Equilibrium Selection in Sequential Games with Imperfect Information

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Abstract

Games with imperfect information often feature multiple equilibria, which depend on beliefs off the equilibrium path. Standard selection criteria such as passive beliefs, symmetric beliefs or wary beliefs rest on ad hoc restrictions on beliefs. We propose a new selection criterion that imposes no restrictions on beliefs: we select the action profile that is supported in equilibrium by the largest set of beliefs. We conduct experiments to test the predictive power of the existing and our novel selection criteria in two applications: a game of vertical multi-lateral contracting, and a game of electoral competition. We find that our selection criterion outperforms the other selection criteria.

Keywords: Equilibrium selection, passive beliefs, symmetric beliefs, vertical contracting, multiple equilibria, imperfect information.

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1 Introduction

We propose a solution to the problem of multiplicity of equilibria in a class of two-stage games in which players who move at the second stage (receivers) are imperfectly informed about the actions played by those who move at the first stage (proposers). These games typically feature multiple Perfect Bayesian and sequential equilibria. Equilibria depend on how, after observing a deviation, receivers update their beliefs about proposers’ actions. Refinements that are useful for signaling games such as the intuitive criterion (Cho and Kreps (10)), D1 or universal divinity (Banks and Sobel (3)) have no bite in this context, because the lack of information is about the proposers’ actions, not about their type (the game is one of imperfect information, not incomplete information; players’ types are known).

Our main application of interest in this class of games is multi-lateral vertical contracting. One or more upstream firms make private offers to each of two or more downstream firms. Each contract signed by a downstream firm affects all downstream firms (contracts generate externalities), but at the time a downstream firm decides whether or not to accept the offer it receives, it does not know what offers other firms have received (downstream firms operate under imperfect information).

The industrial organization literature on multi-lateral vertical contracting has dealt with this multiplicity of equilibria by imposing particular beliefs that are deemed more appealing off the equilibrium path, selecting equilibria that can be supported by these beliefs, and discarding all other equilibria. McAfee and Schwartz (28) propose three possible beliefs to consider: passive beliefs, symmetric beliefs, and wary beliefs.

Passive beliefs, sometimes called “passive conjectures” (Rey and Tirole (35)), are such that a downstream firm that receives an out of equilibrium offer does not update its beliefs on the offers received by all other players; rather, it believes that all the other unobserved actions remain as in equilibrium. The selection criterion based on singling out equilibria that can be supported by such passive beliefs is the one most frequently used in the literature, not only in games of multilateral vertical contracting.
(Hart and Tirole (22); O’Brien and Shaffer (31); Segal (38); Fontenay and Gans (14);
Rey and Tirole (35); Inderst and Ottaviani (24); Caprice (9)) but also in games
of electoral competition in which two candidates make offers to voters in multiple
districts (Gavazza and Lizzeri (16)). The assumption of “passive beliefs” is also
common in the consumer search literature: for instance consumers do not revise their
beliefs about a firm’s quality when it charges an unexpected price (see for instance
recent applications by Bar-Isaac, Caruana and Cuñat (5) or Buehler and Schuett (7)).
However, “in many circumstances the ad hoc restriction to passive beliefs may not
be compelling” (Segal and Whinston (39)). Indeed, while defending the assumption
of passive beliefs in the particular game in which they use it, Rey and Tirole (35)
concede that assuming passive beliefs “is much less appealing in the case of Bertrand
competition, and indeed in many games of contracting with externalities.”

An alternative criterion to solve the multiplicity problem is to select equilibria that
can be supported by symmetric beliefs. These beliefs are such that a downstream
player that receives an out of equilibrium offer believes that all other downstream
players receive this same offer as well. If the equilibrium offer to each downstream
player $i$ is $x_i$, a downstream player who receives an offer of $y \neq x_i$ believes that the
offer to every downstream players is also $y$. Selection by symmetric beliefs is used,
among others, by Pagnozzi and Piccolo (32). However, most of the literature seems
to agree with McAfee and Schwartz’s (28) initial assessment: “[Symmetric] beliefs are
not very compelling.”

