# COMMUNICATION AMONG SHAREHOLDERS
## — *PRELIMINARY* —

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Abstract. This paper studies information transmission among shareholders in an investment venture. An *expert shareholder*, such as an especially knowledgeable active investor, chooses how much information to communicate to a *controlling investor* who controls the investment strategy. The incentives that drive communication are determined by the amounts of shares held by the expert and controlling investors: different share holdings configurations affect information transmission within the venture. In addition, we can discuss how different *share allocation mechanisms* impact information transmission and, through it, welfare. If shares are allocated through a market mechanism, there is a tendency to achieve a welfare-maximal allocation: perfect communication and full risk-sharing. When frictions lead to a welfare-suboptimal outcome, a competitive market for shares fails to reward the positive externalities that an informed investor provides to other investors by purchasing shares. Within a principal-agent setting, the model highlights a trade-off between incentivizing effort provision and promoting information transmission. Also, the model delivers insights about the effects of prudential regulation.

## 1. Introduction

This paper studies information transmission among shareholders in an investment venture. Our protagonist is a shareholder with special expertise, such as an especially knowledgeable active investor. This shareholder has a unique ability to interpret inside information, once the information is revealed to all investors in the venture. Once the inside information is revealed, the expert investor becomes privately informed about the return of a risky investment project, and then she chooses how much of this information to communicate to the shareholder who

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controls the investment strategy. We call this shareholder the controlling investor. After the communication takes place, the controlling investor chooses an investment strategy for the venture and payoffs are realized. All shareholders can trade shares ahead of the release of this inside information (but not after, on the assumption that trading on inside information is prohibited).

This setting allows us to study the amount of cheap-talk information that the expert investor transmits to the controlling investor. Relative to the cheap talk literature, the key innovation is that the incentives that drive communication are determined by the amounts of shares held by the expert and controlling investors. Furthermore, the communication between these two investors has welfare consequences for all other shareholders. Thus the model allows us to study how different *share holdings configurations* affect information transmission within the venture. In addition, we can discuss how different *share allocation mechanisms* impact information transmission and, through it, welfare.

We first study the allocation mechanism where investors acquire shares in a competitive stock market. Investors demand shares based on their expectation of the quality of information transmission between expert and controlling investor; the share price is determined by market clearing. In general, we find that the incentives that govern the market demand for shares are reasonably aligned with those that govern information transmission. Specifically, the allocation that optimizes risk sharing among investors is also the allocation of shares that promotes transparent information transmission. However, an inefficiency lurks in the background: a competitive market for shares fails to reward the positive externalities that an informed investor provides to other investors by purchasing shares. In some scenarios, this leads the expert investor to acquire too few shares, and to transmit too little information to the controlling investor. In our model, this inefficiency arises when the equilibrium price of shares is positive, that is, when the risk that an investor takes on by purchasing a share in the venture is socially valuable at the margin.

The expert agent in our model can be interpreted as an activist shareholder because a key channel for investment activism is persuasion of the controlling shareholder by the activist.[1]

[1]The US Department of Justice gives the following definition of shareholder activism:

> "ValueAct intended to use its position as a major shareholder of these companies to obtain access to management, to learn information about the merger and the companies' strategies in private conversations with senior executives, to influence those executives to improve

Our model posits that the activist's ability to persuade the controlling shareholder is partly a function of the alignment provided by their shareholding. This seems to be the case in practice: "skin in the game," i.e., the alignment (or lack thereof) that shareholdings provide are thought to be an important driver of the communication between activist and controlling investors.[2]

We next study a different allocation mechanism: that in which share holdings are set by a principal as a means of incentivizing an agent. In this setting, the controlling shareholder takes on the role of the principal, and the expert shareholder that of the agent (executive). In addition to communicating information, we assume that the agent also exerts costly effort. Thus the setting features moral hazard, in addition to information transmission. Full efficiency cannot be expected in this setting because a single incentive scheme (shares) for the agent is used to incentivize two different activities: effort provision and information transmission. However, we find that if the space of contracts includes a fixed salary plus shares in the enterprise, the principal can achieve constrained welfare maximization; that is, a maximization of joint welfare conditional on the agent choosing effort and reporting strategy. In this constrained-efficient allocation the principal will own too few shares to efficiently communicate with the agent. Thus, we identify a novel trade-off in the principal-agent literature: the principal chooses the compensation scheme to balance incentives for the agent to exert optimal effort, versus incentives to fully communicate information.

The principal-agent setting described above lends itself to studying the effects of prudential regulation from a risk-taking perspective. Prudential regulation can take two broad forms: (a) direct regulation of risk-taking behavior (e.g., for financial institutions: capital requirements, stress tests, "too big to fail" regulations, etc.); (b) indirect regulation through

> the chances that the merger would be completed, and to influence other business decisions whether or not the merger went forward."

According to this definition, influence exerted by an expert shareholder solely through cheap talk qualifies as activist investing.

[2] "Skin in the game" is commonly invoked during activist investing episodes, often by all sides of the dispute. For example, in a letter to Sotheby's investors, famed activist investor D. Loeb disagreed with the board's long-term strategy referring to "Their lack of 'skin in the game' has led to a dysfunctional corporate culture overly focused on short-term metrics such as auction volumes at the expense of long-term investment in key areas." Quoted from http://www.law360.com/articles/525502/loeb-s-third-point-asks-investors-to-ditch-sotheby-s-board

checks on executive compensation, specifically, reducing its dependence on market valuation.[3] We find that direct regulation, i.e., constraints on the amount of risk that the controlling investor is allowed to take, is beneficial for two reasons: first, a direct effect of making the controlling investor behave in a more risk-averse way, which is the goal of prudential regulation; second, by making the controlling investor behave in a more risk-averse way, prudential regulation effectively more-closely aligns the controlling investor's preferences with those of the executive's, resulting in greater information being transmitted by the executive and therefore a more accurate investment decision. By contrast, reducing the dependence of the executive's compensation on market valuation has no direct effect on the equilibrium strategy's riskiness (because in our model the controlling investor, not the executive, chooses the investment strategy); but, by increasing the alignment between the executive's and the controlling investor's preferences, checks on executive compensation promote greater information transmission from the executive to the controlling investor, leading to more accurate investment decisions.

We conclude by micro-founding the role of the controlling investor. In most of the analysis we assume that one of the investors is given the power to choose the investment strategy. But, if there is disagreement among investors, who is to be chosen as controlling investor? We show that, if the investment action is chosen collectively among shareholders through "one share one vote," then the share-weighted median voter is shown to function as controlling investor.

Technically, our theoretical model builds on Crawford and Sobel (1982). However, because our payoff functions are micro-founded in an investment model, the functional forms that arise are different from the simple example originally developed by Crawford and Sobel (1982), and still prevalent today in analyses of information transmission. The analysis of our model is innovative from a technical perspective and, though it is mostly relegated to the appendix, should be considered a significant technical contribution of this paper. In particular, several results on Chebyshev polynomials in the appendix may be of independent interest and either directly, or through the methods used in their proofs, help in solving problems

[3]The second form of regulation, while not traditional, is now part of the regulatory arsenal; for example, the U.K.'s Financial Conduct Authority has set forth Remuneration Codes designed to curb risk-taking by financial institutions.

involving more general second order difference equations (in particular non-homogeneous equations).

# 2. Model

Investing $a$ into a venture yields returns at a constant (but unknown) rate $X$. It is possible to borrow $a$ at zero interest (this is a normalization), so venture profits are given by $aX$.

The rate of return $X$ is a random variable with mean $\mu$ and variance $\sigma^2$. The parameter $\sigma^2$ is common knowledge. The parameter $\mu$ is not common knowledge among all investors.

## 2.1. Players, information, ownership, and control

An *informed investor* $I$ owns a fraction $\theta_I$ of the venture. He knows $\mu$.

A *controlling investor* $C$ owns a fraction $\theta_C$ and has the right to choose $a$. She does not know $\mu$ : she has a prior belief that $\mu$ is uniformly distributed on $[0, \sigma^2]$ . The support's lower bound being zero implies that every project returns *on average* more than the investment; the upper bound being equal to $\sigma^2$ is just a normalization.

There are a number of *non-controlling investors* $i$, who own fractions $\theta_i$ of the venture. They too do not know $\mu$ and their prior on $\mu$ is uniformly distributed on $[0, \sigma^2]$ .

## 2.2. Preferences and payoffs

All investors have mean-variance preferences. An investor who owns $m$ units of money and a fraction $\theta$ of the corporation, and has risk-aversion parameter $r$ has a payoff equal to:

$$\mathbb{E}\left(\theta a X\right) - \frac{r}{2} Var\left(\theta a X\right) + m.$$

If we denote

$$\frac{\mu}{\sigma^2} \equiv \omega,$$

then the investors' payoffs can be rewritten as:

$$-\frac{\sigma^2}{2r}\left(r\theta a - \omega\right)^2 + m, \tag{2.1}$$

up to an additive constant (refer to Lemma A.1).

Henceforth we normalize the controlling investor's risk parameter $r_C = 1$ and denote the informed and the non-controlling investors' risk parameters by $r_I$ and $r_i$, respectively.

### 2.3. Information transmission

Before the controlling investor chooses the investment level $a$, the informed investor may transmit information to the controlling investor through cheap talk, a' la Crawford and Sobel (1982).[4]

### 2.4. Trading of shares

Shares are traded by all investors in a competitive market before any cheap talk takes places.

### 2.5. Timing of events

(1) All investors trade shares.
(2) The informed investor learns $\mu$ and engages in cheap talk.
(3) The controlling investor directs the venture to invest $a$.
(4) The rate of return $X$ is realized and payoffs accrue.

## 3. Discussion of modeling assumptions

### 3.1. Ownership as "skin in the game"

In our model shares serve two roles. First, they are a source of wealth and risk. Second, shares are "skin in the game" that the active investor needs to have in order to credibly transmit information. How much "skin" is needed will depend, in equilibrium, on the controlling investor's risk aversion and holdings.

### 3.2. Modeling of information and timing of trade

The information we have in mind is venture-specific intelligence. The intelligence could pertain to a business re-organization or managerial opportunity. This information can only be generated by a sophisticated active investor (in our model, the "informed investor") after the investor is able to examine inside information. After the inside information has been examined and the venture-specific intelligence generated, the active investor can no longer trade on this inside information. This is why, in our model, trading takes place before information about $\mu$ is available.

[4] If the non-controlling investors can also listen in to the cheap talk, nothing changes in the equilibrium. See the discussion in Section 3.2.

The assumption that trade takes place before the informed investor learns $\mu$ has several consequences for modeling. First, it ensures that, when purchasing shares, the informed investor cannot engage in insider-trading. If shares were traded after the informed investor learns $\mu$, but before the cheap-talk stage, then non-informed investors would worry about adverse selection. If shares were traded after the cheap-talk stage, then the cheap talk could be used to manipulate the share market. In our framework, instead, investors do not have to worry about trading under adverse selection and cheap talk is used solely to influence the controlling investor's investment choice.

The second consequence of trade taking place before cheap talk is that it does not matter whether the non-controlling investors become privy to cheap-talk communication. This is because the non-controlling investors play no further role in the model after they trade.

### 3.3. Source of Conflict of Interest Among Agents

Consider two investors with risk parameters $r, r'$. If these investors hold shares $\theta, \theta'$ respectively, in the exact ratio $\theta'/\theta = r/r'$, then their payoffs (2.1) are identical functions of $a$ up to a linear affine transformation. Therefore at this holdings ratio there is no conflict of interest regarding the optimal choice of $a$. In addition, the same holdings ratio achieves the optimal risk sharing of the aggregate holdings $\theta + \theta'$ (for details refer to Section 6.1).

This observation indicates that conflict of interest in this model arises solely due to a "misallocation" of ownership relative to the optimal risk-sharing holdings ratio. If risk is shared optimally then there is no conflict of interest regarding the choice of investment, and moreover information can be transmitted perfectly between investors. In this respect our model differs from the standard principal-agent setting, where optimal risk-sharing is undesirable because it conflicts with the need to incentivize effort.

### 3.4. Relationship with Popular Cheap Talk Example from Crawford and Sobel (1982)

Crawford and Sobel's (1982) popular example has the following functional form:

$$(a - \omega - b)^2 .$$

In this functional form, the conflict-of-interest parameter $b$ enters additively. In our model, instead, the conflict-of-interest parameter enters multiplicatively with the action. This makes a difference. Consider for example the payoff function (2.1) for a controlling investor. For the purpose of choosing $a$, this expression is equivalent to:

$$(ab - \omega)^2 ,$$

where $b = r\theta$ denotes the conflict-of-interest parameter. This makes the strategic analysis quite different. For example, if an agent knows $\omega$ then the optimal action is $a^* = \omega + b$ in Crawford and Sobel (1982), but it is $a^* = \omega/b$ in our model. So functional forms matter. It is important, therefore, that our utility function is micro-founded. This way, we can have some confidence that the mechanics of the model meaningfully capture the economic phenomenon described in the model.

### 3.5. Relationship with Models of Authority Allocation (Dessein 2002)

Dessein's (2002) seminal work shows that it can be in the interest of a controlling agent to delegate control (in his terminology, authority) to an informed agent. This is because delegation can help reduce the informational loss caused by cheap talk. Some of the same effects are present in our model, and we explore them in Section 6.7. However, in a corporate context the controlling agent (the board) is required to exercise control and is not allowed to delegate control over strategic matters. Put differently: when ownership and control go together, giving up control requires also giving up some ownership. This observation suggests that delegation could be costly or difficult to achieve. In our model instead, the trading of shares is a strong force driving toward full information transmission, irrespective of who has control. So full information transmission can be achieved even when delegation would not be in the self-interest of the controlling agent. We expand on this point in Section 6.7.

## 4. Information Transmission with Fixed Shareholdings

Fix all share holdings $\theta_C, \theta_I, \{\theta_i\}$. We analyze the case where the controlling investor chooses $a$ after a single round of cheap-talk communication from the informed investor. As in Crawford and Sobel (1982), we will focus on equilibria where the sender communicates in interval partitions with cutoffs $\{\omega_0, \omega_1, ..., \omega_N\}$, where $\omega_0 = 0$ and $\omega_N = 1$.

Because of the functional-form difference between our payoff function (2.1) and Crawford and Sobel's (refer to Section 3.4), solving for the equilibrium and especially computing the equilibrium payoffs will require some technical innovations. These technical innovations are presented in the appendix.

**Definition 1** (Risk-adjusted holdings ratio)**.** *The ratio* $\rho = \theta_C / r_I \theta_I$ *will be called "risk-adjusted holdings ratio."*

The risk-adjusted holdings ratio parameterizes the conflict of interest among investors. The ratio $\rho \in [0, \infty)$. If $\rho = 1$ then there is no conflict of interest (this was discussed in Section 3.3). If $\rho < 1$ the conflict of interest arises from the controlling investor holding "too little" equity relative to the informed investor; if $\rho < 1$ the conflict of interest arises from the controlling investor holding "too much" equity.

**Proposition 1** (Characterization of cheap-talk equilibrium strategies)**.** *Fix a risk-adjusted holdings ratio* $\rho$*. Take any equilibrium with partition cutoffs* $\{\omega_0, \omega_1, ..., \omega_N\}$ *.*

(1) *Upon learning that* $\omega$ *falls into partition* $(\omega_n, \omega_{n+1})$*, the controlling investor's equilibrium action equals* $(\omega_{n+1} + \omega_n) / 2\theta_C$*.*

(2) *Equilibrium cutoffs* $\omega_n$ *solve the following difference equation:*

$$\omega_{n+2} - 2k_\rho \omega_{n+1} + \omega_n = 0, \tag{4.1}$$

*where* $k_\rho = 2\rho - 1,\ \omega_0 = 0,$ *and* $\omega_N = 1$*.*

(3) *Fix an equilibrium with* $N$ *partitions. Equilibrium partitions have the form* $\omega_n = \alpha U_{n-1}(k_\rho)$ *where* $U_n$ *is the order-n Chebyshev polynomial of the second kind, and* $\alpha = 1/U_{N-1}(k_\rho)$ *.*

(4) *Suppose* $\rho > 1$*, that is, the informed investor (sender) has less than the optimal risk-sharing holding relative to the controlling investor (receiver).Then there are equilibria with arbitrarily large number of partitions but none of these equilibria approaches full information transmission.*

(5) *Suppose* $\rho < 1$*, that is, the informed investor (sender) has more than the optimal risk-sharing holding relative to the controlling investor (receiver). Then the maximal number of partitions consistent with an equilibrium is an increasing step function of* $\rho$ *converging to infinity as* $\rho \uparrow 1$*. No information can be communicated when* $\rho < 3/4$*.*

*Proof.* See the Appendix. □

Part 1 of Proposition 1 indicates that the conflict-of-interest parameter enters the controlling investor's equilibrium strategy multiplicatively.[5]

Part 2 introduces the parameter $k_\rho$. This parameter, which is monotonic in $\rho$, encodes the conflict of interest between the controlling and the informed investors regarding the choice of investment. When $k_\rho = 1$ there is no conflict of interest because $\rho = 1$. The parameter $k_\rho$ enters equation (4.1) multiplicatively.[6] Part 2 also provides the difference equation that generates equilibrium partitions. This difference equation is famous because its generates the family of Chebyshev polynomials as its solution. The order-$n$ Chebyshev polynomial of the second kind, $U_n(x)$, is a polynomial function of $x$ defined as the unique solution to the following functional difference equation:

$$\begin{aligned} &U_{-1}(x) \equiv 0, \\ &U_0(x) \equiv 1, \\ &U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \end{aligned} \tag{4.2}$$

The most popular expression for $U_n(x)$, and the one that will best serve our purposes is:

$$U_n(x) = \begin{cases} \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)} & \text{if } |x| \le 1 \\ \frac{\sinh((n+1)\operatorname{arccosh} x)}{\sinh(\operatorname{arccosh} x)} & \text{if } x > 1, \end{cases}$$

where $\operatorname{arccosh} x$ is nonnegative (refer to expressions 1.4 and 1.33b in Mason and Handscomb 2003). Because equation (4.2) which generates the family of Chebyshev polynomials is the same as equation (4.1), there must be a strong connection between the equilibrium of our cheap talk game and the family $U_n(x)$.[7] The only difference lies in the initial conditions. Chebyshev polynomials are generated by specifying two initial conditions, whereas the requirements for a cheap talk equilibrium only specify one initial condition, $\omega_0 \equiv 0$; therefore,

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[5]In the Crawford and Sobel's (1982) example, the conflict-of-interest parameter enters additively. This difference reflects the functional-form difference between our payoff function (2.1) and Crawford and Sobel's (1982): refer to Section 3.4.

[6]Equation (4.1) differs subtly from difference equation (21) in Crawford and Sobel (1982). That equation reads:

$$a_{n+2} - 2a_{n+1} + a_n - 4b = 0. \tag{CS}$$

In equation (CS) the conflict of interest is captured by the parameter $b$, which is additive.

[7]Pursuing this connection, one could view the cheap talk equilibrium thresholds $\omega_n(k_\rho)$ as polynomial functions of the conflict-of-interest parameter $k_\rho$, just as $U_n(x)$ is a polynomial function of $x$.

$\omega_1$ is a free parameter. Varying this free parameter generates the plethora of cheap-talk equilibria. Another way to index the family of cheap talk equilibria is as follows. Equation (4.1) is linear in $\omega$, so if a sequence $\{\omega_n\}$ solves (4.1) then, for any real number $\alpha$, the sequence $\{\alpha\omega_n\}$ also solves (4.1).[8] This means that the plethora of cheap-talk equilibria must, in this model, be indexed by a single scaling factor.

Part 3 confirms that, within a given cheap-talk equilibrium, partition cutoffs are indeed proportional to Chebyshev-polynomial functions of $k_\rho$. The factor $\alpha$ denotes the scaling factor. An intriguing possibility is that, by choosing the scaling factor $\alpha$ small enough, one might be able to generate cheap talk equilibria with an arbitrarily large number of partitions. The next parts of Proposition 1 shows that: (a) it is not always possible to generate cheap talk equilibria with an arbitrary number of partitions; and (b) when it is possible, these equilibria are not arbitrarily informative.

Proposition 1 part 4 indicates that if the conflict of interest arises from the informed investor owning too small a share relative to the controlling investor, there is no upper bound for the number of equilibrium partitions. In this case there are cheap talk equilibria with any number of partitions, but none of these equilibria approaches full communication. In these equilibria low realizations of $\omega$ will be communicated very accurately to the controlling investor(receiver) but high realizations of $\omega$ will not.

Proposition 1 part 5 indicates that if the conflict of interest arises from the informed investor owning too large a share relative to the controlling investor, then no cheap talk equilibrium can have more partitions than an upper bound. This upper bound grows without bound as the conflict of interest vanishes.

We now turn to computing expected utilities for the informed investor (sender) and the controlling investor (receiver).

**Proposition 2** (Payoffs in most informative equilibrium)**.** *Fix a risk-adjusted holdings ratio $\rho$. Equilibrium payoffs for both players increase in the number of equilibrium partitions. Moreover, up to additive constants $\frac{\sigma^2}{2}\mathbb{E}\left(\omega^2\right)$ for the controlling investor and $\frac{\sigma^2}{2r_I}\mathbb{E}\left(\omega^2\right)$ for the informed investor, equilibrium payoffs have the following properties.*

[8]This property does not hold in equation (CS).

