1$, (59) implies that

$$\{-r + \sigma_1[u(x) - z_1]\} + \{-z_2 + \sigma_2[v(y) - z_2]\} \geq 0. \tag{60}$$

Hence, by (60) and $-z_1 + \sigma_1[u(x) - z_1] \geq 0$, $\mathcal{L}$ is also implementable with constant money supply.

Suppose that (21) holds and hence $\mathcal{L}$ is not implementable with constant money supply. The above argument shows that the mechanism must have $C = \{2\}$. Then, for a tax rate $\rho$ to be incentive compatible, one has

$$-(1 + \rho)z_2 + \frac{1}{r}\sigma_2 [v(y) - (1 + \rho)z_2] \geq 0. \tag{61}$$

Buyers are willing to carry $z_1$ units of real balances only if

$$-rz_1 + (1 + r)\tau z_1 + \sigma_1[u(x) - z_1] \geq 0. \tag{62}$$

Combining the above two inequalities, and using (19) to obtain $\sigma_2 \rho z_2 = \tau z_1$, we have

$$\{-r + \sigma_1[u(x) - z_1]\} + \frac{(1 + r)\sigma_2}{\sigma_2 + r} \{-z_2 + \sigma_2[v(y) - z_2]\} \geq 0. \tag{63}$$

Finally, under the PC-implementation, when $C = \{2\}$ and when $\sigma_1 < 1$, in any stage-2 meetings where buyers still have money will pay with money and pay the rest with promises. As $\sigma_1 < 1$ we have $\sigma_2 (1 - \sigma_1)$ measure of such buyers. Now, by (19), this implies that

$$\sigma_2 \rho z_2 > \sigma_2 \sigma_1 \rho z_2 + \sigma_2 (1 - \sigma_1) \rho \max\{z_2 - z_1, 0\} = \tau z_1.$$

As a result, (63) must hold with strict inequality.
Proof of Lemma 9.1

(2) Suppose that \((x^*, y^*)\) does not satisfy (24). First we show that for any \((x', y')\) that satisfies (24) with \(x > x^*\) or \(y' > y^*\), then \((x'', y'') = (\min\{x', x^*\}, \min\{y', y^*\})\) also satisfies (24) and dominates \((x', y')\). Let \(x_0\) and \(y_0\) be such that

\[
u'(x_0) = (1 + r)c'(x_0) \text{ and } \nu'(y_0) = (1 + r)c'(y_0).
\]

Then, \(x_0 < x^*\) and \(y_0 < y^*\). Thus, if \(x' > x^*\), \(u(x') - (1 + r)c(x') < u(x^*) - (1 + r)c(x^*)\) and similar argument holds for \(y'\). So \((x'', y'')\) also satisfies (24) if \((x', y')\) does.

As a result, we can only look for pairs \((x, y) \in [0, x^*] \times [0, y^*]\) that satisfies (24), which consists a compact convex set. As the objective function is strictly concave, there exists a unique solution and is characterized by the Kuhn-Tucker condition: for some \(\lambda \geq 0\),

\[
u'(x^p) - c'(x^p) + \lambda [\nu'(x^p) - (1 + r)c'(x^p)] = 0, \quad (64)
\]

\[
u'(y^p) - c'(y^p) + \lambda [\nu'(y^p) - (1 + r)c'(y^p)] = 0. \quad (65)
\]

If \(\lambda = 0\), then the FOC’s imply that \((x^p, y^p) = (x^*, y^*)\), a contradiction. Thus, \(\lambda > 0\) and hence (24) is binding at the optimum.

Moreover, if \(\nu'(\bar{x})/c'(\bar{x}) = \nu'(\bar{y})/c'(\bar{y})\), then \((\bar{x}, \bar{y})\) satisfies (24) and the FOC’s with

\[
\lambda = \frac{\nu'(\bar{x}) - c'(\bar{x})}{-\nu'(\bar{x}) + (1 + r)c'(\bar{x})} > 0.
\]

The last inequality follows from the fact that \(x^* > \bar{x} > x_0\) and \(y^* > \bar{y} > y_0\). Indeed, if \(x^* \leq \bar{x}\), then \(y^* > \bar{y}\) but this implies that \(\nu'(\bar{x})/c'(\bar{x}) \leq 1 < \nu'(\bar{y})/c'(\bar{y})\).