A third suggestion is to consider equilibria supported by wary beliefs. These
beliefs are such that a downstream player who observes a deviation believes that the
upstream player must have deviated to a strategy that is optimal given the action
that the downstream player observes. This criterion has had scant following in the
literature of vertical contracting (Rey and Vergé (36), Avenel (1)), but Hagiu and
Halaburda (19) apply it in a setup with markets with two-sided network effects.

The problem common to all these criteria is the lack of a convincing argument
for why only one particular set of beliefs should be admissible off the equilibrium path. The existence of several alternative choices of specific beliefs that have received consideration in the literature underscores that none of these beliefs are an obvious choice for all possible games of imperfect information. Eliminating all equilibria that rest on different beliefs is not warranted. We argue that passive beliefs, wary beliefs, or symmetric beliefs, may be plausible in a given particular application, but not in others. A sharp restriction on the set of admissible beliefs to the exclusion of all others is not often appropriate. Martin, Normann and Snyder (27) provide evidence of the weakness of this approach. Using a laboratory experiment that mimics a vertical industry structure with an upstream firm and two competing downstream firms, they find that no specific restriction on beliefs (passive or symmetric beliefs) is consistent with the data.¹

A more cautious or modest approach is to accept that we cannot pin down the exact beliefs off the equilibrium path, beyond the restrictions given by standard refinements such as those implicit in a Perfect Bayesian or a sequential equilibrium (Kreps and Wilson (25)). Any assumption of specific equilibrium beliefs, even a plausible one, is ad hoc and it is difficult to justify as superior to all others.

We solve the multiplicity problem by suggesting that the equilibrium action profile most likely to emerge is the action profile that can be supported in equilibrium by a largest set of different beliefs. In application of Bernoulli (4) and Laplace’s (26) “Principle of Insufficient Reason,” absent any motivation to consider some equilibrium beliefs as more likely to emerge than others, we should treat all equilibrium beliefs as a priori equally probable to emerge.² If so, the strategy profile and action profile most likely to be played in equilibrium are the ones that can be sustained by a largest collection of different equilibrium beliefs. We predict that this action profile will be

¹See as well the contracting experiments by Boone, Müller and Suetens (6), and currently ongoing work by Möllers, Normann and Snyder (29).

²As defined by Wolfram Mathworld, the Principle of Insufficient Reason “states that if we are ignorant of the ways an event can occur (and therefore have no reason to believe that one way will occur preferentially compared to another), the event will occur equally likely in any way.”
the one played in equilibrium.

We do not identify the specific beliefs that support the predicted action profile in equilibrium: we make a prediction only about players’ equilibrium actions. Since actions (unlike beliefs) are directly observable, our predictions are directly testable.\(^3\)

Our selection criterion provides a “stress test” measure of the robustness of an equilibrium path of play to changes in the beliefs for which it is sustained. The criterion explores how likely is it that the particular path of play is still played in equilibrium if beliefs change, and then selects the equilibrium path of play most likely to be played in a new equilibrium after the change of beliefs.

If a profile of actions \(a\) is played in equilibrium only under some narrowly specific beliefs (so only one or few equilibria result in playing \(a\)), while on the other hand action profile \(a'\) can be supported regardless of agents’ off-path beliefs, or for a wide array of possible beliefs (so that lots of different equilibria result in playing \(a'\)), then we predict that in equilibrium, players will play according to \(a'\), not \(a\).

For instance, in Section 3, we consider a vertical contracting game in which a producer can offer a product to each of two retailers at either a high or low price, and each retailer chooses how many units to buy. The producer can price-discriminate, and neither retailer knows the price offered to the other retailer. There are two equilibria in pure strategies in this game, one in which each retailer is offered a high price and buys one unit, and one in which each is offered a low price and buys two units. Both equilibria hold if a retailer who observes a deviation believes that the other retailer is offered a high price with a probability lower than a given threshold. This threshold depends on parameters of the games and differs across the two equilibria. For given specific parameters,\(^4\) the equilibrium with high prices holds

\(^3\)We do not provide a theory of belief formation, and we do not assume that players regard all equilibrium beliefs as equally likely. Our interpretation is that players play one equilibrium, hold this equilibrium as certain, and dismiss all other equilibria and beliefs as entirely unlikely. However, since analysts forecasting how a game will be played do not know how beliefs are formed, a priori we treat all equilibrium beliefs equally in order to predict which equilibrium action profile will be played.