(1) *If $\rho \geq 1$ the supremum of the family of equilibrium payoffs is as follows. For the controlling investor:*

$$\overline{V}_C(\rho) = \frac{\sigma^2}{6} \frac{1-\rho}{(4\rho - 1)}.$$

*For the informed investor:*

$$\overline{V}_I(\rho) = -\frac{\sigma^2}{6} \frac{\rho - 1}{\rho(4\rho - 1)} (4\rho - 3).$$

(2) *If $3/4 \leq \rho \leq 1$ the family of equilibrium payoffs admits the following upper bound. For the controlling investor:*

$$\overline{V}_C(\rho) = \frac{\sigma^2}{6} \frac{(\rho - 1)(3\rho - 1)}{(4\rho - 1)}.$$

*For the informed investor:*

$$\overline{V}_I(\rho) = -\frac{\sigma^2}{3} \frac{1-\rho}{(4\rho - 1)}.$$

*The exact equilibrium payoffs for the controlling and informed investors are given in the appendix, equations B.3 and B.5, respectively. These bounds are tight in the sense that for any $\rho$, there exists a value $\overline{\rho} \in (\rho, 1)$ such that $\overline{V}_I(\overline{\rho})$ is attained.*

(3) ***(communication breakdown)*** *If $\rho < 3/4$ no information can be transmitted. In this case the controlling investor's equilibrium payoff equals $-\sigma^2/24$ and the informed investor's equilibrium payoff equals*

$$-\frac{\sigma^2}{2r_I} \mathbb{E} \left( \frac{1}{2\rho} - \omega \right)^2.$$

*Proof.* See Propositions 10 and 11. □

Proposition 2 provides two surprisingly tractable expressions for the controlling and the informed investors' payoffs in the best equilibrium. Of note, equilibrium payoffs only depend on the holdings ratio $\rho$, not on holding levels. To understand why levels don't matter except through $\rho$, consider two share holding constellations $(\theta_C, \theta_I)$ and $(\theta'_C, \theta'_I)$ such that $\theta_C/\theta_I = \theta'_C/\theta'_I = \rho$. Both investors hold more share in one constellation than in the other, and that should change their payoff. However, recall that investment level $a$ is endogenous; when the controlling investor holds more shares, he will cut down on risk by choosing a

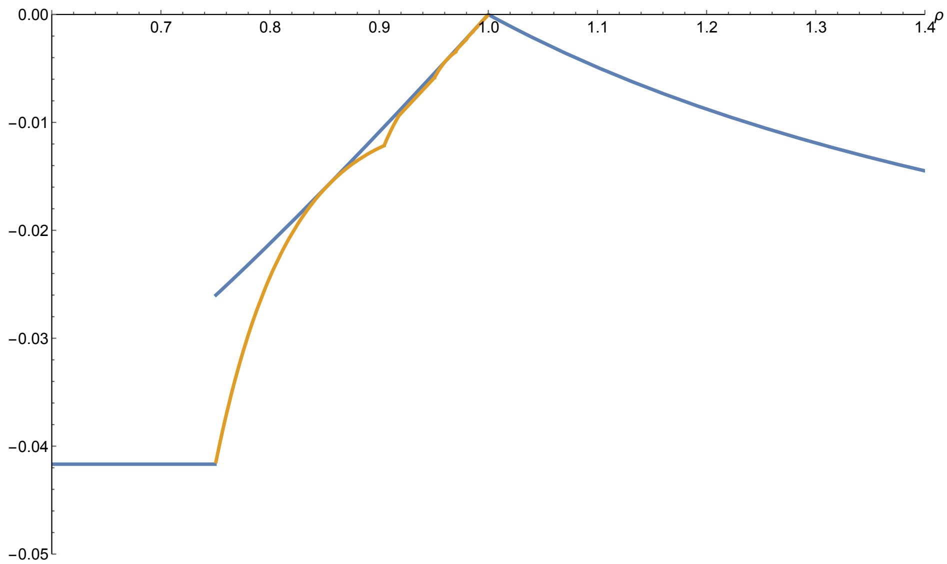


FIGURE 1. Controlling investor's expected utility for different values of $\rho$. Note that for $\rho \in \left(\frac{3}{4}, 1\right)$ proposition 2 gives the approximate payoff. This approximation is an upper bound and is very accurate when $\rho$ close to 1.

lower $a$ (Proposition 1 Part 1). The controlling investor's greater prudence has no effect on equilibrium communication (equation 4.1 is unchanged). So the lower $a$ exactly counteracts the effect of the higher $\theta$ in the payoff function (2.1), and the equilibrium payoff is unchanged.

## 5. Demand for Shares

In this section we compute each investor type's inverse demand for shares. Inverse demand is the marginal utility of increasing the investor's share holdings.

To compute demand in a model with multiple communication equilibria, it is necessary to take a stand on which equilibrium prevails. Consistent with standard practice in the literature on cheap talk, we focus on the most informative equilibrium. This equilibrium is focal because it is the equilibrium that is preferred both by the informed and by the controlling agent, i.e. by both communicating parties (refer to Proposition 2).

**Definition 2.** *Agent's $j = C, I$ inverse demand function for shares is the derivative of $\overline{V}_j(\rho)$ with respect to $\theta_j$.*

Note that inverse demand is computed based on the function $\overline{V}_j(\rho)$ which, for $\rho < 1$, is only an approximation of the actual utility experienced by investor $j$. Refer to the discussion of approximation on page 13.

**Proposition 3** (Demand for shares)**.** *Inverse demand functions for shares are as follows:*

(1) *for the controlling investor:* $\sigma^2\rho(2\rho-1)/r_I\theta_I(4\rho-1)^2$ *for* $\theta_C \in \left(\frac{3}{4}r_I\theta_I, r_I\theta_I\right)$, *which is increasing in* $\theta_C$; *zero for* $\theta_C$ *smaller than* $\frac{3}{4}r_I\theta_I$; *and negative for* $\theta_C$ *larger than* $r_I\theta_I$

(2) *for the informed investor:* $\sigma^2 r_I(8\rho^2-8\rho+1)/2\theta_C(16\rho^2-8\rho+1) > 0$ *for* $\theta_I \in (0, \theta_C/r_I)$, *which is decreasing in* $\theta_I$; *and negative elsewhere*

(3) *for* $i$, *a non-controlling investor:* $\sigma^2(\theta_C - r_i\theta_i)\mathbb{E}\left[a^*(\Omega(\omega))^2\right]$, *where* $a^*(\Omega(\omega))$ *denotes the equilibrium action taken by the controlling agent when the informed agent employs reporting strategy* $\Omega(\omega)$; *this is a decreasing function of* $\theta_i$.

*Proof.*

(1) The expression is the derivative of $\overline{V}_C(\rho)$ from Proposition 2 with respect to $\theta_C$. Lemma C.1 provides the steps to the functional form and Corollary C.2 shows that demand is increasing when positive.

(2) The expression is the derivative of $\overline{V}_I(\rho)$ from Proposition 2 with respect to $\theta_I$. Lemma C.1 provides the steps to the functional form and Corollary C.4 shows that demand is increasing when positive.

(3) Suppose the controlling investor takes optimal action $a^*(\Omega(\omega))$. Then the payoff to non-controlling investor $i$ reads: $-\frac{\sigma^2}{2r_i}\mathbb{E}\left[r_i\theta_i a^*(\Omega(\omega)) - \omega\right]^2$. Differentiating with respect to $\theta_i$ and collecting terms yields the result. The steps are provided in Lemma C.5.

□

The controlling investor's inverse demand is summarized in figure 2. The different regions are noteworthy. First, inverse demand is 0 when $\theta_C < \frac{3}{4}r_I\theta_I$: this is because (1) there is no information transmission in this region and (2) the controlling investor gets to freely choose the level of $a$ and can thus costlessly compensate low share holdings, $\theta_C$, by choosing a high $a$. Inverse demand is zero because, in this model, at the margin, ownership is not valuable to an investor who has control. Second, when $\theta_C \in \left(\frac{3}{4}r_I\theta_I, r_I\theta_I\right)$ information transmission becomes possible; thus, in addition to effect (2) above, adding one more share improves alignment with the informed investor, which means that the controlling investor will receive better information. Thus her demand is positive because of this informational

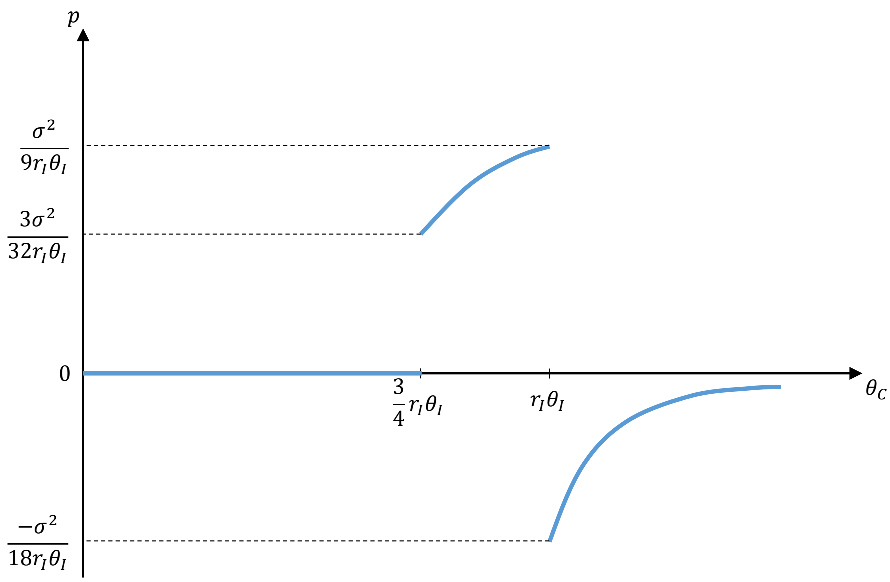


FIGURE 2. Controlling investor's demand.

trading motive. That demand is increasing reflects the increasing informational value of aligning with the informed investor. The discontinuity in demand at $r_I\theta_I$ is a measure of the informational loss from marginal miscoordination when both investors are perfectly aligned. Finally, when $\theta_C > r_I\theta_I$ demand is negative because every additional share worsens the misalignment with the informed investor, thus worsening information transmission.

The informed investor's demand is summarized in Figure 3. Because this investor does not control $a$, he values ownership and has decreasing returns from holding shares. This is why his demand for shares is decreasing. In addition, this investor also has an informational motive for trading; as his holdings align more closely with the controlling investor, he will be able to more credibly communicate information to her, so that her actions will be more informed and also more closely aligned with his own preferences. This informational motive explains the discontinuity at $\theta_C/r_I$, which measures the informational loss from marginal miscoordination when both investors are perfectly aligned.

A non-controlling investor's inverse demand is summarized in figure **??**. It is linear and downward-sloping. This demand function reflects no informational motive for trading: both the action and the information structure are taken as given by this investor. All else equal, inverse demand is increasing in $r_C\theta_C$ because, as the controlling investor becomes less inclined

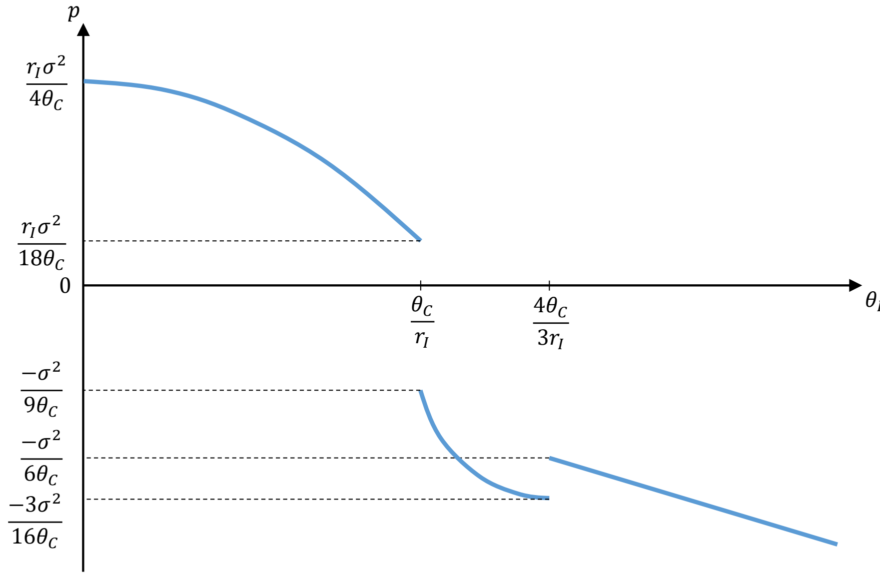


FIGURE 3. Informed investor's demand for shares.

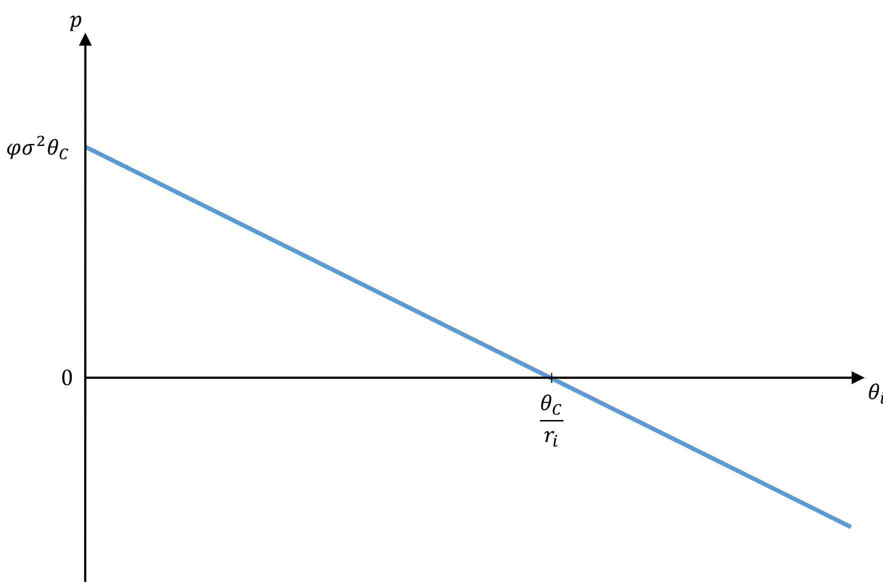


FIGURE 4. Non-controlling Investor's Demand, where $\varphi = \mathbb{E}\left[a^*\left(\Omega\left(\omega\right)\right)^2\right]$.

to take risks (larger risk-aversion parameter $r_C$, or larger holdings $\theta_C$), the venture will become less risky and thus more desirable for all non-controlling investors.

# 6. Applications

In this section we explore several applications of our model. In these applications, unless otherwise specified all investors will acquire shares in a competitive market according to the demand functions in Proposition 3. The competitive equilibrium price obtains when aggregate demand equals 1. It is possible that aggregate demand for shares stays below 1 even at zero price. In that case, for convenience we will say that the equilibrium price of a share is zero. Note that according to this definition when the equilibrium price is zero shares need not be fully allocated.

## 6.1. Welfare Benchmark

In our setting welfare depends on: achieving proper risk sharing; and achieving an informed investment strategy, which requires transparent communication between the informed investor and the controlling investor.

**Definition 3.** ***(notion of optimality)*** *A share allocation* $\theta_C, \theta_I, \{\theta_i\}$ *is welfare-maximizing if it solves:*

$$\max_{\theta_C,\theta_I,\{\theta_i\},a(\omega)} \sum_{j\in C,I,\{i\}} -\frac{\sigma^2}{2r_j}\mathbb{E}\left(\theta_j r_j a\left(\omega\right)-\omega\right)^2 \tag{6.1}$$

$$\textit{subject to} \sum_{j\in C,I,\{i\}} \theta_j \leq 1.$$

Although this notion of optimality pertains to a share allocation, it also requires that the investment strategy be optimized. Note that, in general, investors will differ as to their preferred investment strategy: indeed, the strategy that maximizes an investor's payoff function (2.1) is $a\left(\omega\right) = \omega/r\theta$, which depends on the investor's risk aversion and share holdings.

Conditional on any investment strategy $a\left(\omega\right)$, including the optimal strategy, welfare maximization boils down to an optimal risk-sharing problem. The optimality conditions for this problem obtain from the Lagrangian for problem (6.1). Differentiating the Lagrangian with respect to $\theta_j$ we get a familiar condition:

$$\theta_j^{OPT} = \frac{1}{r_j}\frac{\lambda + \ \mathbb{E}\left[a\left(\omega\right)\omega\right]}{\mathbb{E}\left[a\left(\omega\right)\right]^2}, \tag{6.2}$$

where $\lambda$ is the Lagrange multiplier. In particular, welfare maximization requires that for any two shareholders $j$ and $k$,

$$\frac{\theta_j^{OPT}}{\theta_k^{OPT}} = \frac{r_k}{r_j},$$

a condition which is independent of the investment strategy $a(\omega)$.

The next proposition shows that, despite the fact that investors will differ as to their preferred investment strategy, the non-controlling investor benefit when the controlling and informed investors can communicate more effectively. In this sense, greater information sharing creates a positive externality for the non-controlling investors.

**Proposition 4.** *The non-controlling investor's utility increases as* $\frac{\theta_C}{\theta_I} \rightarrow r_I$ *and is maximal when* $\frac{\theta_C}{\theta_I} = r_I$.

*Proof.* Follows from corollary C.7. □

### 6.2. Shares Allocation Determined By a Competitive Market

This section indicates that, absent trading frictions, the competitive equilibrium in the market for shares is welfare-maximizing, that is, it achieves both the efficient risk allocation and perfect information transmission.

**Proposition 5.** *Suppose all investors trade shares in the competitive market. Then there exists a competitive equilibrium allocation in the share market which is welfare-maximizing. At this equilibrium all investors are unanimous in the choice of investment and information is transmitted perfectly between informed and controlling investor.*

*Proof.* Take any welfare-maximizing share allocation. Since all agents share risk perfectly, condition 6.2 must hold and so $\theta_j^* r_j = K$ for all $j$. At this holdings constellation, investor $j$'s payoff reads (from 2.1):

$$-\frac{\sigma^2}{2r_j}\mathbb{E}\left(\theta_j^* r_j a(\omega) - \omega\right)^2 + m \tag{6.3}$$

$$= \quad -\frac{\sigma^2}{2r_j}\mathbb{E}\left(Ka(\omega) - \omega\right)^2 + m.$$

Therefore, for the purpose of choosing $a$, all investor's preferences are perfectly aligned. Therefore there is no obstacle to information transmission and the unanimous choice of $a$

will be the choice that maximizes the payoff, $a(\omega) = \omega/K$. The share price that supports this equilibrium is found by differentiating (6.3) with respect to $\theta_j^*$. The derivative reads:

$$\begin{aligned} & -\sigma^2 \mathbb{E}\left[a(\omega)\left(\theta_j^* r_j a(\omega) - \omega\right)\right] \\ = \quad & -\sigma^2 \mathbb{E}\left[a(\omega)\left(K a(\omega) - \omega\right)\right]. \end{aligned}$$

This expression is zero because $a(\omega) = \omega/K$. □

It may seem strange that an equilibrium with perfect communication should always exist, even when the non-controlling investors' demand for shares is so large that it threatens to price the informed investor out of the market. The reason the informed investor is not priced out of the market is that, in the welfare-maximizing equilibrium, the informed and the controlling investor can both coordinate on very small ownership fractions while keeping the ratios of their holdings at the level $\rho = 1$. When the controlling investor has a very small share of the enterprise, she will choose a very large investment level, enough to satisfy the non-controlling investors' demand for risk, thus ensuring that the equilibrium share price is low and the informed investor is not priced out.

The welfare maximizing equilibrium price is zero, consistent with the requirement that the socially optimal investment strategy be implemented (in this model risk is not scarce, and so its marginal social value must be zero). It must be noted that there are also equilibria with perfect communication but suboptimal risk sharing. In these equilibria investors $C$ and $I$ coordinate on holding more shares than in the optimal risk-sharing equilibrium, the price is above zero because the non-controlling investors are rationed for risk, and the price is below the level at which the informed and controlling investors are willing to miscoordinate. These equilibria have inefficient risk-sharing because the non-controlling investors are partially rationed whereas the controlling and informed investors are not risk-rationed. In addition, in these equilibria the controlling investor chooses too safe an investment strategy compared to the optimal risk-sharing equilibrium because she holds too many shares.

This proposition indicates that, absent trading frictions, the competitive equilibrium in the share market produces the welfare-maximizing share allocation and investment strategy; information is transmitted perfectly, and the allocation of control rights is irrelevant. It is, perhaps, for this reason that there is relatively little governmental regulation of corporate governance (allocation of control rights), particularly when compared with political governance. In politics, as opposed to the corporation, adjusting one's exposure to the governed entity is typically difficult.

This section's efficiency result can be contrasted with the inefficiency that prevails in Section 6.3. In that section, the necessity to communicate with an entrenched controlling shareholder pulls the informed investor's shareholdings away from full risk sharing with the non-controlling investor.