Suppose that \(\nu'(\bar{x})/c'(\bar{x}) > \nu'(\bar{y})/c'(\bar{y})\). We claim that \(x^p > \bar{x}\) and \(y^p < \bar{y}\). Suppose not and hence \(x^p < \bar{x}\) and \(y^p > \bar{y}\). Then,

\[
\nu'(x^p)/c'(x^p) > \nu'(\bar{x})/c'(\bar{x}) > \nu'(\bar{y})/c'(\bar{y}) > \nu'(y^p)/c'(y^p),
\]

a contradiction to the FOC’s. Thus, \(u(x^p) - (1 + r)c(x^p) < 0\). □

Proof of Theorem 6.1

(1) In this case, in any constrained efficient allocation, \(x^p = x^*\) and \(y^p = y^*\), because \([x^p, z^p_1], (y^p, z^p_2)] = [(x^*, c(x^*)), (y^*, c(y^*))\)] is implementable by Theorem 5.1. If \(u(x^*) - (1 + r)c(x^*) < 0\), then we cannot find \(z_1 \geq c(x^*)\) such that (7) is satisfied, and hence, by Theorem 4.2, we cannot implement any first-best allocation without OMO. So OMO is essential. If \(u(x^*) - (1 + r)c(x^*) \geq 0\), then, by Theorem 4.2, we can implement \([(x^*, c(x^*)), (y^*, c(y^*))\)] by money alone.

(2) First we show that if \((x^p, y^p)\) solves (23)-(24), then \([(x^p, c(x^p)), (y^p, c(y^p))\)] is the unique constrained-efficient allocation. Indeed, for any implementable outcome \([(x, z_1), (y, z_2)]\), it satisfies (11) and \(c(x) \geq z_1\) and \(c(y) \geq z_2\). Hence,

\[
u(x) - (1 + r)c(x) + v(y) - (1 + r)c(y) \geq -rz_1 + [u(x) - z_1] - rz_2 + [v(y) - z_2] \geq 0,
\]

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that is, $(x, y)$ satisfies (24). Hence,

$$u(x) - c(x) + v(y) - c(y) \leq u(x^p) - c(x^p) + v(y^p) - c(y^p).$$

Because $[(x^p, c(x^p)), (y^p, c(y^p))]$ is implementable, it follows that it is the unique constrained efficient allocation.

Now, if $u'(\bar{x})/c'(\bar{x}) > v'(\bar{y})/c'(\bar{y})$, then, by Lemma 9.1, $u(x^p) - (1+r)c(x^p) < 0$, and, by Theorem 4.2, $[(x^p, c(x^p)), (y^p, c(y^p))]$ is not implementable with money alone. Therefore, OMO is essential. If $u'(\bar{x})/c'(\bar{x}) \leq v'(\bar{y})/c'(\bar{y})$, then, by Lemma 9.1, $u(x^p) - (1+r)c(x^p) \geq 0$, and, by Theorem 4.2, $[(x^p, c(x^p)), (y^p, c(y^p))]$ is implementable with money alone and hence OMO is not essential. □

**Proof of Theorem 6.2**

**Proof.** (1) In this case, in any constrained efficient allocation, $x^p = x^*$ and $y^p = y^*$, because $[(x^p, z^p_1), (y^p, z^p_2)] = [(x^*, c(x^*)), (y^*, c(y^*))]$ is PC-implementable by Theorem 5.2. If $u(x^*) - (1+r)c(x^*) < 0$, then we cannot find $z_1 \geq c(x^*)$ such that (7) is satisfied, and hence, by Theorem 4.2, we cannot implement any first-best allocation without OMO. So OMO is essential under PC.