\(^4\)These parameters correspond to Treatment \(H\) in Section 3 (see Table 1).
for any beliefs such that a retailer who observes a deviation (an unexpected offer of a low price) believes that the other retailer is offered a high price with probability in \([0, 0.92]\), while the equilibrium with low prices holds for any beliefs such that a retailer who observes a deviation believes that the other retailer is offered a high price with probability in \([0, 0.16]\). Since the first case encompasses a larger set of beliefs, we predict that the strategy profile played in equilibrium is the one in which the producer offers high prices and retailers buy one unit.

More generally, to identify and select the equilibrium actions that are most robust in the sense that they can be sustained in equilibrium by the largest set of beliefs, we proceed in three steps:

1. We define a measure over sets of out of equilibrium beliefs. In line with the Principle of Insufficient Reason, we use the standard Lebesgue measure.

2. For each strategy profile, we identify the set of beliefs that support this strategy profile in equilibrium. For each action profile, we calculate the measure of the set of beliefs that support at least one equilibrium strategy profile in which this action profile is played.

3. We select the action profile with the largest set of beliefs that support it as an equilibrium action profile.

In any finite game, our criterion yields a non-empty prediction, a desirable property of any solution concept that selection by passive, symmetric or wary beliefs fail to meet.

Above and beyond the appealing theoretical properties of selection by the largest set of beliefs, we are interested in the predictive power of this criterion. Using laboratory experiments, we test our selection by the largest set of beliefs against selection by other criteria in two applications of interest: vertical multilateral contracting, and electoral competition over multiple districts. Our experiments consists in two simple sequential games. Relative to the seminal work by Martin, Normann and Snyder (27),
the simplicity of our games has the advantage that most groups of subjects play some equilibrium; whereas, not a single group of subjects play according to equilibrium in the complex games in the Martin, Normann and Snyder (27) experiments.\footnote{Their experiment is aimed at testing different theories of vertical foreclosure and the analysis of out of equilibrium beliefs and the relative performance of different selection criteria are not the goals of their paper.} We can therefore conduct a more meaningful analysis of equilibrium selection within the large collection of groups of subjects who play according to some equilibrium.

Overall we find that selection according to the largest set of beliefs has better predictive power than any of the other selection criteria. In the application to vertical contracting, in the aggregate over all treatments, 74\% of groups of agents play some equilibrium, and among these, 94\% play the equilibrium selected by the largest set of supporting beliefs, while only 54\% play the equilibrium selected by passive or wary beliefs and only 46\% play the equilibrium supported by symmetric beliefs. In the electoral competition application results are weaker: only 49\% of groups play an equilibrium. Within these groups, selection according to the largest set of supporting beliefs (76\% correct predictions) dominates selection by symmetric beliefs (4\% correct predictions), but it does not predict significantly better or worse than selection by passive beliefs (78\%) or by wary beliefs (67\%). In each of the two applications we run two different treatments. We find strong and significant comparative statics across treatments in each of the two applications. These differences across treatments correspond perfectly with the comparative static predictions of our selection criterion, and clash with the predictions of each of the other equilibrium selection criteria.

Our results show that our criterion correctly predicts which action profile is more likely to emerge, even under the conservative assumption that we lack any information about which beliefs are more salient or plausible so we treat them all as equally likely. While a well-informed prior on which beliefs are more plausible could lead to even better results by weighing beliefs according to such prior (instead of the uniform), we often lack grounds to favor any prior, and assuming that players’ beliefs are passive
or symmetric is not generally warranted; in particular, in our application to vertical contracting, either of these assumptions is detrimental.