### 6.3. Activist Investor Communicates with an Entrenched Controlling Shareholder

An activist investor often seeks to influence the strategy of a controlling shareholder (management, board, controlling owner) by transmitting information. In this scenario, investor $I$ can be interpreted as an activist investor, and investor $C$ as the controlling shareholder. In this section we assume that the controlling shareholder is entrenched, that is, she has an exogenously given share $\theta_C$ of the enterprise. Aside from the controlling shareholder, all other investors trade shares in the competitive market. We now study the equilibrium of this game.

**Proposition 6.** *Suppose the controlling shareholder's share of the enterprise is fixed at $\theta_C$ while all investors trade shares in the competitive market. In equilibrium:*

(1) *no investor ever holds too much "skin in the game" relative to the controlling shareholder: that is,* $\max\{r_I\theta_I^*, r_i\theta_i^*\} \leq \theta_C$;
(2) *there is a threshold $\overline{\theta}_C$ such that: a welfare-maximizing equilibrium exists if and only if $\theta_C \leq \overline{\theta}_C$. This threshold is an increasing function of $r_I$, $r_i$, and a decreasing function of the number of non-controlling investors.*
(3) *there is a threshold $\overline{\overline{\theta}}_C > \overline{\theta}_C$ such that: information is perfectly transmitted between activist investor and controlling shareholder in any equilibrium if and only if $\theta_C \leq \overline{\overline{\theta}}_C$. This threshold is an increasing function of $r_I$, $r_i$, and a decreasing function of the number of non-controlling investors.*

*Proof.*

(1) The demand functions in Proposition 3 are negative for $\theta_I > \theta_C/r_I$ and $\theta_i > \theta_C/r_i$. Because the equilibrium price is nonnegative by assumption, neither investor $I$ nor investors $i$ will want to hold more than $\theta_C/r_I$ and $\theta_C/r_i$, respectively, in equilibrium.

(2) Perfect communication between controlling shareholder and informed activist takes place if and only if $r_I\theta_I^* = \theta_C$. All agents share risk perfectly if and only if $r_j\theta_j^* = \theta_C$ for $j \in I, \{i\}$ (refer to condition 6.2). The demand functions in Proposition 3 indicate that $r_i\theta_i^* = \theta_C$ if and only if the share price is zero. The equilibrium price can be zero only if aggregate demand at that price is below 1, that is, if:

$$\theta_C + \frac{\theta_C}{r_I} + \sum_i \frac{\theta_C}{r_i} \leq 1.$$

$\overline{\theta}_C$ is the solution where the condition holds with equality, whence:

$$\overline{\theta}_C = \frac{1}{1 + \frac{1}{r_I} + \sum_i \frac{1}{r_i}}.$$

(3) Perfect communication between controlling shareholder and informed activist takes place if and only if $\theta_I^* = \theta_C/r_I$; the informed activist is willing to hold this quantity of shares as long as the price does not exceed $r_I\sigma^2/18\theta_C$ (refer to Figure 3). At this price, each non-controlling investor demand is given by equating these investors' demand functions to the price (refer to Proposition 3):

$$\begin{aligned}
\sigma^2\left(\theta_C - r_i\theta_i\right)\frac{1}{3} &= \frac{r_I\sigma^2}{18\theta_C} \\
\theta_C - r_i\theta_i &= \frac{r_I}{6\theta_C} \\
\theta_i &= \frac{\theta_C}{r_i} - \frac{1}{6\theta_C}
\end{aligned}$$

A price below $r_I\sigma^2/18\theta_C$ can be an equilibrium price only if aggregate demand at a price of $r_I\sigma^2/18\theta_C$ does not exceed 1, that is, if:

$$\begin{aligned}
\theta_C + \frac{\theta_C}{r_I} + \sum_i \left(\frac{\theta_C}{r_i} - \frac{1}{6\theta_C}\right) &\leq 1 \\
\theta_C\left(1 + \frac{1}{r_I} + \sum_i \frac{1}{r_i}\right) - \sum_i \frac{1}{6\theta_C} &\leq 1.
\end{aligned}$$

$\overline{\overline{\theta}}_C$ is the solution where the condition holds with equality. Clearly, $\overline{\overline{\theta}}_C > \overline{\theta}_C$. Also, the LHS is an decreasing function of $r_I$, $r_i$, and an increasing function of the number of non-controlling investors, which proves the last statement.

□

This proposition provides conditions under which restricting the controlling shareholder's (manager, board, etc.) ability to trade shares leads to inefficiencies. There inefficiencies are several. First is a direct inefficiency due to the fact that risk may be inefficiently allocated to the controlling shareholder. Further, indirect inefficiencies arise with respect to communication, and to risk allocation between the non-entrenched investors. These indirect inefficiencies are more likely to obtain when: the share owned by the controlling agent is large enough; when there are many non-controlling investors; and when and these investors are not very risk-averse. The intuition behind this result is as follows. If demand for risk is sufficiently large, viz., when the controlling investor owns a large share of the enterprise, when there are many investors and they are not very risk-averse, the equilibrium share price will be positive and therefore the non-controlling investor $i$ will demand less than $\theta_C/r_i$. On the other hand, full communication requires the activist to hold exactly $\theta_C/r_I$. This means that if the share price is positive and full communication holds, then condition (6.2) cannot hold for both $j = i, I$, implying that risk is allocated inefficiently among non-entrenched investors. So, when the demand for risk is large something has to give: either efficient risk-sharing among non-entrenched investors, or full communication. The proposition shows that for values of $\theta_C$ in the interval $\left(\overline{\theta}_C, \overline{\overline{\theta}}_C\right)$ efficient risk-sharing is lost, but full communication is preserved in equilibrium. The reason lies in the discontinuity of the informed activist's demand function. The discontinuity reflects a strong marginal benefit coming from information transmission, at the perfect-coordination share holdings. When $\theta_C > \overline{\overline{\theta}}_C$ the equilibrium is inefficient in both dimensions: risk allocation and information transmission.

Another implication of Proposition 6 is that efficient risk sharing fails more easily than full information transmission. Or, put differently: if we observe imperfect information transmission then we can also infer imperfect risk allocation – but not vice versa.

When contrasted with the result in Section 6.2, Proposition 6 suggests that restricting the controlling agent's (manager, board, etc.) ability to trade shares can lead all other investors to actions that, collectively, are sub-optimal for them. This may have implications in the scenario where the entrenched shareholder represents a manager, and the constraints on trading shares are set as part of the manager's compensation scheme. In this scenario allowing the manager to trade shares may facilitate information transmission to the manager by investors.

Proposition 6 deals with welfare maximization as defined in Definition 3. However, a more stringent optimality benchmark for the equilibrium allocation would be to compare it to the allocation that a social planner would choose conditional on the entrenched investor keeping control over the investment strategy. More formally, we want to know whether the equilibrium allocation coincides with the allocation that solves problem (6.1) subject to the constraint that $a(\omega)$ is chosen by the entrenched investor. The answer is no, as shown in the following proposition.

**Proposition 7.** *When $\theta_C > \bar{\bar{\theta}}_C$ the equilibrium allocation is Pareto-suboptimal within the set of allocations where $a(\omega)$ is chosen by the entrenched investor.*

*Proof.* In the case $\theta_C > \bar{\bar{\theta}}_C$ the equilibrium price $p^*$ is such that the activist investor demands $\theta_I^* < \theta_C / r_I$. Consider the welfare effects of perturbing the equilibrium allocation by taking one share away from a non-controlling investor $i$ and allocating it to the activist investor. The activist's marginal gain from receiving one more share is $p^*$, which takes into account his private benefit from marginally improving communication with the controlling investor; the non-controlling investor's marginal loss from giving up one share given the equilibrium investment strategy equal is $p^*$; and then there is an additional gain for the non-controlling investor which is given by the inframarginal benefit from a better investment strategy, which results from improved communication between the activist and the controlling investor. Therefore, this perturbation results in a net welfare gain both for the non-entrenched investors and for the entrenched investor. Hence the equilibrium allocation is Pareto-suboptimal. □

The roots of the inefficiency featured in Proposition 7 lie in the mechanism that governs share allocation. A competitive market for shares fails to reward the positive externalities that an informed investor (the activist investor, in our scenario) provides by purchasing shares, when the purchase brings his holdings more in line with the controlling investor's holdings. Put differently, the non-controlling investors are free-riding, in equilibrium, on the informed investor's communicative activity.

The inefficiency featured in Proposition 7 suggests that, if the entrenched investor was an entrepreneur selling shares to the non-controlling investors through a competitive market, then the entrepreneur would find that a mechanism to increase the informed investor's share in the company would increase the company's market valuation. One such mechanism could be to make the informed investor an agent of the principal, that is, to hire him as a manager: we show this in Section 6.4.

### 6.4. Share Allocation Determined by a Principal

The inefficiency results in Section 6.3 highlight the limits of the market as a mechanism for allocating shares. The market mechanism does not sufficiently incentivize the informed investor's communicative activity. This observation suggests that sometimes competitive markets for shares can be improved upon as a mechanism for incentive provision. This leads us directly to studying a principal-agent setting. In a principal-agent setting the principal, not the market, sets the incentive scheme. We assume an informed shareholder (a manager, say) whose compensation contract is designed by the controlling shareholder (the principal). We now show that, if the space of contracts includes a fixed salary plus shares in the enterprise, the principal can achieve constrained welfare maximization, that is, a maximization of the objective function in Definition 3 conditional on the agent choosing effort and reporting strategy.

Our setting is as follows. An agent, who is indexed by $I$, has private information about an investment and transmits this information to a principal. The agent's payoff function is as in the informed investor in the previous sections, with an added moral hazard component: the agent exerts effort $e$ at a private cost $c(e)$. Effort $e$ produces output $e$. If compensated with a salary $S$ and shares $\theta_I$, the agent's payoff is:

$$u_I(e;\theta_I,S) = \underbrace{\overline{V}_I\left(\frac{1-\theta_I}{r_I\theta_I}\right)}_{\text{old preferences}} + \underbrace{\theta_I e - c(e)}_{\text{moral hazard part}} + S.$$

Output is owned by a principal, who is indexed by $C$. After communicating with the agent, the principal chooses an investment strategy and has the payoff function of the controlling investor in the previous sections; in addition, the principal owns the output $e$. The principal chooses a compensation scheme for the agent comprised of a fixed salary $S$ plus a share $\theta_I$ in the enterprise. The principal's payoff is then:

$$u_C\left(e;\theta_I,S\right)=\underbrace{\overline{V}_C\left(\frac{1-\theta_I}{r_I\theta_I}\right)}_{\text{old preferences}}+\underbrace{\left(1-\theta_I\right)e}_{\text{worker's output}}-S.$$

**Proposition 8.** *The principal's choice of compensation scheme (salary $S$ plus share $\theta_I$) is welfare-maximizing conditional on the agent choosing effort and reporting strategy. At this compensation scheme the agent receives no fewer shares than $\theta_I = 1/1+r_I$, the amount that they would receive absent the moral hazard problem.*

*Proof.* The argument is standard from the principal agent model. The principal's problem is:

$$\max_{\theta_I,S} u_C\left(e^*\left(\theta_I\right);\theta_I,S\right)\text{ subject to }\begin{array}{r}e^*\left(\theta_I\right)\in\arg\max u_I\left(e;\theta_I,S\right)\text{ (incentive compatibility for effort)}\\ u_I\left(e^*\left(\theta_I\right);\theta_I,S\right)\geq 0\text{ (individual rationality)}\end{array}$$

By standard arguments, the principal's problem can be re-written as:

$$\max_{\theta_I}\overline{V}_C\left(\frac{1-\theta_I}{r_I\theta_I}\right)+\overline{V}_I\left(\frac{1-\theta_I}{r_I\theta_I}\right)+e^*\left(\theta_I\right)-c\left(e^*\left(\theta_I\right)\right).$$

Thus the principal's problem coincides with welfare maximization under the constraint that effort and reporting strategy are chosen by the agent.

Next, note that $e^*\left(\theta_I\right)-c\left(e^*\left(\theta_I\right)\right)$ is an increasing function of $\theta_I$. To see this, differentiate:

$$\begin{aligned}&\frac{\partial}{\partial\theta_I}\left[e^*\left(\theta_I\right)-c\left(e^*\left(\theta_I\right)\right)\right]\\=\;&\frac{\partial}{\partial\theta_I}\left[\left(1-\theta_I\right)e^*\left(\theta_I\right)\right]+\frac{\partial}{\partial\theta_I}\left[\theta_Ie^*\left(\theta_I\right)-c\left(e^*\left(\theta_I\right)\right)\right]\\=\;&\frac{\partial}{\partial\theta_I}\left[\left(1-\theta_I\right)e^*\left(\theta_I\right)\right]+e^*\left(\theta_I\right)\\=\;&-e^*\left(\theta_I\right)+\left(1-\theta_I\right)e^{*\prime}\left(\theta_I\right)+e^*\left(\theta_I\right)\\=\;&\left(1-\theta_I\right)e^{*\prime}\left(\theta_I\right)>0.\end{aligned}$$

Therefore, the principal's problem is the sum of two functions that both peak when $\frac{1-\theta_I}{r_I\theta_I}=1$, that is, when $\theta_I=1/\left(1+r_I\right)$; and a third function which is increasing in $\theta_I$. □

Full efficiency cannot be expected in this setting because a single incentive scheme (shares) for the agent is used to incentivize two different activities: effort provision, and information transmission. The constrained efficiency result is quite standard: when the principal can make herself the residual claimant, her incentives to choose compensation coincide with that of a social planner's who labors under the same constraints as the principal, namely, that effort and reporting strategy are chosen by the agent. What's interesting is that in this constrained-efficient allocation the principal owns too few shares to efficiently communicate with the agent. Thus, in general the principal faces a trade-off between providing the agent with incentives to exert optimal effort, versus incentives to communicate fully.

### 6.5. Prudential Regulation

Prudential regulation has received increasing attention since the global financial crisis as it tries to limit risk-taking by a single firm if this has externalities on other stakeholders. These regulations can take two broad forms: (a) direct regulation of risk-taking behavior (e.g., for financial institutions: capital requirements, stress tests, "too big to fail" regulations, etc.), and (b) indirect regulation through checks on executive compensation.[9] In this section we use the principal-agent model in Section 6.4 to evaluate these two forms of regulation from a risk-taking perspective.

Regulatory form (b) has been recently implemented by the Bank of England. In a press release, the Bank of England explained the goals of this policy as follows:

> "The Prudential Regulation Authority (PRA) and Financial Conduct Authority (FCA) are today publishing new remuneration rules... The new framework aims to further align risk and individual reward in the banking sector, to discourage irresponsible risk-taking and short-termism and to encourage more effective risk management."[10]

The premise of the principal-agent model in Section 6.4 is that the principal, not the agent, chooses the risky investment strategy $a$. This premise is realistic if large strategic bets for a corporation require vetting by the board, i.e., the owners. If this premise is broadly correct, then the executive can advise on and propose large risk-taking strategies, but does

[9]Bebchuk and Spamann (2009) make the case for prudential regulation through checks on executive compensation.

[10]Quote from http://www.bankofengland.co.uk/publications/Documents/news/2015/ps1215.pdf.

not principally decide. The consequence of this observation is that checks on executive compensation will affect risk-taking primarily through information transmission. Our model allows us to examine this channel.

Section 6.4 shows that, within our model, optimal executive compensation, while second-best, falls short of the first best in that the executive (the agent) owns too many shares for efficient communication. This feature will lead the principal to choose an investment strategy based on imperfect information. The principal chooses this incentive scheme because he also wants to incentivize the agent's effort. Society (and the regulator), however, may care less about whether the executive works hard, and may care more about the risk-taking if there are significant risk-taking externalities. Thus, the regulator would prefer the agent to have a less-powerful incentive scheme compared to the one assigned to her by the principal, at the margin. By marginally reducing the share-based amount of compensation, the regulator can force a closer alignment between the principal's and the agent's risk-taking incentives. This leads to a more informed investment decision on the principal's part. If we conceptualize the regulator's objective function as that of a very risk-averse shareholder, then Proposition 4 tells us that the regulator benefits from greater alignment. Of note, the alignment does not result in a more cautious investment strategy: indeed, the strategy is chosen by a controlling investor (owner) whose risk-taking propensity is unaffected by the regulation of executive compensation. In this sense, the benefit for the regulator results from a *more informed* but not *more cautious* investment strategy.

Turning now to regulatory form (a): direct regulation of risk-taking behavior. Let's imagine that the regulators discourage risk-taking by penalizing the controlling investor with $\kappa a^2$ for any investment level $a$. The parameter $\kappa > 0$ captures the tightness of the regulation; the quadratic specification is chosen purely for analytical convenience. The regulatory penalty function $\kappa a^2$ captures in reduced form the regulator's legal capacity to constrain risk-taking behavior. The penalty is not intended as a monetary cost, and thus enterprise value is not affected by it. Let's further assume, at least initially, that the agent's compensation scheme is fixed independent of $\kappa$.

Subtracting the regulatory penalty function from the controlling investor's payoff (2.1) and repeating the steps on page 34 shows that the controlling investor's equilibrium investment strategy takes the form:

$$\frac{\omega_{n+1} + \omega_n}{2\theta_C + \kappa},$$

which corresponds to the investment strategy of a controlling investor with shareholdings $\theta_C + \kappa/2$ (cf. Proposition 1 Part 1). Thus, if risk-taking behavior is directly regulated with intensity $\kappa$, the amount of information transmitted in equilibrium is the same as that in the unregulated cheap talk game with a controlling investor with holdings $\theta_C + \kappa/2$. Increasing $\kappa$ slightly above zero is beneficial in terms of information transmission because it increases alignment between the principal's and the agent's payoffs. Moreover, there is also a direct effect of making the principal (controlling agent) effectively less risk-tolerant and thus less prone to risk-taking. In sum, direct regulation of risk-taking behavior improves matters on two fronts: a direct effect (less risk-taking) and, also, an indirect effect (more accurate investment strategy due to greater information transmission).

The previous paragraph assumes that the agent's compensation scheme is fixed independent of $\kappa$. But what if the agent's compensation scheme is chosen by the principal partly as a function of $\kappa$? Since increasing $\kappa$ is functionally equivalent to having a more risk-averse principal, we conjecture that the optimal compensation scheme from Section 6.4 would transfer more risk to the agent, that is, the compensation scheme would become more stock-based. If this conjecture is true, this would be an intriguing indirect effect of the conventional form of risk-taking regulation. But, to re-emphasize: as mentioned before, in our model giving the agent a more high-powered incentive scheme has no direct effect on risk-taking behavior.

### 6.6. Microfounding the Controlling Shareholder based on Collective Decision-making among Shareholders

Throughout the paper we have assumed that one of the investors is given the power to choose the investment strategy. This assumption turned out to be innocuous in the setting of Section 6.2, where no disagreement existed about the investment strategy after shareholders traded on the share market. But, if there is disagreement among investors, who is to be chosen as controlling investor? In this section we offer a model where the investment action is chosen collectively among shareholders. To allow maximal scope for disagreement, we assume that all the non-informed shareholders' holdings are fixed exogenously. This assumption could be relaxed to some degree (e.g., as done in Section 6.4).

There is a continuum of shareholders with measure 1. Let $i$ index all shareholders except shareholder $I$, the informed shareholder. Shareholder $i$ has risk aversion parameter $r_i$ and is exogenously assigned holdings $\theta_i$. Denote $h_i = r_i \cdot \theta_i$. Both the cumulative distribution $F(i)$ of the index $i$, and the distribution of $h_i$ vary smoothly with $i$.[11] The investment decision is determined collectively as follows: first the informed investor communicates information publicly, then all shareholders simultaneously submit an investment recommendation, and finally the median of the share-weighted distribution of recommendations is implemented.

In this setting it is a dominant strategy for each shareholder to make a sincere recommendation, i.e., to recommend the investment action that he would choose if given the opportunity to choose. These sincere recommendations are ordered by the index $r_i \cdot \theta_i$. To see this, observe that all investors will recommend the action that maximizes their payoff (expression 2.1) conditional on whatever posterior $G(\omega)$ the informed investor's public communication has generated in equilibrium. Thus investor $i$ recommends the investment level $a$ that solves:

$$\max_a - \int \frac{\sigma^2}{2r_i} (h_i a - \omega)^2 \; dG(\omega).$$

The first order conditions for this problem are:

$$- \int (h_i a - \omega) \; dG(\omega) = 0,$$

and it is immediate to see that shareholder $i$'s recommendation is monotonically decreasing in $h_i$. This monotonicity implies that the collectively-chosen policy corresponds to the share-weighted median of the distribution of the $h_i$'s. This is a primitive of our model, which is obtained by ordering investors according to the value of their $h_i$, and then picking the share-weighted median of the distribution. The investor at that median is the controlling shareholder, i.e., the shareholder indexed by $C$ in the rest of the paper.

### 6.7. Delegation

Dessein's (2002) seminal paper shows that it may be in the controlling investor's interest to delegate control (in Dessein's words: authority) to the informed investor. In Desseins' (2002) model, this is the case if the controlling investor's and the informed investor's preferences are sufficiently aligned. A similar result obtains in our model.

[11] In Section 6.2, the distribution of $r_i \cdot \theta_i$ is degenerate and thus does not vary smoothly with $i$.