(2) (a) For each $r$, let the solution to (23)-(24) be denoted by $(x^p(r), y^p(r))$. Then, for $r \leq r^*$, $(x^p(r), y^p(r)) = (x^*, y^*)$, where $r^*$ is the unique $r$ such that

$$u(x^*) - (1+r)c(x^*) + v(y^*) - (1+r)c(y^*) = 0.$$

By Theorem of Maximum, $(x^p(r), y^p(r))$ is continuous in $r$. Therefore, there exists $r_0 < r^*$ such that for all $r \geq r_0$,

$$u(x^p) - c(x^p) + v(y^p) - c(y^p) \geq u(x^*) - c(x^*).$$

Hence, for $r \leq r_0$, $[(x^p(r), c(x^p(r))), (y^p(r), c(y^p(r)))]$ is PC-implementable by Theorem 5.2, and is the unique constrained-efficient allocation under PC by Theorem 6.1. Then, OMO is essential if and only if (26) holds.

(b) Now, suppose that $v(x) \geq u(x)$ and that $c_1(x) \geq c_2(x)$ for all $x$. Let $(x^p, y^p)$ be the solution to (23)-(24).

Consider two cases:

(i) $c_1(x^p) + c_2(y^p) \geq c_1(x^*)$. Now,

$$u(x^p) - c_1(x^p) + v(y^p) - c_2(y^p) \geq r(c_1(x^p) + c_2(y^p)) \geq r c_1(x^*) > u(x^*) - c(x^*),$$

where the last inequality follows from (27). Therefore, $[(x^p, c(x^p)), (y^p, c(y^p))]$ is PC-implementable by Theorem 5.2, and is the unique constrained-efficient allocation under PC by Theorem 6.1. Then, OMO is essential if (26) holds.
(ii) \( c_1(x^p) + c_2(y^p) < c_1(x^*) \). Because \( u \) is concave, it is subadditive, and because \( c_1 \) is convex, it is superadditive. Therefore,

\[
\begin{align*}
   u(x^p) + v(y^p) &\geq u(x^p) + u(y^p) \geq u(x^p + y^p) = u \circ c_1^{-1}(c_1(x^p) + y^p)) \\
   &\geq u \circ c_1^{-1}(c_1(x^p) + c_1(y^p)) \geq u \circ c_1^{-1}(c_1(x^p) + c_2(y^p)).
\end{align*}
\]

Thus, \([(x^p, c(x^p)), (y^p, c(y^p))]\) is PC-implementable by Theorem 5.2, and is the unique constrained-efficient allocation under PC by Theorem 6.1. Then, OMO is essential if (26) holds. □

**Proof of Theorem 6.3**

(1) Assume that \( \sigma_1 = 1 \). If \( r > \tilde{r} \), then either

\[
-r c(x^*) + [u(x^*) - c(x^*)] < 0
\]

or

\[
-r[c(x^*) + c(y^*)] + [u(x^*) - c(x^*)] + \sigma_2[v(y^*) - c(y^*)] < 0.
\]

We claim that the first-best is not implementable with money alone if \( r > \tilde{r} \).

By Theorem 4.2, \([(x^*, y^*), (z_1, z_2)]\) is implementable with money alone only if

\[
-r c(x^*) + [u(x^*) - c(x^*)] \geq -rz_1 + [u(x^*) - z_1] \geq 0
\]

and

\[
-r[c(x^*) + c(y^*)] + [u(x^*) - c(x^*)] + \sigma_2[v(y^*) - c(y^*)] \geq -rz_1 + [u(x^*) - z_1] + \sigma_2[v(y^*) - z_2] \geq 0.
\]

Thus, if \( r > \tilde{r} \), then one of the above two inequalities cannot hold.

Here we show that there exists \( \tilde{r} < \tilde{r} \) such that for all \( r \in (\tilde{r}, \tilde{r}) \), the first-best is implementable with OMO and hence OMO is essential. Consider two cases.

(i) \( \frac{\sigma_1[u(x^*) - c(x^*)]}{c(x^*)} < \frac{\sigma_2[v(y^*) - c(y^*)]}{c(y^*)} \).