For completeness, we have also considered a number of other refinements that, to our knowledge, have not been previously used in our applications of interest (vertical contracting and electoral competition), but are prominent in other game theoretic research. Harsanyi and Selten (20) propose payoff dominance and risk-dominance. Payoff dominance selects a Pareto superior equilibrium (one that yields higher payoffs to every player) over a Pareto inferior one. This criterion may seem plausible when it is applicable, which is only seldom.\(^6\) Risk dominance, extended by Harsanyi (21) and Selten (40) selects the equilibrium that is least risky in the sense that each player minimizes the potential losses if she cannot anticipate which equilibrium will be played by other agents. Risk dominance is defined for normal form games and while it has performed well in laboratory experiments on coordination games (Cabrales, Garcia-Fontes, Motta (8); Schmidt, Shupp, Walker and Ostrom (37)), it is not obvious how to apply the intuition underlying this concept to games with imperfect information in a compelling manner. Peski (33) introduces two variations of risk dominance that apply to any finite normal form game, but the intuition behind these two refinements does not extend well to sequential games and in our application, both refinements deliver an empty prediction. Forward induction as defined by Govindan and Wilson (18) is applicable and makes a non-empty prediction. Alas, this prediction does not solve our multiplicity problem: all equilibria in the vertical contracting game we study satisfy forward induction.\(^7\) Myerson’s (27) proper equilibrium refinement is the only one of these concepts that is applicable and offers a sharper prediction in our games. However, its predictive power is underwhelming: its predictions coincide with those of selection by wary beliefs, and thus in both applications it underperforms relative to equilibrium selection based on identifying the strategy profile that can be supported.

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\(^6\)Payoff dominance only offers a prediction in one of the four treatments in our experiments, and when it does, it coincides with our prediction.

\(^7\)A proof of this statement is available from the authors.
in equilibrium by the largest set of beliefs. Finally, in a more recent research paper, In and Wright (23) propose a new equilibrium concept to refine the set of Perfect Bayesian equilibria in a class of senders-receivers games. They name this concept “RI-equilibrium.” In the application to vertical contracting, the RI-equilibrium coincides with the equilibrium with wary beliefs and with the proper equilibrium (and more generally, the RI-equilibria are a superset of the proper equilibria), and therefore in our applications, the RI-equilibrium does not offer as good a prediction as our selection criterion.

In what follows we first define our selection criterion for a class of two-stage finite games, and then we provide the theory, empirical implications, and experimental tests in the application to vertical contracting. An Appendix contains proofs to the theoretical propositions. The theory, empirical implications and experimental tests in the application to electoral competition is on an online appendix.

2 A New Selection Criterion

Consider a class of finite extensive form games \( G \) with one or more upstream players (proposers), and two or more downstream players (receivers). Let \( P \) denote the set of proposers, \( R \) denote the set of receivers, \( N = P \cup R \) denote the set of players, \( n_P = |P| \) the number of proposers, and \( n_R = |R| \) the number of receivers. Proposers move first, simultaneously: each proposer \( j \) chooses an action \( a_j = (a_{j,1}, a_{j,2}, \ldots, a_{j,n_R}) \in A_j \subset \mathbb{R}^{K \times n_R} \), where \( K \) is a strictly positive integer, \( a_{j,i} \in \mathbb{R}^K \) is a vector for each \( j \in P \) and \( i \in R \), and \( A_j \) is a finite set of feasible actions. A proposer action \( a_j \) is a matrix with \( n_R \) vectors. We interpret each vector \( a_{j,i} \in \mathbb{R}^K \) as an offer made to receiver \( i \). Each receiver \( i \) observes each element of each matrix of offers with some probability in \([0, 1]\). These probabilities may differ across elements of the matrix, they may be correlated across elements and across receivers, and they may be degenerate so that Nature plays no role and \( i \) observes some vectors but not others. We assume that
there is at least one proposer $j$, one receiver $i$, and one element of $a_j$ such that $i$ observes this element of $a_j$ with probability strictly less than one, so that the game is not one of perfect information. Our motivation is that receiver $i$ is more likely to observe the offers made to her ($a_{j,i}$ by each proposer $j$) than the offers made to any other receiver, but we do not need to assume this as a restriction. At the second and last stage, all receivers take simultaneous actions in response to the information they have observed. We do not impose any restriction or assumption (beyond finiteness) on the action set of receivers.