**Proposition 9.** *Fix a risk-adjusted holdings ratio* $\rho$. *If* $1/2 \leq \rho \leq 5/4$ *the controlling investor has a higher equilibrium payoff under delegation than in any cheap talk equilibrium. If* $\rho > 5/4$ *there are cheap talk equilibria in which the investor's payoff is higher than under delegation.*

*Proof.* See Appendix D. □

This proposition indicates that delegation is in the controlling investor's self-interest if and only if $\rho$ is in a neighborhood of 1, that is, if and only if preferences are sufficiently aligned. In our setting, however, $\rho$ is endogenous. If we allow shares to be traded, then in equilibrium there is a strong tendency toward $\rho = 1$. This tendency is present in Proposition 6 and it is strongest in Proposition 5. When $\rho = 1$ communication is perfect and the controlling investor achieves a perfectly informed decision without the need to give up control. The ability to retain control may be valued by investors because, in environments where ownership and control are linked, giving up control may be difficult to execute without giving up some ownership too.

## 7. Conclusion

This paper has studied information transmission about a risky investment prospect when the incentives to communicate are shaped by ownership of shares in the enterprise. We have provided conditions under which share trading delivers a welfare-maximal allocation: perfect communication and full risk-sharing. When frictions lead to a welfare-suboptimal outcome, we have identified a novel source of inefficiency: a competitive market for shares fails to reward the positive externalities that an informed investor provides to other investors by purchasing shares. In general, we have found that the free trading of shares tends to align incentives and thus to facilitate communication.

Much of the literature on information transmission in organizations has focused on optimizing the allocation of control, following Dessein's (2002) pioneering insight. This paper suggests that, in organizations where ownership can be easily adjusted, allowing the market to allocate ownership can also help solve the problem of information transmission. It is, we believe, for this reason that there is relatively little governmental regulation of governance (allocation of control rights) in public corporations, particularly when compared with political governance: in politics, in comparison to the corporation, adjusting one's exposure to the governed entity is typically difficult.

Within a principal-agent setting, the model has highlighted a novel trade-off between incentivizing effort provision and promoting information transmission. Also, the model has delivered insights about the effects of prudential regulation.

These insights have required the analysis of a new theoretical model of information transmission. The model is tailored to financial applications and differs from the functional form originally developed by Crawford and Sobel (1982), and still prevalent today in analyses of information transmission. The analysis of this model is innovative from a technical perspective, and thus should be considered an additional contribution of this paper.

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# Appendix A. Proofs

## A.1. Lemma A.1

**Lemma A.1.** *Up to* $\frac{\sigma^2}{2r}\omega^2$ *(an additive constant that does not depend on* $a$ *or* $\theta$*), the agents' payoffs can be written as:*

$$-\frac{\sigma^2}{2r}\left(r\theta a-\omega\right)^2.$$

*Proof.*

$$\begin{aligned}
&\theta a\mu-\frac{r}{2}\theta^2a^2\sigma^2\\
=\;&-\sigma^2\left(-\theta a\frac{\mu}{\sigma^2}+\frac{r}{2}\theta^2a^2\right)\\
=\;&-\sigma^2\left(-\theta a\frac{\mu}{\sigma^2}+\frac{r}{2}\theta^2a^2+\frac{\mu^2}{2r\sigma^4}-\frac{\mu^2}{2r\sigma^4}\right)\\
=\;&-\frac{\sigma^2}{2r}\left(r^2\theta^2a^2-2r\theta a\frac{\mu}{\sigma^2}+\frac{\mu^2}{\sigma^4}\right)+\sigma^2\frac{\mu^2}{2r\sigma^4}\\
=\;&-\frac{\sigma^2}{2r}\left(ra\theta-\frac{\mu}{\sigma^2}\right)^2+\frac{1}{2}\frac{\mu^2}{r\sigma^2}\\
=\;&-\frac{\sigma^2}{2r}\left(r\theta a-\omega\right)^2+\frac{\sigma^2}{2r}\omega^2.
\end{aligned}$$

□

# Appendix B. Information Transmission with Fixed Shareholdings

We start by providing a new, to our knowledge, characterization of the ratios of two consecutive-ordered Chebyshev polynomial of the second kind.

**Lemma B.1** (Ratios of Chebyshev polynomials of consecutive order)**.**

1. *Suppose* $|x|<1$. *Then* $U_{n-1}(x)/U_n(x)<1$ *if and only if* $n<\frac{1}{2}\left(\frac{\pi}{\arccos x}-1\right)$.
2. *Suppose* $x>1$. *Then* $U_{n-1}(x)/U_n(x)<1$ *for all* $n$, *and* $\lim_{n\to\infty}U_{n-1}(x)/U_n(x)=\exp(-\operatorname{arccosh}x)$.

*Proof.*

(1) Denote $\phi = \arccos x$. Since $|x| < 1$,

$$\begin{aligned} & sgn\left[U_n(x) - U_{n-1}(x)\right] \\ = \; & sgn\left[\sin(n+1)\phi - \sin n\phi\right]. \end{aligned}$$

Now,

$$\begin{aligned} & \sin(n+1)\phi - \sin n\phi \\ = \; & \sin\left(n\phi + \frac{\phi}{2} + \frac{\phi}{2}\right) - \sin\left(n\phi + \frac{\phi}{2} - \frac{\phi}{2}\right) \\ = \; & 2\cos\left(n\phi + \frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right). \end{aligned} \tag{B.1}$$

where we have used the product-to-sum identity $\sin(x+\rho) - \sin(x-\rho) = 2\cos(x)\sin(\rho)$. Let's start by signing the second term; by the half-angle formula,

$$\sin\left(\frac{\phi}{2}\right) = \frac{1-\cos(\phi)}{2} sgn\left(2\pi - \phi + 4\pi\left\lfloor\frac{\phi}{4\pi}\right\rfloor\right).$$

By definition, $\phi = \arccos(x) \in [0, \pi]$, and so the $sgn$ operator returns $+1$. Therefore, the whole right-hand side is positive, which shows that $\sin(\phi/2) > 0$. Thus, expression (B.1) has the same sign as its first term. That term, $\cos(n\phi - \phi/2)$, is positive as long as:

$$n\phi + \frac{\phi}{2} < \frac{\pi}{2}.$$

Solve for $n$ to get:

$$n < \frac{1}{2}\left(\frac{\pi}{\phi} - 1\right).$$

(2) $U_n(x) > U_{n-1}(x)$ because $\operatorname{arccosh} x > 0$ by definition. Denote $\phi = \operatorname{arccosh} x$. Since $x > 1$,

$$\begin{aligned} & \frac{U_{n-2}}{U_{n-1}} \\ = \; & \frac{\sinh(\phi(n-1))}{\sinh(\phi n)} \\ = \; & \frac{\exp(\phi(n-1)) - \exp(-\phi(n-1))}{\exp(\phi n) - \exp(-\phi n)}, \end{aligned}$$

where the second equality is De Moivre's hyperbolic formula. Letting $n \to \infty$ and recalling that $\phi = \operatorname{arccosh} x > 0$ we get

$$\begin{aligned} & \lim_{N\to\infty} \frac{U_{n-2}}{U_{n-1}} \\ = & \frac{\exp\left(\phi\left(n-1\right)\right)}{\exp\left(\phi n\right)} \\ = & \exp\left(-\phi\right). \end{aligned}$$

□

Lemma B.1 characterizes the family of Chebyshev polynomial of the second kind. This family of polynomials is closely related to the equilibrium communication strategy because both solve equation (4.1) .

B.1. **Proof of Proposition 1**

*Proof.* **Part 1.** In equilibrium, after receiving a message the controlling investor (receiver) knows that the state $\omega$ is distributed uniformly over some interval $(\omega_n, \omega_{n+1})$. Then her equilibrium action $a_C^*(\omega_n, \omega_{n+1})$ is given by

$$a^*\left(\omega_n, \omega_{n+1} | \theta_C\right) = \arg\max_a - \int_{\omega_n}^{\omega_{n+1}} \left(a\theta_C - \omega\right)^2 d\omega.$$

The first-order condition w.r.t. $a$ reads:

$$\begin{aligned} 0 &= \int_{\omega_n}^{\omega_{n+1}} \left(a\theta_C - \omega\right) d\omega \\ &= a\theta_C\left(\omega_{n+1} - \omega_n\right) - \frac{\omega_{n+1}^2 - \omega_{n+1}^2}{2} \\ &= a\theta_C - \frac{\omega_{n+1} + \omega_n}{2}. \end{aligned}$$

Thus, the investor's optimal action is:

$$a_C^*\left(\omega_n, \omega_{n+1}\right) = \frac{\omega_{n+1} + \omega_n}{2\theta_C}.$$

**Part 2.** Let's now turn to the informed investor's strategy. At a cutpoint between different intervals, the sender needs to be indifferent between inducing the equilibrium action associated to either interval. Thus if the sender knows that $\omega = \omega_n$ it must be that:

$$\left(\frac{\omega_{n+1} + \omega_n}{2\theta_C}\theta_I r_I - \omega_n\right)^2 = \left(\frac{\omega_n + \omega_{n-1}}{2\theta_C}\theta_I r_I - \omega_n\right)^2.$$

Taking square roots on both sides of the equation while switching the sign of the right-hand side yields:

$$\frac{\omega_{n+1} + \omega_n}{2\theta_C}\theta_I r_I - \omega_n = -\frac{\omega_n + \omega_{n-1}}{2\theta_C}\theta_I r_I + \omega_n$$

Rearranging we get the following difference equation:

$$\omega_{n+1} = 2\left(2\frac{\theta_C}{r_I\theta_I} - 1\right)\omega_n - \omega_{n-1} \tag{B.2}$$

which, when incremented, reads as in (4.1).

**Part 3.** For any $k_\rho$, all cheap talk equilibrium partitions satisfy equation (4.1) with initial condition $\omega_0 = 0$. Chebyshev polynomials $U_n(k_\rho)$ are the unique solutions to (4.1) with initial conditions $U_{-1}(k_\rho) = 0, U_0(k_\rho) = 1$. Thus if we denote $\omega_1 = \alpha$ then all cheap talk equilibria must take the form $\{\omega_n(k_\rho)\}_n = \{\alpha U_{n-1}(k_\rho)\}_n$. That $\alpha = U_{N-1}(k_\rho)$ follows from the equilibrium requirement that $\omega_N = 1$.

**Part 4.** $\rho > 1$ is equivalent to $k_\rho > 1$. In this case $\omega_n(k_\rho) = \alpha U_{n-1}(k_\rho)$ is monotonic in $n$. Furthermore, for any $N$ one can choose $\alpha$ such that $\alpha U_{N-1}(k_\rho) = 1$, and thus construct a communication equilibrium with exactly $N$ partitions. But, although the number of partitions can be arbitrarily large, it does not follow that all partitions are arbitrarily small. To see this, fix any equilibrium and compute the width of the topmost partition:

$$\begin{aligned} & \omega_N - \omega_{N-1} \\ = \; & 1 - \alpha U_{N-2}(k_\rho) \\ = \; & 1 - \frac{U_{N-2}(k_\rho)}{U_{N-1}(k_\rho)}, \end{aligned}$$

where the equalities make repeated use of the identity $1 = \omega_N = \alpha U_{N-1}$. Letting $N \to \infty$ implies taking the limit across different equilibria. Using Lemma B.1 part 2 we have:

$$\lim_{N\to\infty} \omega_N - \omega_{N-1} = 1 - \exp(-\operatorname{arccosh} k_\rho) > 0.$$

The strict inequality holds because by definition $\operatorname{arccosh} k_\rho > 0$. This shows that as we move across equilibria with an increasing numbers of partitions $N$, even though $N$ grows without bound, the width of the topmost partition converges to a positive number and, in particular, it does not vanish. Therefore, the extent of communication in this class of equilibria is uniformly bounded away from full communication.

**Part 5.** $\rho < 1$ is equivalent to $|k_\rho| < 1$. In this case the Chebyshev polynomial $U_n(k_\rho)$ is not monotonic in $n$, and so we need to worry about the monotonicity constraint $\omega_n > \omega_{n-1}$ which is required by the equilibrium definition. (Trying to construct an equilibrium with too many partitions will run into the monotonicity constraint, and the maximal number of partitions $N_\rho$ consistent with equilibrium is found by making the constraint bind.) The monotonicity constraint requires that, for all $n = 0, ..., N$,

$$\begin{aligned} 0 &< sgn\left[\omega_n - \omega_{n-1}\right] \\ &= sgn\left[U_{n-1}(k_\rho) - U_{n-2}(k_\rho)\right]. \end{aligned}$$

In Lemma B.1 part 1, we saw that the above holds if $n - 1 < \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right)$, or $n < \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} + 1\right)$. Thus, the maximal integer $N_\rho$ that is consistent with this inequality is:

$$\begin{aligned} N_\rho &= \left\lceil \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} + 1\right)\right\rceil - 1 \\ &= \left\lceil \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right)\right\rceil. \end{aligned}$$[12]

This $N_\rho$ is increasing in $k_\rho$ because $arccos(\cdot)$ is a decreasing function over the interval $[-1, 1]$, and has a vertical asymptote at $k_\rho = 1$ because $arccos(1) = 0$. No information can be communicated for $k_\rho < 1/2$ because in this range $\arccos(k_\rho) > \pi/3$ and so $N_\rho = 1$. □

### B.2. Expected Utility Calculations

The order-$n$ Chebyshev polynomial of the first kind, $T_n(x)$, is a polynomial function of $x$ defined as the unique solution to the following functional difference equation:

$$T_0(x) \equiv 0,$$

$$T_1(x) \equiv 1,$$

$$T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0.$$

The recursion that gives rise to the polynomials is the same as for polynomial of the second kind, but the initial conditions are different. Nevertheless, the two classes of polynomials are closely related. The relation we will use is the following:

$$T_n(x) = U_n(x) - xU_{n-1}(x).$$

We start by providing new, to our knowledge, explicit expressions for some useful sums of series of Chebyshev polynomials.

**Lemma B.2** (Sums Involving Cubes of Chebyshev Polynomials)**.** *We note the following identities concerning Chebyshev polynomials:*

(1) $\sum_{n=0}^{N-1}\left[U_n(x) - U_{n-1}(x)\right]^3 = \begin{cases} \frac{1-x}{2x+1}\left(2 - 3\frac{1+x}{\sin^2(N\arccos x)}\right)U_{N-1}(x)^3 & \text{if } |x| \leq 1 \\ \frac{1-x}{2x+1}\left(2 + 3\frac{1+x}{\sinh^2(N\operatorname{arccosh} x)}\right)U_{N-1}(x)^3 & \text{if } x \geq 1. \end{cases}$

(2) $\sum_{n=0}^{N-1}\left[\left(U_n(x) - U_{n-1}(x)\right)\left(U_n(x) + U_{n-1}(x)\right)^2\right]$

$= \begin{cases} \frac{2+2x}{1+2x}\left(1 - \frac{1+x}{2\sin^2(N\arccos x)}\right)U_{N-1}(x)^3 & \text{if } |x| \leq 1 \\ \frac{2+2x}{1+2x}\left(1 + \frac{1+x}{2\sinh^2(N\operatorname{arccosh} x)}\right)U_{N-1}(x)^3 & \text{if } x \geq 1. \end{cases}.$

(3) $\sum_{n=0}^{N-1} T_{n+1}(x)^3 + T_n(x)^3 = \begin{cases} \frac{(x+1)^2(x-1)}{2x+1}\left(2x - 1 - 3\frac{x}{\sin^2(N\arccos x)}\right)U_{N-1}(x)^3 & \text{if } |x| \leq 1 \\ \frac{(x+1)^2(x-1)}{2x+1}\left(2x - 1 + 3\frac{x}{\sinh^2(N\operatorname{arccosh} x)}\right)U_{N-1}(x)^3 & \text{if } x \geq 1. \end{cases}$

*Proof.*

(1) We will rewrite Chebyshev polynomials in terms of exponentials. Note that by De Moivre's theorem:

$$\sin n\phi = \frac{e^{n\iota\phi} - e^{-n\iota\phi}}{2\iota} \text{ and } \sinh n\phi = \frac{e^{n\phi} - e^{-n\phi}}{2},$$

where we define $\phi = \arccos k_\rho$. Thus we may write:

$$\begin{aligned} U_{n-1}(k_\rho) &= \frac{e^{n\phi\iota^{\mathbf{1}[k_\rho<1]}} - e^{-n\phi\iota^{\mathbf{1}[k_\rho<1]}}}{e^{\phi\iota^{\mathbf{1}[k_\rho<1]}} - e^{-\phi\iota^{\mathbf{1}[k_\rho<1]}}} \\ &= \frac{e^{\xi n\phi} - e^{-\xi n\phi}}{e^{\xi\phi} - e^{-\xi\phi}}, \end{aligned}$$

where $\xi := \iota^{\mathbf{1}_{[k_\rho<1]}}$. Now (dropping $\theta$ subscripts for simplicity):

$$\begin{aligned}
\sum_{n=0}^{N-1}\left(U_{n-1}\left(k\right)-U_n\left(k\right)\right)^3 &= \sum_{n=0}^{N-1}\left(\frac{e^{\xi n\phi}-e^{-\xi n\phi}}{e^{\xi\phi}-e^{-\xi\phi}}-\frac{e^{\xi(n+1)\phi}-e^{-\xi(n+1)\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3 \\
&= \sum_{n=0}^{N-1}\left(\frac{1}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\left(e^{\xi n\phi}-e^{-\xi n\phi}-e^{\xi(n+1)\phi}+e^{-\xi(n+1)\phi}\right)^3 \\
&= \sum_{n=0}^{N-1}\left(\frac{1}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\left(e^{\xi n\phi}-e^{\xi(n+1)\phi}+e^{-\xi(n+1)\phi}-e^{-\xi n\phi}\right)^3 \\
&= \sum_{n=0}^{N-1}\left(\frac{1}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\left(e^{\xi n\phi}-e^{\xi(n+1)\phi}+e^{-\xi(n+1)\phi}-e^{-\xi n\phi}\right)^3 \\
&= \sum_{n=0}^{N-1}\left(\frac{1}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\left(e^{\xi n\phi}\left(1-e^{\xi\phi}\right)+e^{-\xi(n+1)\phi}\left(1-e^{\xi\phi}\right)\right)^3 \\
&= \sum_{n=0}^{N-1}\left(\frac{1-e^{\xi\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\left(e^{\xi n\phi}+e^{-\xi(n+1)\phi}\right)^3 \\
&= \left(\frac{1-e^{\xi\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\sum_{n=0}^{N-1}\left(e^{\xi n\phi}+e^{-\xi(n+1)\phi}\right)^3.
\end{aligned}$$

Let us work on computing the sum and we'll add the coefficient later:

$$\begin{aligned}
& \sum_{n=0}^{N-1}\left(e^{\xi n\phi}+e^{-\xi(n+1)\phi}\right)^3 \\
= & \sum_{n=0}^{N-1}\left(e^{3\xi n\phi}+3e^{\xi(n-1)\phi}+3e^{-\xi(n+2)\phi}+e^{-3\xi(n+1)\phi}\right) \\
= & \frac{e^{3\xi N\phi}-1}{e^{3\xi\phi}-1}+3\frac{e^{\xi(N-1)\phi}-e^{-\xi\phi}}{e^{\xi\phi}-1} \\
& +3\frac{e^{-\xi(N+2)\phi}-e^{-2\xi\phi}}{e^{-\xi\phi}-1}+\frac{e^{-3\xi(N+1)\phi}-e^{-3\xi\phi}}{e^{-3\xi\phi}-1} \\
= & \frac{e^{3\xi N\phi}-1}{e^{3\xi\phi}-1}+\frac{e^{-3\xi N\phi}-1}{1-e^{3\xi\phi}}+3\frac{e^{\xi(N-1)\phi}-e^{-\xi\phi}}{e^{\xi\phi}-1}+3\frac{e^{-\xi(N+1)\phi}-e^{-\xi\phi}}{1-e^{\xi\phi}} \\
= & \frac{e^{3\xi N\phi}-1+1-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}+3\frac{e^{\xi(N-1)\phi}-e^{-\xi\phi}+e^{-\xi\phi}-e^{-\xi(N+1)\phi}}{e^{\xi\phi}-1} \\
= & \frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}+3\frac{e^{\xi(N-1)\phi}-e^{-\xi(N+1)\phi}}{e^{\xi\phi}-1}.
\end{aligned}$$

So that:

$$\sum_{n=0}^{N-1}\left(U_{n-1}\left(k\right)-U_{n}\left(k\right)\right)^{3}=\left(\frac{1-e^{\xi\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^{3}\left(\frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}+3\frac{e^{\xi(N-1)\phi}-e^{-\xi(N+1)\phi}}{e^{\xi\phi}-1}\right).$$

Now:

$$\begin{aligned}\left(\frac{1-e^{\xi\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^{3}3\frac{e^{\xi(N-1)\phi}-e^{-\xi(N+1)\phi}}{e^{\xi\phi}-1} &= -3\frac{\left(1-e^{\xi\phi}\right)^{2}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^{3}}\left(e^{\xi(N-1)\phi}-e^{-\xi(N+1)\phi}\right)\\ &= -3\frac{\left(1-e^{\xi\phi}\right)^{2}e^{-\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^{2}}\frac{e^{\xi N\phi}-e^{-\xi N\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\\ &= -3\frac{\left(e^{\xi\phi}+e^{-\xi\phi}-2\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^{2}}U_{N-1}\left(k_{\rho}\right)\\ &= -6\frac{\left(k_{\rho}-1\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^{2}}U_{N-1}\left(k_{\rho}\right),\end{aligned}$$

where the last line follows by $e^{\xi\phi}+e^{-\xi\phi}=2k_{\rho}$, since $2\cos\phi=e^{\iota\phi}+e^{-\iota\phi}$ and $2\cosh\phi=e^{\phi}+e^{-\phi}$.