In this case, \( \tilde{r} = r_0 < r_1 \). By Theorem 5.3, \([(x^*, y^*), (c(x^*), c(y^*))]\) is implementable (under PC) for all \( r \leq r_1 = r_2 \). We may simply let \( \tilde{r} = r_1 \). However, we may be able to implement (under PC) the first-best for slightly larger \( r \)'s by using Theorem 5.4.

(ii) \( \frac{\sigma_1[u(x^*) - c(x^*)]}{c(x^*)} > \frac{\sigma_2[v(y^*) - c(y^*)]}{c(y^*)} \).

In this case, \( \tilde{r} = r_1 = r_2 < r_0 \). However, when \( r = r_1 \), \( -r_1 c(x^*) + [u(x^*) - c(x^*)] > 0 \), and

\[
-r_1[c(x^*) + c(y^*)] + [u(x^*) - c(x^*)] + \sigma_2[v(y^*) - c(y^*)] = 0.
\]

Hence, \( -r_1 c(y^*) + \sigma_2[v(y^*) - c(y^*)] < 0 \), and, because \( (1 + r_1)\sigma_2/(r_1 + \sigma_2) < 1 \),

\[
\{ -r_1 c(x^*) + [u(x^*) - c(x^*)] \} + \frac{(1 + r_1)\sigma_2}{r_1 + \sigma_2} \{ -r_1 c(y^*) + \sigma_2[v(y^*) - c(y^*)] \} > 0.
\]

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By continuity, there exists \( r < \tilde{r} \) such that for all \( r \geq r \),
\[
\{-r c(x^*) + [u(x^*) - c(x^*)]\} + \frac{(1 + r) \sigma_2}{r + \sigma_2} \{ -r c(y^*) + \sigma_2 [v(y^*) - c(y^*)] \} \geq 0.
\]
By Theorem 5.4, for such \( r \)'s, \( [(x^*, y^*), (c(x^*), c(y^*))] \) is implementable (under PC).

(2) Assume that \( \sigma_1 < 1 \). In this case, \( \tilde{r} \leq r_1 < r_2 \). For all \( r \in (\tilde{r}, r_2] \), \( [(x^*, y^*), (c(x^*), c(y^*))] \) is not implementable with money alone but is implementable (under PC) with OMO, and hence OMO is essential under (PC). \( \square \)

**Proof of Corollary 6.2**

First note that \( r^* \) exists by Theorem 6.3 (2). Indeed, if
\[
\frac{\sigma_1[u(x^*) - c(x^*)]}{c(x^*)} \leq \frac{\sigma_2[v(y^*) - c(y^*)]}{c(y^*)},
\]
then \( r^* = r_2 \). Otherwise, let \( r' \) be the largest number no greater than \( r_1 \) such that for all \( r \leq r' \),
\[
\{-r c(x^*) + [u(x^*) - c(x^*)]\} + \frac{(1 + r) \sigma_2}{r + \sigma_2} \{ -r c(y^*) + \sigma_2 [v(y^*) - c(y^*)] \} \geq 0.
\]
Then, \( r^* = \max\{r_2, r'\} \).

Then, by Theorem 6.3 (2), when \( r = r^* \), the first-best is implementable with OMO but not with money. Let \( W \) be the largest welfare achievable by allocations that are implementable with money alone at \( r = r^* \), and let \( W^* \) be the welfare of the first-best allocation. Now, let \( W^*(r) \) be the welfare achievable for symmetric allocations that are implementable with OMO. Then, \( W^*(r) \) is continuous in \( r \) and \( W^*(r^*) = W^* > W \). Thus, there exists \( \hat{r} > r^* \) such that for all \( r < \hat{r} \), \( W^*(r) > W \). Then, for those \( r \)'s, OMO is essential. We can make them implementable under PC as well for \( r \)'s not too far from \( r^* \). \( \square \)

**Proof of Theorem 6.4**

First we give a lemma.