For any finite extensive form game $\Gamma \in \mathcal{G}$, let $s_P$ denote a proposers’ pure strategy profile, $s_R$ a receivers’ pure strategy profile, and $s = (s_P, s_R)$ a pure strategy profile. Let $s_i$ denote the strategy for agent $i$, and let $s_{-i}$ denote a strategy profile for every other agent, so $s = (s_i, s_{-i})$. Let $S_P, S_R$ and $S = S_P \times S_R$ be the set of all proposers’ pure strategy profiles, the set of all receivers’ strategy profiles, and the set of all strategy profiles. Let $I$ be an information set in game $\Gamma$, let $\mathcal{I}$ be the collection of all information sets, so that $|\mathcal{I}|$ denotes the number of information sets in this collection, and for each $I \in \mathcal{I}$, let $i(I)$ be the player who moves at information set $I$. Let $\mathcal{I}_P \equiv \{ I \in \mathcal{I} : i(I) \in P \}$ be the collection of information sets in which the mover is a proposer. For each information set $I \in \mathcal{I}$, let $a(I)$ denote an action taken at $I$. For each information set $I \in \mathcal{I}$ and each strategy profile $s$, let $s(I)$ be the action (or behavioral strategy) taken by $i(I)$ at $I$ according to strategy profile $s$. Let a proposers’ action profile, denoted $a_P$, be a list specifying the action taken at each information set in which the mover is a proposer. That is, a proposers’ action profile is a list of all the actions taken by the set of proposers.

Given any proposers’ action profile $a_P$, let $\mathcal{I}(a_P) \subset \mathcal{I}$ be the collection of all information sets contained in the branches of the game tree that follow after proposers play $a_P$. That is, each $I \in \mathcal{I}(s_P)$ is reached after proposers play $a_P$ for some realization of Nature’s moves, and no node outside $\mathcal{I}(a_P)$ is reached after proposers play $a_P$, regardless of Nature’s moves.
Given a proposer’s action profile $a_P$, let $\mathcal{O}(a_P) \subset \mathcal{I}$ be the collection of all information sets that:

i) cannot be reached if proposers play $a_P$, i.e. the intersection $\mathcal{O}(a_P) \cap \mathcal{I}(a_P)$ is empty,

ii) are reached for some realization of Nature’s moves if exactly one proposer $j$ deviates from $a_P$, and

iii) there are at least two actions $a'_j$ and $a''_j$ such that information set $I$ can be reached if $j$ plays $a'_j$ or $a''_j$ (hence receiver $i(I)$ must form beliefs about proposer $j$’s actions).

We interpret $\mathcal{O}(a_P)$ as the collection of information sets one step out of the path of $a_P$. Since the game $\Gamma$ is finite, the size of $\mathcal{O}(a_P)$, denoted $|\mathcal{O}(a_P)|$, is also finite for any $a_P$. At each information set $I$, let beliefs $\mu_I$ be a probability distribution over the nodes contained in $I$. Let $\mu = (\mu_1,...,\mu_{|\mathcal{I}|})$ be a list of such beliefs, one per information set.

Since each proposer’s set of pure strategies is the set of actions available to the proposer, the proposers’ strategy profile and proposers’ action profile coincide: given any proposers’ strategy profile $s_P$, the proposers’ action profile is $a_P = s_P$. Given a proposers’ action profile $a_P$, let a receivers’ action profile be a list specifying the action to be taken at each information set $I \in \mathcal{I}(a_P)$, and let this receivers’ action profile be denoted $a^R_P$. We then refer to the pair $a = (a_P, a^R_P)$ as an action profile. Each equilibrium strategy profile $s$ is associated with one equilibrium action profile $a$, which specifies the actions taken along the equilibrium paths. Notice, however, that if there is uncertainty, not all information sets along the equilibrium paths are reached along the path of play in every realization of the game, as Nature’s resolution of uncertainty directs the game along one or another path of play. Let $A$ be the set of all possible action profiles, and let $a \in A$ be an arbitrary action profile.