Similarly, we find that:

$$\begin{aligned}
&\left(\frac{1-e^{\xi\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3 \frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1} \\
= \quad & \frac{\left(1-e^{\xi\phi}\right)^3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \frac{1}{e^{3\xi\phi}-1} \frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{\xi\phi}-e^{-\xi\phi}} \\
= \quad & \frac{1-3e^{\xi\phi}+3e^{2\xi\phi}-e^{3\xi\phi}}{\left(e^{3\xi\phi}-1\right)\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)\left(e^{2\xi N\phi}+e^{-2\xi N\phi}+1\right)}{e^{\xi\phi}-e^{-\xi\phi}} \\
= \quad & \frac{1-e^{3\xi\phi}-3e^{\xi\phi}\left(1-e^{\xi\phi}\right)}{\left(e^{3\xi\phi}-1\right)} \frac{U_{N-1}\left(k_\rho\right)\left(e^{2\xi N\phi}-2+e^{-2\xi N\phi}+3\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \\
= \quad & \left(-1+\frac{3e^{\xi\phi}\left(1-e^{\xi\phi}\right)}{\left(1-e^{\xi\phi}\right)\left(1+e^{\xi\phi}+e^{2\xi\phi}\right)}\right) U_{N-1}\left(k_\rho\right) \frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2+3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \\
= \quad & \left(-1+\frac{3e^{\xi\phi}}{1+e^{\xi\phi}+e^{2\xi\phi}}\right) U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \left(-\frac{1-2e^{\xi\phi}+e^{2\xi\phi}}{1+e^{\xi\phi}+e^{2\xi\phi}}\right) U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \left(-\frac{e^{\xi\phi}e^{-\xi\phi}-2e^{\xi\phi}+e^{2\xi\phi}}{e^{\xi\phi}e^{-\xi\phi}+e^{\xi\phi}+e^{2\xi\phi}}\right) U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \left(-\frac{e^{-\xi\phi}-2+e^{\xi\phi}}{e^{-\xi\phi}+1+e^{\xi\phi}}\right) U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \frac{2-2k_\rho}{2k_\rho+1} U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right),
\end{aligned}$$

where we have used that $2k_\rho = e^{\iota\phi}+e^{-\iota\phi}$.

Putting everything together:

$$\begin{aligned}
& \sum_{n=0}^{N-1} \left(U_{n-1}\left(k_\rho\right) - U_n\left(k_\rho\right)\right)^3 \\
= \; & \frac{2-2k_\rho}{2k_\rho+1} U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2 + \frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) - 6\frac{\left(k_\rho-1\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} U_{N-1}\left(k_\rho\right) \\
= \; & U_{N-1}\left(k_\rho\right)\left(\frac{1-k_\rho}{k_\rho+1/2}\left(U_{N-1}\left(k_\rho\right)^2 + \frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) + 6\frac{\left(1-k_\rho\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \; & \frac{1-k_\rho}{k_\rho+1/2}\left(1 + \frac{3}{U_{N-1}\left(k_\rho\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} + 6\frac{k_\rho+1/2}{U_{N-1}\left(k_\rho\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) U_{N-1}\left(k_\rho\right)^3 \\
= \; & \frac{1-k_\rho}{k_\rho+1/2}\left(1 + 6\frac{k_\rho+1}{U_{N-1}\left(k_\rho\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) U_{N-1}\left(k_\rho\right)^3 \\
= \; & \frac{1-k_\rho}{k_\rho+1/2}\left(1 + 6\frac{1+k_\rho}{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2}\right) U_{N-1}\left(k_\rho\right)^3
\end{aligned}$$

The result follows after noting that:

$$\begin{aligned}
\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2 &= \begin{cases} \left(2\iota \sin N\phi\right)^2 & \text{if } \left|k_\rho\right| \le 1 \\ \left(2\sinh N\phi\right)^2 & \text{if } k_\rho \ge 1 \end{cases} \\
&= \begin{cases} -4\sin^2 N\phi & \text{if } \left|k_\rho\right| \le 1 \\ 4\sinh^2 N\phi & \text{if } k_\rho \ge 1 \end{cases} .
\end{aligned}$$

(2) Note that we may write:

$$\begin{aligned}
U_{n-1}\left(k_\rho\right) &= \frac{e^{n\phi\iota^{\mathbf{1}\left[k_\rho<1\right]}} - e^{-n\phi\iota^{\mathbf{1}\left[k_\rho<1\right]}}}{e^{\phi\iota^{\mathbf{1}\left[k_\rho<1\right]}} - e^{-\phi\iota^{\mathbf{1}\left[k_\rho<1\right]}}} \\
&= \frac{e^{\xi n\phi}-e^{-\xi n\phi}}{e^{\xi\phi}-e^{-\xi\phi}},
\end{aligned}$$

where $\xi := \iota^{\mathbf{1}_{[k_\rho<1]}}$. Thus (dropping $\theta$ subscripts for simplicity):

$$
\begin{aligned}
& \sum_{n=0}^{N-1} \left(U_n(x) - U_{n-1}(x)\right)\left(U_n(x) + U_{n-1}(x)\right)^2 \\
= & \sum_{n=0}^{N-1} \left(\frac{e^{\xi(n+1)\phi} - e^{-\xi(n+1)\phi}}{e^{\xi\phi} - e^{-\xi\phi}} - \frac{e^{\xi n\phi} - e^{-\xi n\phi}}{e^{\xi\phi} - e^{-\xi\phi}}\right)\left(\frac{e^{\xi(n+1)\phi} - e^{-\xi(n+1)\phi}}{e^{\xi\phi} - e^{-\xi\phi}} + \frac{e^{\xi n\phi} - e^{-\xi n\phi}}{e^{\xi\phi} - e^{-\xi\phi}}\right)^2 \\
= & \sum_{n=0}^{N-1} \left(\frac{1}{e^{\xi\phi} - e^{-\xi\phi}}\right)^3 \left(e^{\xi(n+1)\phi} - e^{-\xi(n+1)\phi} - e^{\xi n\phi} + e^{-\xi n\phi}\right)\left(e^{\xi(n+1)\phi} - e^{-\xi(n+1)\phi} + e^{\xi n\phi} - e^{-\xi n\phi}\right)^2 \\
= & \sum_{n=0}^{N-1} \left(\frac{1}{e^{\xi\phi} - e^{-\xi\phi}}\right)^3 \left(e^{\xi(n+1)\phi} - e^{\xi n\phi} + e^{-\xi n\phi} - e^{-\xi(n+1)\phi}\right)\left(e^{\xi(n+1)\phi} + e^{\xi n\phi} - e^{-\xi n\phi} - e^{-\xi(n+1)\phi}\right)^2 \\
= & \sum_{n=0}^{N-1} \left(\frac{1}{e^{\xi\phi} - e^{-\xi\phi}}\right)^3 \left(e^{\xi n\phi}\left(e^{\xi\phi} - 1\right) + e^{-\xi(n+1)\phi}\left(e^{\xi\phi} - 1\right)\right)\left(e^{\xi n\phi}\left(e^{\xi\phi} + 1\right) - e^{-\xi(n+1)\phi}\left(e^{\xi\phi} + 1\right)\right)^2 \\
= & \sum_{n=0}^{N-1} \frac{\left(e^{\xi\phi} - 1\right)\left(e^{\xi\phi} + 1\right)^2}{\left(e^{\xi\phi} - e^{-\xi\phi}\right)^3}\left(e^{\xi n\phi} + e^{-\xi(n+1)\phi}\right)\left(e^{\xi n\phi} - e^{-\xi(n+1)\phi}\right)^2 \\
= & \frac{\left(e^{\xi\phi} - 1\right)\left(e^{\xi\phi} + 1\right)^2}{\left(e^{\xi\phi} - e^{-\xi\phi}\right)^3} \sum_{n=0}^{N-1} \left(e^{\xi n\phi} + e^{-\xi(n+1)\phi}\right)\left(e^{\xi n\phi} - e^{-\xi(n+1)\phi}\right)^2
\end{aligned}
$$

Now, we have that:

$$
\begin{aligned}
& \sum_{n=0}^{N-1} \left(e^{\xi n\phi} + e^{-\xi(n+1)\phi}\right)\left(e^{\xi n\phi} - e^{-\xi(n+1)\phi}\right)^2 \\
= & \sum_{n=0}^{N-1} \left(e^{2\xi n\phi} - e^{-2\xi(n+1)\phi}\right)\left(e^{\xi n\phi} - e^{-\xi(n+1)\phi}\right) \\
= & \sum_{n=0}^{N-1} \left(e^{3\xi n\phi} - e^{2\xi n\phi}e^{-\xi(n+1)\phi} - e^{\xi n\phi}e^{-2\xi(n+1)\phi} + e^{-3\xi(n+1)\phi}\right) \\
= & \frac{e^{3\xi N\phi} - 1}{e^{3\xi\phi} - 1} - \frac{e^{-2\xi\phi} - e^{\xi(N-2)\phi}}{e^{-\xi\phi} - 1} - \frac{e^{-\xi(N+2)\phi} - e^{-2\xi\phi}}{e^{-\xi\phi} - 1} + \frac{1 - e^{-3\xi N\phi}}{e^{3\xi\phi} - 1} \\
= & \frac{e^{3\xi N\phi} - e^{-3\xi N\phi}}{e^{3\xi\phi} - 1} - \frac{e^{-2\xi\phi} - e^{\xi(N-2)\phi} + e^{-\xi(N+2)\phi} - e^{-2\xi\phi}}{e^{-\xi\phi} - 1} \\
= & \frac{e^{3\xi N\phi} - e^{-3\xi N\phi}}{e^{3\xi\phi} - 1} - \frac{e^{-\xi(N+2)\phi} - e^{\xi(N-2)\phi}}{e^{-\xi\phi} - 1}.
\end{aligned}
$$

Thus:

$$\sum_{n=0}^{N-1}\left(U_n\left(x\right)-U_{n-1}\left(x\right)\right)\left(U_n\left(x\right)+U_{n-1}\left(x\right)\right)^2$$
$$=\quad\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\left(\frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}-\frac{e^{-\xi(N+2)\phi}-e^{\xi(N-2)\phi}}{e^{-\xi\phi}-1}\right).$$

We will work on computing the two parts in turn:

$$\begin{aligned}
&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}\\
=\;&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)\left(e^{2\xi N\phi}+1+e^{-2\xi N\phi}\right)}{\left(e^{\xi\phi}-1\right)\left(e^{2\xi\phi}+e^{\xi\phi}+1\right)}\\
=\;&\frac{e^{2\xi\phi}+2e^{\xi\phi}+1}{e^{2\xi\phi}+e^{\xi\phi}+1}\frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)\left(e^{2\xi N\phi}+1+e^{-2\xi N\phi}\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\\
=\;&\frac{e^{\xi\phi}+2+e^{-\xi\phi}}{e^{\xi\phi}+1+e^{-\xi\phi}}U_{N-1}\left(k_\rho\right)\frac{\left(e^{2\xi N\phi}-2+e^{-2\xi N\phi}\right)+3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\\
=\;&\frac{2+2k_\rho}{1+2k_\rho}U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)\\
=\;&\frac{2+2k_\rho}{1+2k_\rho}U_{N-1}\left(k_\rho\right)\left(U_{N-1}\left(k_\rho\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)
\end{aligned}$$

The second part is:

$$\begin{aligned}
\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\frac{e^{-\xi(N+2)\phi}-e^{\xi(N-2)\phi}}{e^{-\xi\phi}-1}\quad&=\quad-\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2e^{-2\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\frac{e^{\xi N\phi}-e^{-\xi N\phi}}{e^{-\xi\phi}-1}\\
&=\quad\frac{\left(1-e^{\xi\phi}\right)\left(e^{\xi\phi}+1\right)^2e^{-2\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(e^{-\xi\phi}-1\right)}U_{N-1}\left(k_\rho\right)\\
&=\quad\frac{\left(e^{\xi\phi}+1\right)^2e^{-\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}U_{N-1}\left(k_\rho\right)\\
&=\quad\frac{2k_\rho+2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}U_{N-1}\left(k_\rho\right)
\end{aligned}$$

where we have used $e^{\xi\phi}+e^{-\xi\phi}=2k_\rho$ in the last step.

Putting this together, we have that:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\left(U_n(x)-U_{n-1}(x)\right)\left(U_n(x)+U_{n-1}(x)\right)^2 \\
=\;& \frac{2+2k_\rho}{1+2k_\rho}U_{N-1}(k_\rho)\left(U_{N-1}(k_\rho)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)+\frac{2k_\rho+2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}U_{N-1}(k_\rho) \\
=\;& \frac{2+2k_\rho}{1+2k_\rho}U_{N-1}(k_\rho)\left(U_{N-1}(k_\rho)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}+\frac{1+2k_\rho}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
=\;& \frac{2+2k_\rho}{1+2k_\rho}U_{N-1}(k_\rho)^3\left(1+2\frac{1+k_\rho}{U_{N-1}(k_\rho)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
=\;& \frac{2+2k_\rho}{1+2k_\rho}U_{N-1}(k_\rho)^3\left(1+2\frac{1+k_\rho}{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2}\right),
\end{aligned}$$

where the result follows after noting that:

$$\begin{aligned}
\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2 &= \begin{cases}(2\iota\sin N\phi)^2 & \text{if } |k_\rho|\leq 1\\ (2\sinh N\phi)^2 & \text{if } k_\rho\geq 1\end{cases} \\
&= \begin{cases}-4\sin^2 N\phi & \text{if } |k_\rho|\leq 1\\ 4\sinh^2 N\phi & \text{if } k_\rho\geq 1\end{cases}.
\end{aligned}$$

(3) Note that by the product-to-sum identity for Chebyshev polynomials $2T_I(k_\rho)T_n(k_\rho) = T_{I+n}-T_{|I-n|}$ and hence:

$$\begin{aligned}
T_n(k_\rho)^3 &= T_n(k_\rho)T_n(k_\rho)^2 \\
&= T_n(k_\rho)\left(\frac{T_{2n}(k_\rho)+1}{2}\right) \\
&= \frac{1}{2}T_{2n}(k_\rho)T_n(k_\rho)+\frac{1}{2}T_n(k_\rho) \\
&= \frac{1}{4}\left(T_{3n}(k_\rho)+T_n(k_\rho)\right)+\frac{1}{2}T_n(k_\rho) \\
&= \frac{1}{4}T_{3n}(k_\rho)+\frac{3}{4}T_n(k_\rho).
\end{aligned}$$

Thus we want to evaluate:

$$\sum_{n=0}^{N_\rho-1}T_{n+1}(k_\rho)^3+T_n(k_\rho)^3=\frac{1}{4}\sum_{n=0}^{N_\rho-1}T_{3n+3}(k_\rho)+3T_{n+1}(k_\rho)+T_{3n}(k_\rho)+3T_n(k_\rho).$$

Now we can again rewrite the Chebyshev polynomials in terms of exponentials, i.e., De Moivre's theorem implies:

$$\cos n\phi = \frac{e^{\iota n\phi} + e^{-\iota n\phi}}{2} \text{ and } \cosh n\phi = \frac{e^{n\phi} + e^{-n\phi}}{2},$$

where we define $\phi = \arccos k_\rho$. This means that Chebyshev polynomials of the first kind may be written as:

$$\begin{aligned} T_n\left(k_\rho\right) &= \frac{e^{\iota^{\mathbf{1}[k_\rho<1]} n\phi} + e^{-\iota^{\mathbf{1}[k_\rho<1]} n\phi}}{2} \\ &= \frac{e^{\xi n\phi} + e^{-\xi n\phi}}{2}, \end{aligned}$$

where $\xi := \iota^{\mathbf{1}[k_\rho<1]}$. Thus:

$$\begin{aligned} &\sum_{n=0}^{N_\rho-1} T_{n+1}\left(k_\rho\right)^3 + T_n\left(k_\rho\right)^3 \\ = &\ \frac{1}{8}\sum_{n=0}^{N_\rho-1}\left\{\begin{array}{c} e^{\xi(3n+3)\phi} + e^{-\xi(3n+3)\phi} + 3e^{\xi(n+1)\phi} + 3e^{-\xi(n+1)\phi} \\ +e^{\xi 3n\phi} + e^{-\xi 3n\phi} + 3e^{\xi n\phi} + 3e^{-\xi n\phi} \end{array}\right\} \\ = &\ \frac{1}{8}\sum_{n=0}^{N_\rho-1}\left\{\begin{array}{c} e^{\xi 3n\phi}\left(e^{\xi 3\phi}+1\right) + e^{-\xi 3n\phi}\left(e^{-\xi 3\phi}+1\right) \\ +3e^{\xi n\phi}\left(e^{\xi\phi}+1\right) + 3e^{-\xi n\phi}\left(e^{-\xi\phi}+1\right) \end{array}\right\} \\ = &\ \frac{1}{8}\left(\begin{array}{c} \frac{e^{\xi 3\phi N_\rho}-1}{e^{\xi 3\phi}-1}\left(e^{\xi 3\phi}+1\right) + \frac{e^{-\xi 3\phi N_\rho}-1}{e^{-\xi 3\phi}-1}\left(e^{-\xi 3\phi}+1\right) \\ +3\frac{e^{\xi\phi N_\rho}-1}{e^{\xi\phi}-1}\left(e^{\xi\phi}+1\right) + 3\frac{e^{-\xi\phi N_\rho}-1}{e^{-\xi\phi}-1}\left(e^{-\xi\phi}+1\right) \end{array}\right). \end{aligned}$$

Now, we'll look at these in turn:

$$
\begin{aligned}
& 3\frac{e^{\xi\phi N_\rho}-1}{e^{\xi\phi}-1}\left(e^{\xi\phi}+1\right)+3\frac{e^{-\xi\phi N_\rho}-1}{e^{-\xi\phi}-1}\left(e^{-\xi\phi}+1\right) \\
= & \; 3\frac{\left(e^{\xi\phi N_\rho}-1\right)\left(e^{-\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)}{\left(e^{\xi\phi}-1\right)\left(e^{-\xi\phi}-1\right)}+3\frac{\left(e^{-\xi\phi N_\rho}-1\right)\left(e^{-\xi\phi}+1\right)\left(e^{\xi\phi}-1\right)}{\left(e^{-\xi\phi}-1\right)\left(e^{\xi\phi}-1\right)} \\
= & \; -3\frac{\left(e^{\xi\phi N_\rho}-1\right)\left(e^{\xi\phi}-e^{-\xi\phi}\right)}{\left(e^{\xi\phi}-1\right)\left(e^{-\xi\phi}-1\right)}+3\frac{\left(e^{-\xi\phi N_\rho}-1\right)\left(e^{\xi\phi}-e^{-\xi\phi}\right)}{\left(e^{-\xi\phi}-1\right)\left(e^{\xi\phi}-1\right)} \\
= & \; -3\frac{\left(e^{\xi\phi}-e^{-\xi\phi}\right)\left(e^{\xi\phi N_\rho}-1-e^{-\xi\phi N_\rho}+1\right)}{2-e^{\xi\phi}-e^{-\xi\phi}} \\
= & \; 3\frac{\left(e^{\xi\phi/2}-e^{-\xi\phi/2}\right)\left(e^{\xi\phi/2}+e^{-\xi\phi/2}\right)\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)}{e^{\xi\phi}-2+e^{-\xi\phi}} \\
= & \; 3\frac{\left(e^{\xi\phi/2}+e^{-\xi\phi/2}\right)\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)}{\left(e^{\xi\phi/2}-e^{-\xi\phi/2}\right)} \\
= & \; 3\frac{\left(e^{\xi\phi/2}+e^{-\xi\phi/2}\right)^2\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)}{e^{\xi\phi}-e^{-\xi\phi}} \\
= & \; 3\left(e^{\xi\phi}+e^{-\xi\phi}+2\right)U_{N-1}\left(k_\rho\right) \\
= & \; 6\left(2k_\rho+2\right)U_{N-1}\left(k_\rho\right),
\end{aligned}
$$

where we have used $e^{\xi\phi}+e^{-\xi\phi}=2k_\rho$.