**Lemma 9.1.** Consider the following optimization problem.
\[
\max_{(x,y)} [u(x) - c(x)] + \sigma_2 [v(y) - c(y)] \quad (66)
\]
subject to
\[
[u(x) - (1 + r)c(x)] + \sigma_2 v(y) - (\sigma_2 + r)c(y) \geq 0. \quad (67)
\]
Its solution exists and is unique, denoted by \((x^p, y^p)\), and can be characterized as follows. Let \( \bar{x} > 0 \) and \( \bar{y} > 0 \) be such that
\[
u(\bar{x}) - (1 + r)c(\bar{x}) = 0 = \sigma_2 v(\bar{y}) - (\sigma_2 + r)c(\bar{y}). \quad (68)
\]
1. The solution, \((x^p, y^p)\), is equal to \((x^*, y^*)\) if and only if \((x^*, y^*)\) satisfies (67).

2. Suppose that \((x^*, y^*)\) does not satisfy (67).

- \((x^p, y^p)\) is the unique pair such that
  \[
  - \frac{u'(x^p) - c'(x^p)}{u'(x^p) - (1+r)c'(x^p)} = \frac{\sigma_2[u'(y^p) - c'(y^p)]}{\sigma_2v'(y^p) - (\sigma_2 + r)c'(y^p)}; \\
  - u(x^p) - (1+r)c(x^p) + \sigma_2v(y^p) - (\sigma_2 + r)c(y^p) = 0.
  \]

- \(u(x^p) - (1+r)c(x^p) \geq 0\) if and only if
  \[
  \frac{u'(\bar{x}) - c'(\bar{x})}{u'(\bar{x}) - (1+r)c'(\bar{x})} \leq \frac{\sigma_2[u'(\bar{y}) - c'(\bar{y})]}{\sigma_2v'(\bar{y}) - (\sigma_2 + r)c'(\bar{y})}.
  \]

**Proof.** (1) Let \((x^p, y^p)\) be the solution to (66)-(67). Consider two cases.

(i) \(\frac{u'(\bar{x}) - c'(\bar{x})}{u'(\bar{x}) - (1+r)c'(\bar{x})} > \frac{\sigma_2[u'(\bar{y}) - c'(\bar{y})]}{\sigma_2v'(\bar{y}) - (\sigma_2 + r)c'(\bar{y})}\).

In this case, \(-rc(x^p) + [u(x^p) - c(x^p)] < 0\). By Theorem 4.2, the allocation \([x^p, y^p, (z_1, z_2)]\) is not implementable with money alone for any \((z_1, z_2)\), but the allocation \([x^p, y^p, (c(x^p), c(y^p))]\) is implementable with OMO. Thus, OMO is essential.

(ii) \(\frac{u'(\bar{x}) - c'(\bar{x})}{u'(\bar{x}) - (1+r)c'(\bar{x})} < \frac{\sigma_2[u'(\bar{y}) - c'(\bar{y})]}{\sigma_2v'(\bar{y}) - (\sigma_2 + r)c'(\bar{y})}\).

In this case, \(-rc(x^p) + [u(x^p) - c(x^p)] > 0\). By Theorem 4.2, for any allocation \([x^p, y^p, (z_1, z_2)]\) that is implementable with money alone satisfies (67). Thus, among allocations that are implementable with money alone, the welfare is bounded above the optimum to (66), denoted by \(W^0\). However, at \((x^p, y^p)\), because \(\sigma_2 < 1\), we have

\[
\{-rc(x^p) + [u(x^p) - c(x^p)]\} + \frac{(1+r)\sigma_2}{\sigma_2v(y^p) - (\sigma_2 + r)c(y^p)}\{(-rc(y^p) + \sigma_2v(y^p) - c(y^p))\} > 0.
\]

By continuity, there exists \((x', y') \neq (x^p, y^p)\) such that \(x' \in [x^p, x^*], y' \in [y^p, y^*]\) and such that

\[
\{-rc(x') + [u(x') - c(x')]\} + \frac{(1+r)\sigma_2}{\sigma_2v(y') - (\sigma_2 + r)c(y')}\{(-rc(y') + \sigma_2v(y') - c(y'))\} > 0.
\]

By Theorem 5.4, \([x', y'], (c(x'), c(y'))\) is implementable with OMO and \(C = \{1\}\), and it has higher welfare than \(W^0\). So OMO is essential.