A pure Perfect Bayesian Equilibrium (PBE) of game $\Gamma$ is a pair $(s, \mu)$ such that given $\mu$ and given $s_{-i}$, each player at each information set maximizes her expected
utility playing according to $s_i$, and such that given $s$, beliefs $\mu$ are correct along the equilibrium path, and satisfy Bayes Rule and the usual consistency requirements off the equilibrium path.\footnote{See, for instance, Fudenberg and Tirole (15) p. 331-332.}

Given any Perfect Bayesian Equilibrium strategy profile $s^*$ with associated action profile $a^*$, beliefs along the path of equilibrium play are pinned down by equilibrium strategies and Bayes rule. Beliefs about the actions of other receivers at information sets that follow an observed deviation by one (or more) proposers are also pinned down by receiver’s equilibrium strategies, as are the beliefs about the proposers that have not been observed to deviate from $s^*$. There is, however, some indeterminacy in the beliefs held by receivers about the actions taken by any proposer $j$ who is observed to have deviated from $s^*$. For any information set $I \in \mathcal{O}(a^*_P)$, let $j^I$ be the proposer who must deviate from $a^*_P$ in order for $I$ to be reached.\footnote{If more than one proposer deviates, beliefs about each deviating proposer are determined independently. Therefore, a specification of beliefs in $\mathcal{O}(a^*_P)$ suffices to construct a complete belief system everywhere in $I$.} Then $A_{j^I}$ is the set of all feasible actions for $j^I$. Let $B_{j^I} \subset A_{j^I}$ be the subset of actions for proposer $j^I$ such that information set $I$ can be reached if every proposer in $P \setminus \{j^I\}$ plays according to $a^*_P$.

Let $\Delta_I$ be the set of all probability distributions over $B_{j^I}$ and let $\omega_I \in \Delta_I$ be a probability distribution over $B_{j^I}$. Set $\Delta_I$ is the set of all beliefs that $i(I)$ might hold at $I$ about what proposer $j^I$ did to make the game unexpectedly reach information set $I$. Hereafter we refer to $\omega_I$ as the belief by $i$ at $I$. The standard “belief” $\mu_I$ over the nodes in information set $I$ can be calculated directly from the belief $\omega_I$.

Let $\Delta_{\mathcal{O}(a_P)} = \prod_{I \in \mathcal{O}(a_P)} \Delta_I$ be the set of all possible belief profiles over the collection of information sets $\mathcal{O}(a_P)$, and let $\omega = (\omega_1, ..., \omega_{|\mathcal{O}(a_P)|}) \in \Delta_{\mathcal{O}(a_P)}$ be a belief profile, which specifies beliefs at each information set in $\mathcal{O}(a_P)$.

For each equilibrium strategy profile $s = (s_P, s_R)$ and for each information set $I \in \mathcal{O}(a_P)$, let $\Delta^*_I \subseteq \Delta_I$ be the set of beliefs of $i(I)$ such that a Perfect Bayesian
Equilibrium with strategy profile \( s \) and with beliefs \( \omega_I \) at each \( I \in \mathcal{O}(a_P) \) exists if and only if \( \omega_I \in \Delta_I^s \) for each \( I \in \mathcal{O}(a_P) \). Then, \( \prod_{I \in \mathcal{O}(a_P)} \Delta_I^s \) is the set of belief profiles that sustain a PBE with strategy profile \( s \).

Let \( \Delta_{\mathcal{O}(a_P)}^s = \prod_{I \in \mathcal{O}(a_P)} \Delta_I^s \). A PBE in which agents play \( s \) and hold belief profile \( \omega \) exists if and only if \( \omega \in \Delta_{\mathcal{O}(a_P)}^s \).