The other part is a little more complicated:

$$
\begin{aligned}
& \frac{e^{\xi 3\phi N_\rho}-1}{e^{\xi 3\phi}-1}\left(e^{\xi 3\phi}+1\right)+\frac{e^{-\xi 3\phi N_\rho}-1}{e^{-\xi 3\phi}-1}\left(e^{-\xi 3\phi}+1\right) \\
= & \frac{\left(e^{\xi 3\phi N_\rho}-1\right)\left(e^{\xi 3\phi}+1\right)\left(e^{-\xi 3\phi}-1\right)}{\left(e^{\xi 3\phi}-1\right)\left(e^{-\xi 3\phi}-1\right)}+\frac{\left(e^{-\xi 3\phi N_\rho}-1\right)\left(e^{-\xi 3\phi}+1\right)\left(e^{\xi 3\phi}-1\right)}{\left(e^{-\xi 3\phi}-1\right)\left(e^{\xi 3\phi}-1\right)} \\
= & -\frac{\left(e^{\xi 3\phi N_\rho}-1\right)\left(e^{\xi 3\phi}-e^{-\xi 3\phi}\right)}{\left(e^{\xi 3\phi}-1\right)\left(e^{-\xi 3\phi}-1\right)}+\frac{\left(e^{-\xi 3\phi N_\rho}-1\right)\left(e^{\xi 3\phi}-e^{-\xi 3\phi}\right)}{\left(e^{-\xi 3\phi}-1\right)\left(e^{\xi 3\phi}-1\right)} \\
= & \frac{\left(1-e^{\xi 3\phi N_\rho}\right)\left(e^{\xi 3\phi}-e^{-\xi 3\phi}\right)}{\left(e^{\xi 3\phi}-1\right)\left(e^{-\xi 3\phi}-1\right)}+\frac{\left(e^{-\xi 3\phi N_\rho}-1\right)\left(e^{\xi 3\phi}-e^{-\xi 3\phi}\right)}{\left(e^{-\xi 3\phi}-1\right)\left(e^{\xi 3\phi}-1\right)} \\
= & -\frac{\left(e^{\xi 3\phi}-e^{-\xi 3\phi}\right)\left(e^{\xi 3\phi N_\rho}-1-e^{-\xi 3\phi N_\rho}+1\right)}{2-e^{\xi 3\phi}-e^{-\xi 3\phi}} \\
= & -\frac{\left(e^{\xi 3\phi/2}-e^{-\xi 3\phi/2}\right)\left(e^{\xi 3\phi/2}+e^{-\xi 3\phi/2}\right)\left(e^{\xi 3\phi N_\rho}-e^{-\xi 3\phi N_\rho}\right)}{2-e^{\xi 3\phi}-e^{-\xi 3\phi}} \\
= & \frac{\left(e^{\xi 3\phi/2}-e^{-\xi 3\phi/2}\right)\left(e^{\xi 3\phi/2}+e^{-\xi 3\phi/2}\right)\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)\left(e^{2\xi\phi N_\rho}+e^{-2\xi\phi N_\rho}+1\right)}{\left(e^{\xi 3\phi/2}-e^{-\xi 3\phi/2}\right)^2} \\
= & \frac{\left(e^{\xi 3\phi/2}+e^{-\xi 3\phi/2}\right)}{\left(e^{\xi 3\phi/2}-e^{-\xi 3\phi/2}\right)}\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)\left(e^{2\xi N_\rho\phi}-2+e^{-2\xi N_\rho\phi}+3\right) \\
= & \frac{\left(e^{\xi 3\phi/2}+e^{-\xi 3\phi/2}\right)^2\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)\left(\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)^2+3\right)}{\left(e^{\xi 3\phi/2}+e^{-\xi 3\phi/2}\right)\left(e^{\xi 3\phi/2}-e^{-\xi 3\phi/2}\right)} \\
= & \frac{\left(e^{\xi 3\phi}+e^{-\xi 3\phi}+2\right)\left(e^{\xi\phi N_\rho}-e^{-\xi\phi N_\rho}\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}\left(k_\rho\right)^2+3\right)}{e^{\xi 3\phi}-e^{-\xi 3\phi}} \\
= & \frac{\left(e^{\xi 3\phi}+3e^{\xi\phi}+3e^{-\xi\phi}+e^{-\xi 3\phi}-3e^{\xi\phi}-3e^{-\xi\phi}+2\right)U_{N-1}\left(k_\rho\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}\left(k_\rho\right)^2+3\right)}{\left(e^{2\xi\phi}+1+e^{-\xi 2\phi}\right)} \\
= & \frac{\left(\left(e^{\xi\phi}+e^{-\xi\phi}\right)^3-3\left(e^{\xi\phi}+e^{-\xi\phi}\right)+2\right)U_{N-1}\left(k_\rho\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}\left(k_\rho\right)^2+3\right)}{\left(\left(e^{\xi\phi}+e^{-\xi\phi}\right)^2-1\right)} \\
= & \frac{\left(\left(2k_\rho\right)^3-3\left(2k_\rho\right)+2\right)U_{N-1}\left(k_\rho\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}\left(k_\rho\right)^2+3\right)}{\left(2k_\rho\right)^2-1} \\
= & 2\frac{\left(k_\rho+1\right)\left(2k_\rho-1\right)^2 U_{N-1}\left(k_\rho\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}\left(k_\rho\right)^2+3\right)}{\left(2k_\rho-1\right)\left(2k_\rho+1\right)} \\
= & 2\frac{\left(k_\rho+1\right)\left(2k_\rho-1\right)U_{N-1}\left(k_\rho\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}\left(k_\rho\right)^2+3\right)}{2k_\rho+1}
\end{aligned}
$$

Putting this together, we have:

$$\begin{aligned}
&\sum_{n=0}^{N_\rho-1} T_{n+1}(k_\rho)^3 + T_n(k_\rho)^3 \\
= &\ \frac{1}{8}\left(6(k_\rho+1)U_{N-1}(k_\rho) + 2\frac{(k_\rho+1)(2k_\rho-1)U_{N-1}(k_\rho)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}(k_\rho)^2+3\right)}{2k_\rho+1}\right) \\
= &\ \frac{1}{4}(k_\rho+1)U_{N-1}(k_\rho)\left(3\frac{2k_\rho+1}{2k_\rho+1} + \frac{(2k_\rho-1)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 U_{N-1}(k_\rho)^2+3\right)}{2k_\rho+1}\right) \\
= &\ \frac{1}{4}\frac{k_\rho+1}{2k_\rho+1}\left(\frac{3(2k_\rho+1)}{U_{N-1}(k_\rho)^2} + (2k_\rho-1)\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 + \frac{3(2k_\rho-1)}{U_{N-1}(k_\rho)^2}\right) \\
= &\ \frac{1}{4}\frac{k_\rho+1}{2k_\rho+1}\left(\frac{12k_\rho}{U_{N-1}(k_\rho)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} + 2k_\rho - 1\right)U_{N-1}(k_\rho)^3 \\
= &\ \frac{1}{4}\frac{k_\rho+1}{2k_\rho+1}\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(\frac{12k_\rho}{U_{N-1}(k_\rho)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} + 2k_\rho - 1\right)U_{N-1}(k_\rho)^3 \\
= &\ \frac{1}{4}\frac{k_\rho+1}{2k_\rho+1}\left(\left(e^{\xi\phi}+e^{-\xi\phi}\right)^2-4\right)\left(2k_\rho-1+\frac{12k_\rho}{U_{N-1}(k_\rho)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)U_{N-1}(k_\rho)^3 \\
= &\ \frac{(k_\rho+1)^2(k_\rho-1)}{2k_\rho+1}\left(2k_\rho-1+\frac{12k_\rho}{U_{N-1}(k_\rho)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)U_{N-1}(k_\rho)^3
\end{aligned}$$

As in the proof of part 1, the result follows since:

$$\begin{aligned}
\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2 &= \begin{cases} (2\iota\sin n\phi)^2 & \text{if } |k_\rho|\le 1 \\ (2\sinh n\phi)^2 & \text{if } k_\rho \ge 1 \end{cases} \\
&= \begin{cases} -4\sin^2 n\phi & \text{if } |k_\rho|\le 1 \\ 4\sinh^2 n\phi & \text{if } k_\rho \ge 1 \end{cases}.
\end{aligned}$$

□

Regardless of whether the series being summed involves polynomials of the first or the second kind, their sums are expressed as a function of $U_{N-1}(x)$, a single polynomial of the second kind. This polynomial happens to be the scaling factor $\alpha$ identified in Proposition 1 part 3. We now leverage Lemma B.2 to compute the investor's equilibrium payoff.

**Proposition 10** (Closed-form solutions for controlling investor's equilibrium payoff)**.** *Fix $\rho$. Up to the additive constant $\frac{\sigma^2}{2}\mathbb{E}\left(\omega^2\right)$, the controlling investor's equilibrium payoffs are as follows:*

(1) *If $\rho < 3/4$ the controlling investor's equilibrium payoff equals $-\sigma^2/24$.*

(2) *If $3/4 \leq \rho \leq 1$ the controlling investor's payoff in an equilibrium with $N > 1$ partitions is:*

$$\begin{aligned} V_C\left(\rho, N\right) &= \frac{\sigma^2}{6}\frac{\rho - 1}{(4\rho - 1)}\left(\frac{3\rho}{\sin^2\left(N \arccos k_\rho\right)} - 1\right) \\ &\leq \overline{V}_C\left(\rho\right) = \frac{\sigma^2}{6}\frac{\left(\rho - 1\right)\left(3\rho - 1\right)}{\left(4\rho - 1\right)}. \end{aligned} \tag{B.3}$$

*The payoff $V_C\left(\rho, N\right)$ is increasing in $N$. For any $\rho$, there exists a value $\overline{\rho} \in [\rho, 1)$ such that $V_C\left(\overline{\rho}, N_{\overline{\rho}}\right)$ attains the bound $\overline{V}_C\left(\overline{\rho}\right)$.*

(3) *If $\rho \geq 1$ the controlling investor's payoff in an equilibrium with $N > 1$ partitions is:*

$$\begin{aligned} V_C\left(\rho, N\right) &= \frac{\sigma^2}{6}\frac{1 - \rho}{4\rho - 1}\left(\frac{3\rho}{\sinh^2\left(N \operatorname{arccosh} k_\rho\right)} + 1\right) \\ &\leq \overline{V}_C\left(\rho\right) = \frac{\sigma^2}{6}\frac{1 - \rho}{\left(4\rho - 1\right)}. \end{aligned}$$

*The payoff $V_C\left(\rho, N\right)$ is increasing in $N$ and $\lim_{N\to\infty} V_C\left(\rho, N\right) = \overline{V}_C\left(\rho\right)$.*

*Proof.* In light of Lemma A.1, the investor's expected utility in an equilibrium when $N$ intervals are being communicated is:

$$\begin{aligned} V_C(\rho, N) &= -\frac{\sigma^2}{2}\sum_{n=0}^{N-1}\int_{\omega_n}^{\omega_{n+1}} \left(\omega - a_C^*(\omega_n, \omega_{n+1}|\theta_C)\,\theta_C\right)^2 \,\mathrm{d}\omega \\ &= -\frac{\sigma^2}{6}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1} - a_C^*(\omega_n, \omega_{n+1}|\theta_C)\,\theta_C\right)^3 - \left(\omega_n - a_C^*(\omega_n, \omega_{n+1}|\theta_C)\,\theta_C\right)^3\right] \\ &= -\frac{\sigma^2}{6}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1} - \frac{\omega_{n+1}+\omega_n}{2}\right)^3 - \left(\omega_n - \frac{\omega_{n+1}+\omega_n}{2}\right)^3\right] \\ &= -\frac{\sigma^2}{6}\sum_{n=0}^{N-1}\left[\frac{1}{4}\left(\omega_{n+1} - \omega_n\right)^3\right] \\ &= -\frac{\sigma^2}{24}\sum_{n=0}^{N-1}\left[\left(\frac{U_n(k_\rho)}{U_{N-1}(k_\rho)} - \frac{U_{n-1}(k_\rho)}{U_{N-1}(k_\rho)}\right)^3\right] \\ &= -\frac{\sigma^2}{24}\frac{1}{\left[U_{N-1}(k_\rho)\right]^3}\sum_{n=0}^{N-1}\left[U_n(k_\rho) - U_{n-1}(k_\rho)\right]^3, \end{aligned} \tag{B.4}$$

where to get the second equality we have substituted for $a^*(\omega_n, \omega_{n+1}|\theta_C)$ using Lemma 1.

(1) If $\rho < 3/4$ no information can be communicated. In this case the investor's equilibrium payoff equals $-\sigma^2/24$. would choose action $1/2\theta_C$ (see Lemma 1) giving rise to an expected payoff which, by Lemma A.1, equals

$$-\frac{\sigma^2}{2}\mathbb{E}\left(\frac{1}{2} - \omega\right)^2 = -\frac{\sigma^2}{24}.$$

(2) In the case $3/4 \le \rho \le 1$ we have $|k_\rho| \le 1$, and substituting the relevant expression from Lemma B.2 into (B.4) we get:

$$\begin{aligned} V_C(\rho, N) &= \frac{\sigma^2}{24}\frac{1-k_\rho}{2k_\rho+1}\left(2 - 3\frac{1+k_\rho}{\sin^2(N\arccos k_\rho)}\right) \\ &= \frac{\sigma^2}{6}\frac{\rho-1}{4\rho-1}\left(3\frac{\rho}{\sin^2(N\arccos k_\rho)} - 1\right), \end{aligned}$$

where the last equality follows from substituting $2\rho - 1$ for $k_\rho$ and collecting terms.

Note that the payoff $V_C(\rho, N)$ is increasing in $N$, since:

$$\begin{aligned} V_C(\rho, N) - V_C(\rho, N-1) &= \frac{\sigma^2}{6}\frac{\rho-1}{4\rho-1}3\rho\left(\frac{1}{\sin^2(N\arccos k_\rho)} - \frac{1}{\sin^2((N-1)\arccos k_\rho)}\right) \\ &= \frac{\sigma^2}{6}\frac{\rho-1}{4\rho-1}3\rho\left(\frac{\sin^2((N-1)\arccos k_\rho) - \sin^2(N\arccos k_\rho)}{\sin^2(N\arccos k_\rho)\sin^2((N-1)\arccos k_\rho)}\right) \end{aligned}$$

and since in an equilibrium where $N > 1$ it must be $\rho > 3/4$ (Proposition 1 part 5), so the factor multiplying the parenthesis is negative and thus $V_C(\rho, N) \geq V_C(\rho, N-1)$ if and only if:

$$\begin{aligned} \sin^2(N\arccos k_\rho) &\geq \sin^2((N-1)\arccos k_\rho) \\ \sin(N\arccos k_\rho) &\geq \sin((N-1)\arccos k_\rho), \end{aligned}$$

where the second line follows since sin is positive on the relevant range. Let $\phi = \arccos k_\rho$ and note that:

$$\begin{aligned} \sin(N\phi) - \sin((N-1)\phi) &= \sin\left(N\phi - \frac{\phi}{2} + \frac{\phi}{2}\right) - \sin\left(N\phi - \frac{\phi}{2} - \frac{\phi}{2}\right) \\ &= 2\cos\left(N\phi - \frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right), \end{aligned}$$

and since $\sin(\phi/2) > 0$ we have that $\sin(N\phi) \geq \sin((N-1)\phi)$ if and only if $\cos(N\phi - \phi/2)$, is positive, i.e., if and only if:

$$N\phi - \frac{\phi}{2} \leq \frac{\pi}{2},$$

or after solving for $N$, if:

$$\begin{aligned} N &\leq \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right) \\ &\leq \left\lceil \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right)\right\rceil \\ &= N_\rho. \end{aligned}$$

To establish the tightness of the upper bound, note that:

$$V_C(\overline{\rho}, N_{\overline{\rho}}) = \frac{\sigma^2}{6}\frac{\overline{\rho}-1}{(4\overline{\rho}-1)}\left(\frac{3\overline{\rho}}{\sin^2(N_{\overline{\rho}}\arccos k_{\overline{\rho}})} - 1\right).$$

Now, the approximation is tight whenever:

$$\left\lceil \frac{1}{2}\left(\frac{\pi}{\arccos k_{\overline{\rho}}}-1\right)\right\rceil \arccos k_{\overline{\rho}}=\frac{\pi}{2},$$

or:

$$\left\lceil \frac{1}{2}\frac{\pi}{\arccos k_{\overline{\rho}}}-\frac{1}{2}\right\rceil =\frac{1}{2}\frac{\pi}{\arccos k_{\overline{\rho}}},$$

which holds whenever $\frac{\pi}{\arccos k_{\overline{\rho}}}\in\mathbb{Z}$ and this occurs arbitrarily often as $\lim_{\overline{\rho}\to 1}\arccos k_{\overline{\rho}}=0$.

(3) In the case $\rho\geq 1$ we have $|k_{\rho}|\geq 1$, and substituting the relevant expression from Lemma B.2 into (B.4) we get:

$$\begin{aligned} V_C(\rho,N) &= \frac{\sigma^2}{24}\frac{1-k_\rho}{2k_\rho+1}\left(2+3\frac{1+k_\rho}{\sinh^2\left(N\operatorname{arccosh} k_\rho\right)}\right)\\ &= \frac{\sigma^2}{6}\frac{1-\rho}{4\rho-1}\left(1+3\frac{\rho}{\sinh^2\left(N\operatorname{arccosh} k_\rho\right)}\right). \end{aligned}$$

This expression is negative and is increasing in $N$ because the function $\sinh(\cdot)$ is monotonically increasing. As $N\to\infty$ this converges to $\frac{\sigma^2}{6}\frac{1-\rho}{(4\rho-1)}$.

□

Proposition 10 provides a convenient expression for the investor's equilibrium payoff in any equilibrium, and a tight upper bound $\overline{V}_C(\rho)$ for the payoff in the best equilibrium for the investor. The best equilibrium is achieved at the maximal number of partitions compatible with equilibrium.

**Proposition 11** (Closed-form solutions for informed investor's equilibrium payoff)**.** *Fix $\rho$. Up to the additive constant $\frac{\sigma^2}{2r_I}\mathbb{E}(\omega^2)$, the informed investor's equilibrium payoffs are as follows:*

(1) *If $\rho<3/4$ no information can be communicated. In this case the informed investor's equilibrium payoff equals*

$$-\frac{\sigma^2}{2r}\mathbb{E}\left(\frac{1}{2\rho}-\omega\right)^2.$$

(2) *If* $3/4 \leq \rho \leq 1$ *the informed investor's payoff in an equilibrium with* $N > 1$ *partitions is:*

$$\begin{aligned} V_I(\rho, N) &= -\frac{\sigma^2}{6} \frac{1-\rho}{\rho(4\rho-1)} \left( 3 \frac{2\rho - 1}{\sin^2(N \arccos k_\rho)} + 3 - 4\rho \right) \\ &\leq \overline{V}_I(\rho) = -\frac{\sigma^2}{3} \frac{1-\rho}{(4\rho-1)} \end{aligned} \tag{B.5}$$

*The payoff* $V_I(\rho, N)$ *is increasing in* $N$*. For any* $\rho$*, there exists a value* $\overline{\rho} \in [\rho, 1)$ *such that* $V_I(\overline{\rho}, N_{\overline{\rho}})$ *attains the bound* $\overline{V}_I(\overline{\rho})$*.*

(3) *If* $\rho \geq 1$ *the informed investor's payoff in an equilibrium with* $N > 1$ *partitions is:*

$$\begin{aligned} V_I(\rho, N) &= -\frac{\sigma^2}{6\rho} \frac{(\rho-1)}{4\rho-1} \left( 3 \frac{2\rho - 1}{\sinh^2(N \operatorname{arccosh} k_\rho)} + 4\rho - 3 \right) \\ &\leq \overline{V}_I(\rho) = -\frac{\sigma^2}{6} \frac{\rho-1}{\rho(4\rho-1)} (4\rho - 3), \end{aligned}$$

*The payoff* $V_I(\rho, N)$ *is increasing in* $N$ *and* $\lim_{N \to \infty} V_I(\rho, N) = \overline{V}_I(\rho)$.