(1) Consider two cases.

(i) \(\frac{u'(\bar{x}) - c'(\bar{x})}{u'(\bar{x}) - (1+r)c'(\bar{x})} > \frac{\sigma_2[u'(\bar{y}) - c'(\bar{y})]}{\sigma_2v'(\bar{y}) - (\sigma_2 + r)c'(\bar{y})}\).

Consider the following two maximization problems:

\[
\max_{(x,y)} \sigma_1[u(x) - c(x)] + \sigma_2[v(y) - c(y)] \quad \text{(69)}
\]

subject to

\[
\sigma_1[u(x) - (1+r)c(x)] + \sigma_2v(y) - (\sigma_2 + r)c(y) \geq 0. \quad \text{(70)}
\]

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and

$$\max_{(x,y)} \sigma_1[u(x) - c(x)] + \sigma_2 \sigma_1[v(y) - c(y)] + \sigma_2(1 - \sigma_1)[v(y^*) - c(y^*)]$$

(71)

subject to

$$\sigma_1[u(x) - c(x)] + \sigma_2[v(y) - c(y)] + \sigma_2(1 - \sigma_1)[v(y^*) - c(y^*)] \geq r[c(x) + c(y)]$$

(72)

$$-rc(x) + \sigma_1[u(x) - c(x)] + (1 - \sigma_1)\sigma_2[v(y^*) - c(y^*)] \geq 0.$$  

(73)

Both problems have unique solutions characterized by the Kuhn-Tucker conditions. Let $W^1(\sigma_1)$ and $W^0(\sigma_1)$ denote the maximal welfare corresponding to the first and the second problem.

Now, when $\sigma_1 = 1$, the objective functions (69) and (71) are identical and the constraints (70) and (73) are identical. So $W^1(1) \geq W^0(1)$. But by the proof of (1) (i), we know that $W^1(1) > W^0(1)$. By the Theorem of Maximum, there exists $\tilde{\sigma}_1 < 1$ (which may depend on $r$) such that if $\sigma_1 > \tilde{\sigma}_1$, then $W^1(\sigma_1) > W^0(\sigma_1)$. Now, by Theorem 4.2, for any allocation $[(x, y^0, y^1), (z_1, z_2^0, z_2)]$ that is implementable by money alone, $(x, y^1)$ satisfies (73) and (73), and its welfare is no larger than $(x, y^*, y^1)$. Thus, $W^0(\sigma_1)$ is an upper bound on the welfare achievable by allocations implementable by money alone. However, for any $(x, y)$ that satisfies (70), $[(x, y), (c(x), c(y))]$ is implementable with OMO. As a result, if $\sigma_1 > \tilde{\sigma}_1$, OMO is essential.

(ii) $\frac{u(x) - c(x)}{u(x) - (1 + r)c(x)} < \frac{\sigma_2v(y) - c(y)}{\sigma_2v(y) - (\sigma_2 + r)c(y)}$.

Consider the following maximization problem:

$$\max_{(x,y)} \sigma_1[u(x) - c(x)] + \sigma_2[v(y) - c(y)]$$

(74)

subject to

$$[u(x) - (\sigma_1 + r)c(x)] + \frac{(1 + r)\sigma_2}{r + \sigma_2} \{\sigma_2v(y) - (\sigma_2 + r)c(y)\} \geq 0.$$  

(75)

The problem has a unique solution and its maximal value is denoted $W^2(\sigma_2)$. By (1) (ii), we have $W^2(1) > W^1(1) = W^0(1)$. By the Theorem of Maximum, there exists $\tilde{\sigma}_1 < 1$ (which may depend on $r$) such that if $\sigma_1 > \tilde{\sigma}_1$, then $W^2(\sigma_1) > W^0(\sigma_1)$. By Theorem 5.4, for any $(x, y)$ that satisfies (70), $[(x, y), (c(x), c(y))]$ is implementable with OMO. As a result, if $\sigma_1 > \tilde{\sigma}_1$, OMO is essential. □