For any action profile \( a = (a_P, a_R^a) \), let \( S^a = \{ s \in S : s_P = a_P \text{ and } s_R(I) = a_R(I) \} \) for each \( I \in \mathcal{I}(a_P) \) be the subset of strategy profiles such that according to any strategy profile in \( S^a \), agents play action profile \( a \) along the equilibrium path.

Let \( \Delta_{\mathcal{O}(a_P)}^a = \bigcup_{s \in S^a} \Delta_{\mathcal{O}(a_P)}^s \). Then \( \Delta_{\mathcal{O}(a_P)}^a \) is the set of belief profiles such that a PBE in which agents play \( a \) and hold belief profile \( \omega \) exists if and only if \( \omega \in \Delta_{\mathcal{O}(a_P)}^a \).

Let \( L \) be the Lebesgue measure over \( \Delta_{\mathcal{O}(a_P)} \), where \( \Delta_{\mathcal{O}(a_P)} \) is the Cartesian product of \( |\mathcal{O}(a_P)| \) simplexes and hence is itself a subset of a unit hypercube. Then, \( L(\Delta_{\mathcal{O}(a_P)}) \) represents the size of all the set of belief profiles over the collection of information sets \( \mathcal{O}(a_P) \), and \( L(\Delta_{\mathcal{O}(a_P)}^a) \) is the size of the subset of these that support action profile \( a \) in a PBE. The fraction \( \frac{L(\Delta_{\mathcal{O}(a_P)}^a)}{L(\Delta_{\mathcal{O}(a_P)})} \) captures the relative size of the belief profiles that support action profile \( a \) in a Perfect Bayesian Equilibrium, over all belief profiles. We refer to this size as the “Size of Supporting Beliefs of \( a \)” or \( SSB(a) \).

**Definition 1** The size of beliefs that support an action profile \( a \) in equilibrium is \( SSB(a) = \frac{L(\Delta_{\mathcal{O}(a_P)}^a)}{L(\Delta_{\mathcal{O}(a_P)})} \). An equilibrium action profile \( a^* \) has a largest set of supporting beliefs if \( SSB(a^*) \geq SSB(a) \) for any \( a \in A \).

We refer to \( a^* \) as the equilibrium action profile with the largest set of beliefs.\(^{10}\) The motivation for this selection criterion is agnosticism about beliefs off the equilibrium path. Selection criteria based on passive, symmetric or wary beliefs assume that off-path beliefs take a particular form, and discard any equilibria not supported by these particular beliefs. But the focality of these beliefs to the exclusion of all others is

\(^{10}\)Note that there can be more than one equilibrium action profiles with a largest set of beliefs. In the Appendix, however, we show that the equilibrium supported by a largest set of beliefs is most often unique. Hence, we refer to \( a^* \) as the equilibrium action profile with *the* largest set of beliefs.
often difficult to justify, as we discuss in the introduction. We take a more open-minded approach toward off-path beliefs. Following traditional pure game theory, we conjecture that agents may have any beliefs off the equilibrium path, and that we are unable to predict which of these beliefs agents will hold. If we have a uniform prior over these off-equilibrium beliefs, any equilibrium may hold.

However, we highlight that if all equilibria are equally likely, some equilibrium strategy profiles or equilibrium action profiles are more likely than others. An equilibrium is a pair \((s, \mu)\), composed of a strategy profile and beliefs. If an equilibrium action profile \(a\) is played according to several different equilibrium strategy profiles, each of which holds for many different beliefs, action profile \(a\) holds in many equilibria. If action profile \(a'\) is played according to a few different strategy profiles, each of which holds for a few beliefs, \(a'\) holds in a few equilibria.

Consider the following algorithm: Randomly draw an action profile \(a\) from a uniform distribution over \(A\). Then, randomly draw a profile of off-path beliefs \(\omega\) from a uniform distribution over \(\Delta_{\mathcal{O}(a_P)} = \prod_{I \in \mathcal{O}(a_P)} \Delta_I\). If a PBE with action profile \(a\) and beliefs \(\omega\) exists, stop. If not, restart the algorithm. The action profile most likely to be the outcome of this algorithm is the one supported by the largest set of beliefs.