*Proof.* In light of Lemma A.1, the informed investor's expected utility in an equilibrium when $N$ intervals are being communicated is:

$$\begin{aligned}
V_I(\rho,N) &= -\frac{\sigma^2}{2}\sum_{n=0}^{N-1}\int_{\omega_n}^{\omega_{n+1}}\left(\omega-\theta_I r a^*(\omega_n,\omega_{n+1}|\theta_C)\right)^2\,\mathrm{d}\omega \\
&= -\frac{\sigma^2}{2}\sum_{n=0}^{N-1}\int_{\omega_n}^{\omega_{n+1}}\left(\omega-\theta_I r a^*(\omega_n,\omega_{n+1}|\theta_C)\right)^2\,\mathrm{d}\omega \\
&= -\frac{\sigma^2}{6}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1}-\theta_I r a^*(\omega_n,\omega_{n+1}|\theta_C)\right)^3-\left(\omega_n-\theta_I r a^*(\omega_n,\omega_{n+1}|\theta_C)\right)^3\right] \\
&= -\frac{\sigma^2}{6}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1}-\theta_I r_I\frac{\omega_{n+1}+\omega_n}{2\theta_C}\right)^3-\left(\omega_n-\theta_I r_I\frac{\omega_{n+1}+\omega_n}{2\theta_C}\right)^3\right] \\
&= -\frac{\sigma^2}{6}\sum_{n=0}^{N-1}\left[\left(\frac{U_n(k_\rho)}{U_{N-1}(k_\rho)}-\frac{U_n(k_\rho)+U_{n-1}(k_\rho)}{(k_\rho+1)U_{N-1}(k_\rho)}\right)^3-\left(\frac{U_{n-1}(k_\rho)}{U_{N-1}(k_\rho)}-\frac{U_n(k_\rho)+U_{n-1}(k_\rho)}{(k_\rho+1)U_{N-1}(k_\rho)}\right)^3\right] \\
&= -\frac{\sigma^2}{6U_{N-1}(k_\rho)^3}\sum_{n=0}^{N-1}\left[\left(U_n(k_\rho)-\frac{U_n(k_\rho)+U_{n-1}(k_\rho)}{k_\rho+1}\right)^3-\left(U_{n-1}(k_\rho)-\frac{U_n(k_\rho)+U_{n-1}(k_\rho)}{k_\rho+1}\right)^3\right] \\
&= -\frac{\sigma^2}{6(k_\rho+1)^3U_{N-1}(k_\rho)^3}\sum_{n=0}^{N-1}\left[\left(k_\rho U_n(k_\rho)-U_{n-1}(k_\rho)\right)^3-\left(k_\rho U_{n-1}(k_\rho)-U_n(k_\rho)\right)^3\right] \\
&= \frac{-\sigma^2}{6(k_\rho+1)^3U_{N-1}(k_\rho)^3}\sum_{n=0}^{N-1}\left[T_{n+1}(k_\rho)^3+T_n(k_\rho)^3\right], \qquad \text{(B.1)}
\end{aligned}$$

where the last line follows by an identity linking the Chebyshev polynomials of the first and second kinds, $T_n(k_\rho)=U_n(k_\rho)-k_\rho U_{n-1}(k_\rho)$, and by incrementing this identity and applying the defining linear difference equation:

$$\begin{aligned}
T_{n+1}(k_\rho) &= U_{n+1}(k_\rho)-k_\rho U_n(k_\rho) \\
&= 2k_\rho U_n(k_\rho)-U_{n-1}(k_\rho)-k_\rho U_n(k_\rho) \\
&= k_\rho U_n(k_\rho)-U_{n-1}(k_\rho).
\end{aligned}$$

(1) If $\rho<3/4$ no information can be communicated. In this case the controlling investor would choose action $1/2\theta_C$ (see Lemma 1) giving rise to an expected payoff which, by Lemma A.1, equals

$$-\frac{\sigma^2}{2r}\mathbb{E}\left(\frac{r\theta_I}{2\theta_C}-\omega\right)^2=-\frac{\sigma^2}{2r}\mathbb{E}\left(\frac{1}{2\rho}-\omega\right)^2.$$

(2) In the case $3/4 \leq \rho \leq 1$ we have $|k_\rho| \leq 1$, and substituting the relevant expression from Lemma B.2 into (B.6) we get:

$$\begin{aligned} V_I(\rho, N) &= -\frac{\sigma^2}{6(k_\rho+1)} \frac{(k_\rho-1)}{2k_\rho+1}\left(2k_\rho - 1 - 3\frac{k_\rho}{\sin^2(N \arccos k_\rho)}\right) \\ &= -\frac{\sigma^2}{6}\frac{1-\rho}{\rho(4\rho-1)}\left(3\frac{2\rho-1}{\sin^2(N\arccos k_\rho)} + 3 - 4\rho\right), \end{aligned}$$

where the last equality follows from substituting $2\rho - 1$ for $k_\rho$ and collecting terms. When $\sin^2 = 1$ the expression reduces to $\overline{V}_I(\rho)$. Monotonicity in $N$ and tightness of the bound are proved, mutatis mutandis, as in Proposition 10 part 1.

(3) In the case $\rho \geq 1$ we have $|k_\rho| \geq 1$, and substituting the relevant expression from Lemma B.2 into (B.6) we get:

$$\begin{aligned} V_I(\rho, N) &= -\frac{\sigma^2}{6(k_\rho+1)} \frac{(k_\rho-1)}{2k_\rho+1}\left(2k_\rho - 1 + 3\frac{k_\rho}{\sinh^2(N \operatorname{arccosh} k_\rho)}\right) \\ &= -\frac{\sigma^2}{6\rho}\frac{(\rho-1)}{4\rho-1}\left(3\frac{2\rho-1}{\sinh^2(N\operatorname{arccosh} k_\rho)} - 3 + 4\rho\right), \end{aligned}$$

where the last equality follows from substituting $2\rho - 1$ for $k_\rho$ and collecting terms. The upper bound $\overline{V}_I(\rho)$ is approached for $N \to \infty$.

□

## Appendix C. Demand for Shares

**Lemma C.1** (Controlling Investor's Demand)**.** *Assume the action $a_C^*(\Omega)$ is optimally chosen by controlling investor $C$ under the advice of a informed investor with $r_I, \theta_I$. Let $D_C(\theta_C | r_C, a^*)$ denote the controlling investor $C$'s demand function. Then $D_C(\theta_C | r_C, a_C^*) = \frac{\sigma^2}{r_I\theta_I}\frac{\rho(2\rho-1)}{(4\rho-1)^2}$ for $\theta_C \in \left(\frac{3}{4}r_I\theta_I, r\theta_I\right)$. For levels of $\theta_C$ smaller than $\frac{3}{4}r_I\theta_I$ demand equals zero, and it is negative for levels of $\theta_C$ larger than $r_I\theta_I$.*

*Proof.* From Proposition 10, the investor's equilibrium highest payoff $\overline{V}_C(\rho)$ equals approximately:

$$\overline{V}_C(\rho) \simeq \begin{cases} -\frac{\sigma^2}{24} & \text{if } 0 \leq \rho \leq \frac{3}{4} \\ \frac{\sigma^2}{6}\frac{(\rho-1)(3\rho-1)}{(4\rho-1)} & \text{if } \frac{3}{4} \leq \rho \leq 1 \\ \frac{\sigma^2}{6}\frac{1-\rho}{(4\rho-1)} & \text{if } 1 \leq \rho \end{cases}$$

Differentiating the above, we have that:

$$\frac{\partial \overline{V}_C(\rho)}{\partial \rho} \simeq \begin{cases} 0 & \text{if } 0 \leq \rho \leq \frac{3}{4} \\ \frac{\sigma^2}{6}\left[3\frac{\rho-1}{4\rho-1} + \frac{3\rho-1}{4\rho-1} - 4(\rho-1)\frac{3\rho-1}{(4\rho-1)^2}\right] & \text{if } \frac{3}{4} \leq \rho \leq 1 \\ \frac{\sigma^2}{6}\left[-\frac{1}{4\rho-1} - 4\frac{1-\rho}{(4\rho-1)^2}\right] & \text{if } 1 \leq \rho \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 \leq \rho \leq \frac{3}{4} \\ \sigma^2 \frac{\rho(2\rho-1)}{(4\rho-1)^2} & \text{if } \frac{3}{4} \leq \rho \leq 1 \\ -\frac{\sigma^2}{2}\frac{1}{(4\rho-1)^2} & \text{if } 1 \leq \rho \end{cases}.$$

Since

$$D_C(\theta_C | r_C, a_C^*) = \frac{\partial \overline{V}_C(\rho)}{\partial \rho} \cdot \frac{\partial \rho}{\partial \theta_C},$$

demand of the controlling investor is given by:

$$D_C(\theta_C | r_C, a_C^*) \simeq \begin{cases} 0 & \text{if } 0 \leq \theta_C \leq \frac{3}{4} r_I \theta_I \\ \frac{\sigma^2}{r_I \theta_I}\frac{\rho(2\rho-1)}{(4\rho-1)^2} & \text{if } \frac{3}{4} r\theta_I \leq \theta_C \leq r_I\theta_I \\ -\frac{\sigma^2}{2 r_I \theta_I}\frac{1}{(4\rho-1)^2} & \text{if } r_I \theta_I \leq \theta_C \end{cases}$$

□

**Corollary C.2** (Features of controlling investor's demand)**.** *The controlling investor's demand is zero on* $\left(0, \frac{3}{4} r\theta_I\right)$*, it is increasing on* $\left(\frac{3}{4} r\theta_I, r\theta_I\right)$*, and it is negative on* $(r\theta_I, \infty)$*. The controlling investor purchases a positive amount of shares only if prices are in* $\left[0, \frac{5\sigma^2}{192 r\theta_I}\right]$*.*

*Proof.* $D_C(\theta_C|r_C, a_C^*)$ is constant on $\left(0, \frac{3}{4}r\theta_I\right)$. $D_C(\theta_C|r_C, a_C^*)$ is increasing on $\left(\frac{3}{4}r\theta_I, r\theta_I\right)$ since:

$$\begin{aligned}
\frac{\partial}{\partial\rho}D_C(\theta_C|r_C, a_C^*) &\propto 2\frac{\rho}{(4\rho-1)^2} + \frac{2\rho-1}{(4\rho-1)^2} - 8\rho\frac{2\rho-1}{(4\rho-1)^3} \\
&= \frac{1}{(4\rho-1)^2}\left[2\rho + 2\rho - 1 - 8\rho\frac{2\rho-1}{(4\rho-1)}\right] \\
&= \frac{1}{(4\rho-1)^2}\left[4\rho - 1 - 8\rho\frac{2\rho-1}{(4\rho-1)}\right] \\
&= \frac{1}{(4\rho-1)^3}\left[(4\rho-1)^2 - 8\rho(2\rho-1)\right] \\
&= \frac{1}{(4\rho-1)^3}\left[16\rho^2 - 8\rho + 1 - 8\rho(2\rho-1)\right] \\
&= \frac{1}{(4\rho-1)^3} > 0.
\end{aligned}$$

The perfect-communication conclusion reflects perfect alignment of preferences between controlling investor and informed investor. Clearly there can be no equilibrium at if the controlling investor owns an amount different to $r\theta_I$ as his demand is weakly increasing up to this point and negative after it.

To calculate the maximum willingness to pay $p$ we need to ensure that:

$$\begin{aligned}
0 &\leq \int_0^{r\theta_I}\left(D_C(\theta_C|r_C, a_C^*) - p\right)\,\mathrm{d}\theta_C \\
&= \int_{\frac{3}{4}r\theta_I}^{r\theta_I}\frac{\sigma^2}{r\theta_I}\frac{\rho(2\rho-1)}{(4\rho-1)^2}\,\mathrm{d}\theta_C - \int_0^{r\theta_I}p\,\mathrm{d}\theta_C \\
&= \int_{\frac{3}{4}r\theta_I}^{r\theta_I}\frac{\sigma^2}{r\theta_I}\frac{\frac{\theta_C}{r\theta_I}\left(2\frac{\theta_C}{r\theta_I} - 1\right)}{\left(4\frac{\theta_C}{r\theta_I} - 1\right)^2}\,\mathrm{d}\theta_C - pr\theta_I \\
&= \int_{\frac{3}{4}r\theta_I}^{r\theta_I}\frac{\sigma^2}{r\theta_I}\theta_C\frac{2\theta_C - r\theta_I}{(4\theta_C - r\theta_I)^2}\,\mathrm{d}\theta_C - pr\theta_I \\
&= \frac{5\sigma^2}{192} - pr\theta_I,
\end{aligned}$$

or if $p \leq \frac{5\sigma^2}{192r\theta_I}$. □

**Lemma C.3** (Informed investor's demand)**.** *Assume the action $a_C^*(\omega_n)$ is optimally chosen by controlling investorC under the advice of a informed investor with $r_I, \theta_I$. Let $D_I(\theta_I|r_I, a_C^*)$*

*denote the informed investor's demand function. For* $\theta_I \in (0, \theta_C/r_I)$ *we have* $D_I(\theta_I | r_I, a_C^*) = \frac{\sigma^2}{2}\frac{r_I}{\theta_C}\left[\frac{8\rho^2-8\rho+1}{16\rho^2-8\rho+1}\right] > 0$ *. Demand is negative for* $\theta_I \in (\theta_C/r_I, 4\theta_C/3r_I)$*. Demand is also negative for* $\theta_I \in (4\theta_C/3r_I, 1)$*.*

*Proof.* From Proposition 10, the investor's equilibrium highest payoff $\overline{V}_C(\rho)$ equals approximately:

$$\overline{V}_I(\rho) \simeq \begin{cases} -\frac{\sigma^2}{2r}\mathbb{E}\left(\frac{1}{2\rho}-\omega\right)^2 & \text{if } 0 \le \rho \le \frac{3}{4} \\ -\frac{\sigma^2}{3}\frac{1-\rho}{(4\rho-1)} & \text{if } \frac{3}{4} \le \rho \le 1 \\ -\frac{\sigma^2}{6}\frac{\rho-1}{\rho(4\rho-1)}(4\rho-3) & \text{if } 1 \le \rho \end{cases}$$

where $\rho = \theta_C/r_I\theta_I$.

**Case** $\rho < 3/4$. This case corresponds to $\theta_I > 4\theta_C/3r_I$.

$$\begin{aligned} D_I(\theta_I | r_I, a_C^*) &= \frac{\partial \overline{V}_I(\rho)}{\partial \theta_I} \\ &= -\frac{\sigma^2}{2r_I}\frac{\partial}{\partial \theta_I}\mathbb{E}\left(\frac{r_I\theta_I}{2\theta_C}-\omega\right)^2 \\ &= -\frac{\sigma^2}{2\theta_C}\mathbb{E}\left(\frac{r\theta_I}{2\theta_C}-\omega\right) \\ &= -\frac{\sigma^2}{2\theta_C}\left(\frac{1}{2\rho}-\frac{1}{2}\right) < 0. \end{aligned}$$

**Case** $\frac{3}{4} \le \rho \le 1$. Since $\frac{\partial}{\partial\rho}\left(\frac{-\sigma^2}{3}\frac{1-\rho}{(4\rho-1)}\right) = \frac{\sigma^2}{(4\rho-1)^2}$ by what we found in the controlling-investor $\rho \ge 1$ case, we have that:

$$\begin{aligned} D_I(\theta_I | r_I, a_C^*) &= \frac{\partial \overline{V}_I(\rho)}{\partial \rho}\cdot\frac{\partial \rho}{\partial \theta_I} \\ &= \frac{\sigma^2}{(4\rho-1)^2}\left(-\rho\frac{1}{\theta_I}\right) \\ &= \frac{-\sigma^2\rho}{\theta_I(4\rho-1)^2}. \end{aligned}$$

**Case** $1 \le \rho$. This case corresponds to $\theta_I \le \theta_C/r_I$. On the range $1 \le \rho$ we have:

$$\begin{aligned} \frac{\partial \overline{V}_I(\rho)}{\partial \rho} &= -\frac{\sigma^2}{6}\left(\frac{\partial}{\partial \rho}\frac{\rho-1}{\rho(4\rho-1)}(4\rho-3)\right) \\ &= -\frac{\sigma^2}{6}\left(\frac{3(8\rho^2-8\rho+1)}{\rho^2(4\rho-1)^2}\right) \end{aligned}$$

Now:

$$\begin{aligned} D_I\left(\theta_I|r_I,a_C^*\right) &= \frac{\partial \overline{V}_I\left(\rho\right)}{\partial\rho}\cdot\frac{\partial\rho}{\partial\theta_I} \\ &= \frac{\partial \overline{V}_I\left(\rho\right)}{\partial\rho}\cdot\left(-\frac{\theta_C}{\left(r\theta_I\right)^2}r_I\right) \\ &= \frac{\partial \overline{V}_I\left(\rho\right)}{\partial\rho}\cdot\left(-\rho\frac{1}{\theta_I}\right) \end{aligned}$$

So

$$\begin{aligned} D_I\left(\theta_I|r_I,a_C^*\right) &= \frac{\sigma^2}{2\rho\theta_I}\left(\frac{8\rho^2-8\rho+1}{\left(4\rho-1\right)^2}\right) \\ &= \frac{\sigma^2}{2}\frac{r_I}{\theta_C}\left(\frac{8\rho^2-8\rho+1}{16\rho^2-8\rho+1}\right). \end{aligned}$$

Which means that:

$$D_I\left(\theta_I|r_I,a_C^*\right)\simeq\begin{cases} 0 & \text{if } 0\leq\rho\leq\frac{3}{4} \\ \frac{-\sigma^2\rho}{\theta_I\left(4\rho-1\right)^2} & \text{if } \frac{3}{4}\leq\rho\leq 1 \\ \frac{\sigma^2}{2}\frac{r_I}{\theta_C}\left(\frac{8\rho^2-8\rho+1}{\left(4\rho-1\right)^2}\right)>0 & \text{if } 1\leq\rho \end{cases}$$

□

**Corollary C.4** (Features of informed investor's demand)**.** *Informed investor's demand is positive and decreasing in* $\theta_I$ *over the interval* $\left(0,\theta_C/r_I\right)$. $D_I\left(0|r_I,a_C^*\right)=\frac{\sigma^2}{4}\frac{r_I}{\theta_C}$ *and* $D_I\left(\frac{\theta_C}{r_I}|r_I,a_C^*\right)=\frac{r_I\sigma^2}{18\theta_C}$. *Since informed investor's demand is negative or zero elsewhere, if the informed investor purchases any positive amount of shares in equilibrium he purchases within the interval* $\left(0,\theta_C/r_I\right)$. *The informed investor purchases* $\theta_C/r_I$ *only if prices are in* $\left[0,\frac{\sigma^2}{18}\frac{r_I}{\theta_C}\right]$.

*Proof.* The function $\frac{8\rho^2-8\rho+1}{16\rho^2-8\rho+1}$ is positive if $\rho>1$; it is also increasing in $\rho$ since $\frac{d}{d\rho}\left(\frac{8\rho^2-8\rho+1}{16\rho^2-8\rho+1}\right)=\frac{16\rho}{\left(4\rho-1\right)^3}>0$ since $\rho>1/4$. The function has value 1/9 at $\rho=0$ and it asymptotes horizontally to 1/2 as $\rho\to\infty$. Therefore, since $\rho$ is a decreasing function of $\theta_I$, we have that $D_I\left(\theta_I|r_I,a_C^*\right)$ is decreasing in $\theta_I$ on $\left(0,\theta_C/r_I\right)$. $D_I\left(0|r_I,a_C^*\right)=\frac{\sigma^2}{4}\frac{r_I}{\theta_C}$ and $D_I\left(\frac{\theta_C}{r_I}|r_I,a_C^*\right)=\frac{\sigma^2}{18}\frac{r_I}{\theta_C}$. □

**Lemma C.5** (Non-controlling investor's demand)**.** *Assume the action* $a^*\left(\Omega\right)$ *is optimally chosen by the controlling agent under information set* $\left\{\Omega\right\}$. *Let* $D_i\left(\theta_i|r_i,a^*\right)$ *denote the demand function of a non-controlling investor with risk aversion* $r_i$. *Then* $D_i\left(\theta_i|r_i,a^*\right)=\sigma^2\left(r_C\theta_C-r_i\theta_i\right)\mathbb{E}\left[a^*\left(\Omega\right)^2\right]$.

*Proof.*

$$\begin{aligned} D_i\left(\theta_i | r_i, \overline{a}^*\right) &= -\frac{\partial}{\partial \theta_i}\frac{\sigma^2}{2r_i}\mathbb{E}\left[r_i\theta_i a^*\left(\omega\right) - \omega\right]^2 \\ &= -\sigma^2\mathbb{E}\left[a^*\left(\omega\right)\left(r_i\theta_i a^*\left(\omega\right) - \omega\right)\right] \\ &= \sigma^2\mathbb{E}\left[\mathbb{E}\left[a^*\left(\omega\right)\left(\omega - r_i\theta_i a^*\left(\Omega\right)\right)|\Omega\right]\right] \\ &= \sigma^2\mathbb{E}\left[a^*\left(\Omega\right)\mathbb{E}\left[\omega - r_C\theta_C a^*\left(\Omega\right) + r_C\theta_C a^*\left(\Omega\right) - r_i\theta_i a^*\left(\Omega\right)|\Omega\right]\right]. \quad \text{(C.1)} \end{aligned}$$

Now, $a^*\left(\Omega\right)$ must solve the following maximization problem:

$$\max_a -\frac{\sigma^2}{2r_C}\mathbb{E}\left[\left(r_C\theta_C a - \omega\right)^2|\Omega\right],$$

and the first order conditions w.r.t. $a$ read $\mathbb{E}\left[\omega - r_C\theta_C a^*\left(\Omega\right)|\Omega\right] = 0$. Use this expression to simplify (C.1), then isolate $a^*\left(\Omega\right)$ to get the desired expression. □

**Lemma C.6** (Controlling investor's optimal action)**.** *Let $a^*\left(\rho,\Omega\right)$ denote the action chosen by the controlling agent if players are playing the cheap talk equilibrium with the finest partition given $\rho$ and the controlling investor knows information set $\{\Omega\}$. We have that*

$$\mathbb{E}\left[a^*\left(\rho,\Omega\right)^2\right] = \begin{cases} \frac{\theta_C^2+\theta_I r_I\theta_C}{16\theta_C\theta_I^3 r_I - 4\theta_I^4 r_I^2} & \text{if } \theta_C \le r_I\theta_I \\ \frac{\theta_C}{4\theta_C\theta_I^2 - \theta_I^3 r_I} & \text{if } r_I\theta_I \le \theta_C \end{cases}.$$

*Proof.* Note that:

$$\omega_n = \frac{U_{n-1}\left(k_\rho\right)}{U_{N-1}\left(k_\rho\right)} = \frac{e^{n\phi i^{\mathbf{1}[k_\theta<1]}} - e^{-n\phi i^{\mathbf{1}[k_\theta<1]}}}{e^{N\phi i^{\mathbf{1}[k_\theta<1]}} - e^{-N\phi i^{\mathbf{1}[k_\theta<1]}}} = \frac{e^{\xi n\phi} - e^{-\xi n\phi}}{e^{\xi N\phi} - e^{-\xi N\phi}}$$

where $\xi := \iota^{\mathbf{1}[k_\rho<1]}$ and that:

$$a^*\left(\rho,\left[\omega_n,\omega_{n+1}\right]\right) = \frac{\omega_n+\omega_{n+1}}{2\theta_I}.$$

Now since:

$$\begin{aligned} \mathbb{E}\left[a^*\left(\rho,\Omega\right)^2\right] &= \sum_{n=0}^{N-1}\left(\omega_{n+1}-\omega_n\right)\left(\frac{\omega_n+\omega_{n+1}}{2\theta_I}\right)^2 \\ &= \frac{1}{4\theta_I^2}\sum_{n=0}^{N-1}\left(\omega_{n+1}-\omega_n\right)\left(\omega_n+\omega_{n+1}\right)^2, \end{aligned}$$

we need to evaluate (at the highest possible value of $N$, more on this later):

$$\sum_{n=0}^{N-1} (\omega_{n+1} - \omega_n)(\omega_n + \omega_{n+1})^2$$
$$= \frac{1}{U_{N-1}(k_\rho)^3} \sum_{n=0}^{N-1} (U_n(k_\rho) - U_{n-1}(k_\rho))(U_n(k_\rho) + U_{n-1}(k_\rho))^2 .$$

Note that lemma B.2 part 2 gives exactly the sum part of this expression. Dividing into two regimes we have that for $\rho > 1$ (where $N \to \infty$):

$$\begin{aligned}
\mathbb{E}\left[a^*(\rho,\Omega)^2\right] &= \begin{cases} \frac{1}{\theta_I^2}\frac{\rho}{4\rho-1}\left(1+\frac{\rho}{\sin^2(N_\rho \arccos 2\rho)}\right) & \text{if } 0 \le \rho \le 1 \\ \frac{1}{\theta_I^2}\frac{\rho}{4\rho-1} & \text{if } 1 \le \rho \end{cases} \\
&\le \begin{cases} \frac{1}{4\theta_I^2}\frac{\rho(1+\rho)}{4\rho-1} & \text{if } 0 \le \rho \le 1 \\ \frac{1}{\theta_I^2}\frac{\rho}{4\rho-1} & \text{if } 1 \le \rho \end{cases} \\
&= \begin{cases} \frac{\theta_C^2+\theta_I r_I \theta_C}{16\theta_C\theta_I^3 r_I - 4\theta_I^4 r_I^2} & \text{if } \theta_C \le r_I\theta_I \\ \frac{\theta_C}{4\theta_C\theta_I^2 - \theta_I^3 r_I} & \text{if } r_I\theta_I \le \theta_C \end{cases} .
\end{aligned}$$

□

**Corollary C.7.** *The demand function of a non-controlling investor is increasing if the controlling investor's preferences are more aligned with the informed investor's preferences, i.e., if $\rho \to 1$, or $\theta_C$ gets closer to $r_I\theta_I$.*

*Proof.* By lemma C.5 the demand of the non-controlling investor is increasing in $\mathbb{E}\left[a^*(\rho,\Omega)^2\right]$. From lemma C.6 we have that:

$$\frac{d}{d\theta_C}\mathbb{E}\left[a^*(\rho,\Omega)^2\right]\Big|_{r_I\theta_I\le\theta_C} = -\frac{r_I}{\theta_I}\frac{1}{(4\theta_C-\theta_I r_I)^2} < 0.$$

and

$$\frac{d}{d\theta_C}\mathbb{E}\left[a^*(\rho,\Omega)^2\right]\Big|_{\theta_C\le r_I\theta_I} = \frac{4\theta_C^2 - 2\theta_C\theta_I r_I - \theta_I^2 r_I^2}{4\theta_I^3 r_I (4\theta_C-\theta_I r_I)^2},$$

which is positive iff $\theta_C > \frac{1}{4}\theta_I r_I\left(\sqrt{5}+1\right)$ or if $\rho > \frac{1}{4}\left(\sqrt{5}+1\right)$. The lower-bound on $\rho$ above can be significantly tightened by not using the approximate actions, since these approximations are only very accurate when $\rho$ is close to 1. □

# Appendix D. Delegation

## D.1. Comparing Expected Utilities under Cheap Talk and Delegation

Suppose the authority to choose $a$ is delegated by the controlling investor to the informed investor, making him an *informed controlling investor*. The informed controlling investor knows the realized value of $\omega$.

**Lemma D.1.** *Under delegation, the informed controlling investor is allowed to choose his preferred full-information optimal action $a_I^* = \omega/r_I\theta_I$, and his equilibrium payoff is independent of his holdings $\theta_I$. The investor's equilibrium payoff equals:*

$$-\frac{\sigma^2}{2r_I}(\rho-1)^2\,\mathbb{E}\left(\omega^2\right),$$

*up to an additive constant which does not depend on the agents' holdings $\theta_C$,$\theta_I$.*

*Proof.* The investor chooses strategy $a^*$ to maximize the following function up to linear affine transformations (refer to Lemma A.1):

$$-\left(\theta_I a r_I - \omega\right)^2.$$

This function is concave in $a$. Taking first order conditions w.r.t. $a$ we the optimal action:

$$a_I^* = \frac{\omega}{r_I\theta_I}.$$

This strategy has the property that $\left(\theta_I a r_I - \omega\right)^2 = 0$, which by Lemma A.1 implies that in equilibrium the informed investor's payoff is independent of $\theta_I$. Let's plug in this informed investor's strategy into the investor's expected payoff function given by Lemma A.1. We get, up to additive constants,

$$\begin{aligned}
& -\frac{\sigma^2}{2r}\mathbb{E}\left(\theta_C a_I^* - \omega\right)^2 \\
= & -\frac{\sigma^2}{2r}\mathbb{E}\left(\theta_C\frac{\omega}{r_I\theta_I} - \omega\right)^2 \\
= & -\frac{\sigma^2}{2r}\left(\frac{\theta_C}{r_I\theta_I} - 1\right)^2\mathbb{E}\left(\omega^2\right).
\end{aligned}$$

□

Lemma D.1 highlights an important feature of the model. If the informed investor is given control, ownership does not matter to him: the informed investor can perfectly compensate for a less-than-ideal ownership level $\theta_I$ by increasing or decreasing $a$ as needed. For example, if the informed investor is given fewer shares, then the informed investor can compensate by choosing a larger value of $a$, that is, by amping up the risky strategy. Formally, this can be seen from Lemma A.1: in the strategically relevant component of the payoff function, $a$ and $\theta$ are perfect substitutes. Ownership matters to the investor, however. If the ownership structure is not well aligned, such that $\rho \neq 1$, the controlling investor suffers a utility loss.

**Proof of Proposition 9.**

*Proof.* From Proposition 10 the investor's payoff under any cheap talk equilibrium with $N$ partitions is:

$$V_C(\rho, N) \leq \begin{cases} -\frac{\sigma^2}{24} & \text{if } 0 \leq \rho \leq \frac{3}{4} \\ \frac{\sigma^2}{6}\frac{(\rho-1)(3\rho-1)}{(4\rho-1)} & \text{if } \frac{3}{4} \leq \rho \leq 1 \\ \frac{\sigma^2}{6}\frac{1-\rho}{(4\rho-1)} & \text{if } 1 \leq \rho \end{cases}$$

**Case** $0 \leq \rho \leq \frac{3}{4}$ **:** From Lemma D.1, because $\mathbb{E}(\omega^2) = 1/3$, the investor's payoff under delegation is:

$$-\frac{\sigma^2}{6}(\rho - 1)^2 .$$

The controlling investor does better under delegation than under cheap talk if and only if this inequality is satisfied:

$$-\frac{\sigma^2}{6}(\rho - 1)^2 \geq -\frac{\sigma^2}{24},$$

(the "only if" part holds because in this region $V_C(\rho, N)$ actually equals $-\sigma^2/24$). This inequality holds so long as $\rho \geq 1/2$.

**Case** $3/4 \leq \rho \leq 1$ **:** If $3/4 \leq \rho \leq 1$, the controlling investor does better under delegation than under cheap talk if this inequality is satisfied:

$$-\frac{\sigma^2}{6}(\rho - 1)^2 \geq \frac{\sigma^2}{6}\frac{(\rho - 1)(3\rho - 1)}{(4\rho - 1)}.$$

This inequality is equivalent to:

$$-(\rho - 1) \leq \frac{(3\rho - 1)}{4\rho - 1}. \tag{D.1}$$

Since $\rho > 3/4$, (D.1) rewrites as $\rho \geq 1/2$, which is true when $\rho > 3/4$. So delegation is better than cheap talk for the controlling investor in the entire interval $3/4 \leq \rho \leq 1$.

**Case** $\rho \geq 1$ : If $\rho \geq 1$ the controlling investor does better under delegation than under cheap talk if and only if this inequality is satisfied:

$$-\frac{\sigma^2}{6}(\rho-1)^2 \geq \frac{\sigma^2}{6}\frac{1-\rho}{(4\rho-1)}, \tag{D.2}$$

Condition (D.2) is equivalent to $\rho \leq 5/4$. If $\rho > 5/4$ then (D.2) holds with the reverse inequality; since there are values of $N$ large enough such that $V_C(\rho, N)$ is arbitrarily close to the left-hand side of (D.2), for those values of $N$ the investor's payoff is higher under cheap talk. □

## D.2. Competitive Equilibrium in the Market for Shares under Delegation

We study the case in which the informed investor is delegated the power to choose $a$ no matter the size of his holdings $\theta_I$. In this case, the informed investor will choose his preferred full-information optimal action $a_I^* = \omega/r\theta_I$.

Working out the competitive equilibrium requires computing the agent's demand functions for shares. Demand functions must be defined carefully because in our setting the value of the corporation depends on the action $a$. Under delegation, the action $a$ is chosen by the informed investor and it affects an investor's demand function. In the spirit of competitive equilibrium, we define an agent's inverse demand function as the marginal contribution of one additional share to that agent's equilibrium payoff, keeping fixed the actions of all other agents.

**Lemma D.2.** *Under delegation, the informed investor has a payoff of* $\frac{\sigma^2}{2r}\omega^2$ *for any positive share holdings, and thus has no demand for shares. Fixing the informed investor's holdings at* $\theta_I > 0$, *an investor's inverse demand for shares under delegation is given by* $D_C(\theta_C|\textit{delegation}) = \frac{\sigma^2}{r\theta_I}(1-\rho)\mathbb{E}(\omega^2)$.

*Proof.* The informed investor sets his strategy at:

$$a_I^* = \frac{\omega}{r\theta_I}. \tag{D.3}$$

Plugging into the informed investor's expected utility function yields the equilibrium expected payoff:

$$v_I(\theta_I|\text{delegation}) = -\frac{\sigma^2}{2r}\left(a_I^*\theta_I r_I - \omega\right)^2 + \frac{\sigma^2}{2r}\omega^2 = \frac{\sigma^2}{2r}\omega^2.$$

This expression is independent of $\theta_I$, so the informed investor has no value for stock.

Let's plug the informed investor's optimal action into the investor's expected payoff function. We get

$$v_C(\theta_C|\text{delegation}) = -\frac{\sigma^2}{2}\mathbb{E}\left(a_I^*\theta_C - \omega\right)^2 + \frac{\sigma^2}{2}\mathbb{E}\left(\omega^2\right).$$

The derivative of this value function represents the inverse demand for shares on the controlling investor part.

$$\begin{aligned} D_C(\theta_C|\text{delegation}) &= \frac{\partial}{\partial\theta_C}v_C(\theta_C|\text{delegation}) \\ &= -\sigma^2\mathbb{E}a_I^*\left(a_I^*\theta_C - \omega\right) \\ &= -\sigma^2\mathbb{E}\frac{\omega}{r\theta_I}\left(\frac{\omega}{r\theta_I}\theta_C - \omega\right) \\ &= \frac{\sigma^2}{r\theta_I}\left(1 - \frac{\theta_C}{r\theta_I}\right)\mathbb{E}\left(\omega^2\right). \end{aligned}$$

□

The fact that the informed investor does not value shares may seem counterintuitive – after all, the enterprise is valuable. Indeed, the informed investor finds value in ownership, as witnessed by the fact that the informed investor's expected payoff is strictly positive $\frac{\sigma^2}{2r}\mathbb{E}\left(\omega^2\right) > 0$. However, the informed investor's payoff is independent of the size of its holdings. In our model, this happens because the informed investor can fully compensate for a low share allocation $\theta_I$ by costlessly scaling up the enterprise (increasing $a$). Put differently, having control of the enterprise makes the informed investor indifferent about ownership.

We see that the investors' inverse demand function is a decreasing function of $\theta_C$ which becomes negative when $\theta_C > r\theta_I$. This means that no controlling investor would want to hold more than $r_I \geq 1$ times the informed investor's holding. This observation raises the question of whether shares are scarce (i.e., valuable) in this economy. The answer depends on how large $N$ is.

**Proposition 12.** *Fix the informed investor's holding $\theta_I > 0$. Under delegation, the investors' demand for shares exceeds supply, and so the market-clearing price of a share is positive, if and only if the number of investors $N > (1-\theta_I)/r\theta_I$. In this case the market-clearing price equals $\frac{\sigma^2}{r\theta_I}\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)\mathbb{E}\left(\omega^2\right)$.*

*Proof.* To compute individual controlling investor demand at zero price set $D_C(\theta_C|\text{delegation}) = 0$ and solve for $\theta_C$. This yields an individual demand of $r\theta_I$ and thus a market demand of $N \cdot r\theta_I$. The amount of available shares for the investors equals $1-\theta_I$. When $N \cdot r\theta_I > 1-\theta_I$, shares are valuable.

When shares have a positive market-clearing price, every controlling investor purchases the same quantity $(1-\theta_I)/N$ in equilibrium. Plugging this quantity into the inverse demand function yields the equilibrium price:

$$D_C\left(\frac{1-\theta_I}{N}|\text{delegation}\right) = \frac{\sigma^2}{r\theta_I}\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)\mathbb{E}\left(\omega^2\right).$$

□

**Proposition 13.** *Under delegation, the market-clearing price varies non-monotonically with the informed investor's holding $\theta_I$. When the market-clearing price is zero, however, the market-clearing price is locally increasing in $\theta_I$.*

*Proof.* The sign of the derivative of the market clearing price with respect to the informed investor's holding $\theta_I$ is ambiguous:

$$\begin{aligned}
&\frac{\partial}{\partial\theta_I}\frac{\sigma^2}{r\theta_I}\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)\mathbb{E}\left(\omega^2\right) \\
= \; & \frac{\sigma^2}{r_I}\mathbb{E}\left(\omega^2\right)\frac{\partial}{\partial\theta_I}\left[\frac{1}{\theta_I}\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)\right] \\
= \; & \frac{\sigma^2}{r_I}\mathbb{E}\left(\omega^2\right)\left[-\frac{1}{(\theta_I)^2}\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)+\frac{1}{\theta_I}\frac{1}{Nr\left(\theta_I\right)^2}\right] \\
= \; & \frac{\sigma^2}{r_I}\mathbb{E}\left(\omega^2\right)\frac{1}{(\theta_I)^2}\left[-\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)+\frac{1}{Nr\theta_I}\right] \\
= \; & \frac{\sigma^2}{r_I}\mathbb{E}\left(\omega^2\right)\frac{1}{(\theta_I)^2}\left[-1+\frac{1-\theta_I}{Nr\theta_I}+\frac{1}{Nr\theta_I}\right] \\
> \; & \frac{\sigma^2}{r_I}\mathbb{E}\left(\omega^2\right)\frac{1}{(\theta_I)^2}\left[-1+\frac{1-\theta_I}{Nr\theta_I}\right] < 0
\end{aligned}$$

Therefore, increasing $\theta_I$ does not necessarily guarantee that the market clearing price increases. However, when the market-clearing price equals zero then

$$\frac{\partial}{\partial \theta_I}\frac{\sigma^2}{r\theta_I}\left(1-\frac{1-\theta_I}{Nr\theta_I}\right)\mathbb{E}\left(\omega^2\right)=\frac{\sigma^2}{r_I}\mathbb{E}\left(\omega^2\right)\frac{1}{\left(\theta_I\right)^2}\left[\frac{1}{Nr\theta_I}\right]>0.$$

□

We now examine optimal allocations.

**Definition 4.** *An optimal risk-sharing allocation is a triple of functions* $\left(\theta_I^{OPT}(\omega), \theta_C^{OPT}(\omega), a^{OPT}(\omega)\right)$ *such that the sum of utilities of informed investor and investors is maximized for each* $\omega$.

This notion of optimal risk-sharing is informationally demanding in that it assumes that the social planner is able to choose all the variables conditional on the realized $\omega$. The next notion of constrained-optimal risk-sharing is comparatively undemanding, in that ownership shares do not depend on $\omega$.

**Definition 5.** *A constrained-optimal risk-sharing allocation is a pair of numbers* $\left(\theta_I^{OPT}, \theta_C^{OPT}\right)$ *such that the sum of expected utilities of informed investor and investors is maximized, conditional on* $a = a_I^*$.

**Proposition 14.** *Fix* $N$. *The constrained-optimal risk allocation coincides with the optimal risk allocation. At this allocation we have:* $\theta_I^{OPT} = 1/(1+Nr)$; $\theta_C^{OPT} = r_I/(1+Nr)$; $a^{OPT}(\omega) = \omega(1+Nr)/r_I$; *and the market-clearing price equals zero.*

*Proof.* The optimal risk-sharing allocation solves:

$$\begin{aligned}&\arg\max_{a(\omega),\theta_I(\omega),\theta_C(\omega)} -\frac{1}{r_I}\left(a(\omega)r\theta_I(\omega)-\omega\right)^2-N\cdot\left(a(\omega)\theta_C(\omega)-\omega\right)^2\\&\text{s.t. } \theta_I(\omega)+N\cdot\theta_C(\omega)=1.\end{aligned}$$

Form the Lagrangian:

$$L\left(\theta_I(\omega),\theta_C(\omega),a(\omega)\right)=-\frac{1}{r_I}\left[a(\omega)r\theta_I(\omega)-\omega\right]^2-N\left[a(\omega)\theta_C(\omega)-\omega\right]^2+\lambda(\omega)\left[1-\theta_I(\omega)-N\cdot\theta_C(\omega)\right]$$

The first order conditions are: for all $\omega$,

$$\begin{aligned} -2a(\omega)\left[a(\omega) r\theta_I(\omega) - \omega\right] &= \lambda(\omega) \\ -Na(\omega)\left[a(\omega)\theta_C(\omega) - \omega\right] &= \lambda(\omega) \\ -\theta_I(\omega) \cdot \left[a(\omega) r\theta_I(\omega) - \omega\right] &= N\theta_C(\omega) \cdot \left[a(\omega)\theta_C(\omega) - \omega\right]. \end{aligned}$$

All three equations are satisfied if: $\lambda(\omega) \equiv 0$; $\theta_C(\omega) \equiv \theta_C; \theta_I(\omega) \equiv \theta_I; \theta_C = r\theta_I \equiv \omega / a(\omega)$ for all $\omega$; and the resource constraint $\theta_I + N \cdot \theta_C = 1$ holds. Using $\theta_C = r\theta_I$ to substitute into the resource constraint we get:

$$\theta_I + Nr\theta_I = 1,$$

and hence

$$\theta_I^{OPT} = \frac{1}{1+Nr}.$$

Then

$$\theta_C^{OPT} = \frac{r_I}{1+Nr},$$

and

$$a^{OPT}(\omega) = \omega \frac{1+Nr}{r_I}.$$

Let's now turn to the constrained-optimal risk allocation. By construction, that sum of payoffs in that allocation is bounded above by the sum of payoffs in the optimal risk allocation. We now show that the bound is achieved. To see this, note that if we set $\theta_I = \theta_I^{OPT}$, then the informed investor's optimal action solves (see D.3):

$$a_I^* = \frac{\omega}{r\theta_I^{OPT}} = a^{OPT}(\omega).$$

This means that the constraint imposed by the fact that $a$ is chosen by the informed investor is no constraint at all when $\theta_I = \theta_I^{OPT}$. This shows that the triple $\theta_I^{OPT}, \theta_C^{OPT}, a^{OPT}(\omega)$ also solves the constrained-optimal problem.

Finally, note that

$$\frac{1-\theta_I^{OPT}}{r\theta_I^{OPT}} = N,$$

which from Proposition 12 is exactly the condition under which the competitive price equals zero. □

The intuition for why the optimal risk sharing requires a zero competitive price of the stock is as follows. The planner can use the share allocation $\theta_I, \theta_C$ to perfectly align the objective functions of investors and informed investor (typically, this instrument will be used to compensate for the informed investor's risk aversion parameter $r_I$). Once this alignment is achieved, either the planner or the informed investor himself are free to set $a$ in a way that maximizes societal welfare. This intuition explains why constrained and non-constrained optimal allocations coincide. Note, in addition, that in the constrained optimal allocation the informed investor's choice of $a_I^*$ guarantees a riskless payoff regardless of the value of $\theta_I$. Because marginal utilities must be equated at the optimum, the investors' payoffs must also be insensitive to the allocation of $\theta_C$. But this means that the inverse demand function for stock must equal zero at the optimum.