We predict an equilibrium action profile. We do not specify the particular equilibrium that supports this equilibrium action profile. We seek to explain agents’ behavior and choices, and an action profile contains a full prediction over these.

The purpose of any equilibrium refinement or selection criterion is to solve the problem of multiplicity under a preferred solution concept by imposing additional restrictions to yield a sharper, ideally unique, prediction. An essential property of a useful refinement or selection criterion is that the criterion must make a non-empty selection. Unfortunately, selection by passive or symmetric beliefs fail this basic requirement even for finite games: in some finite games with pure strategy equilibria, requiring beliefs to be passive, or to be symmetric, eliminates all pure strategy equilibria (see Rey and Tirole (35) for lack of existence of pure equilibria with passive
beliefs). In fact, in applications with capacity constraints, passive or symmetric beliefs are untenable, as they would imply a belief in non-feasible actions (see Avenel (1) and (2)).

In contrast, given a non-empty set of pure equilibria in a finite game, our selection criterion always relies on feasible beliefs, and it always makes a non-empty selection.

**Proposition 1** Given any game $\Gamma \in \mathcal{G}$ with a non-empty set of pure Perfect Bayesian equilibria, there exists an equilibrium action profile with the largest set of supporting beliefs.

More generally, in any game with a finite set of equilibria, selecting the equilibrium action profile supported by the largest set of beliefs guarantees delivery of a well-defined prediction, while the alternative criteria of selection based on passive or symmetric beliefs do not.

In addition to existence of a non-empty prediction, sharpness is a desirable property of a selection criterion. In most games, selection by the size of supporting beliefs yields a unique prediction; the subset of equilibrium action profiles with a largest set of supporting beliefs is a singleton. An exception are games in which two or more equilibrium action profiles are supported by all off-path beliefs: in this class of games, all equilibria supported by any beliefs appear to be equally compelling, and they are both selected according to passive beliefs, symmetric beliefs, wary beliefs, the largest set of beliefs, or any other belief-based notion: there is no belief-based reason to favor an equilibrium action profile over the other. However, in a vast collection of games, each equilibrium holds for some beliefs and not for others. Within this class of games, selection by the largest set of beliefs obtains uniqueness generically (while selection using passive or symmetric beliefs does not attain generic uniqueness). We formalize

---

11For lack of existence with symmetric beliefs, consider a game in which the proposer privately makes an offer $A$ or $B$ to each of two receivers, and receivers accept or reject. Let the proposer prefer an outcome in which one receiver accepts $A$ and the other $B$ to an outcome in which both accept $A$, and prefer both offers rejected to any other outcome; and let receivers prefer both to accept $A$ more than to both accept $B$, and both to reject their offers more than any other combination. The only pure equilibrium is for both to accept $A$, and it is not sustained by symmetric beliefs.
and prove this claim in the Appendix. We also note that our selection criterion is robust (i.e. delivers an invariant prediction) if we relabel agents or actions, and it is also invariant in affine transformations of the payoff matrix. Like Sequential Equilibrium, the equilibrium action profile with the largest set of beliefs is sensitive to changes in the extensive form tree resulting from the addition of strategically superfluous moves, the elimination of dominated strategies, or the addition of cloned (payoff-equivalent) strategies.

Having established that our selection criterion better satisfies desirable theoretical properties (existence and uniqueness), in the next two sections we test how its predictive power compares to that of other alternatives in two applications of interest: vertical contracting, and electoral competition.

3 Application: Vertical Contracting

The main application to test our selection criterion is a theory of vertical contracting with imperfect information and externalities. The game form we present is adapted from Segal (38) and Rey and Tirole (35) (see also Hart and Tirole (22); McAfee and Schwartz (28); or Rey and Vergé (36) among others).

3.1 Theory and Predictions

We present a game form of a simplified multi-lateral vertical contracting game that allows us to generate multiple equilibria and different predictions according to different selection criteria, by varying parameter values.

A proposer, interpreted as a producer or upstream firm, and labeled 0, makes independent offers to each of two receivers \{1, 2\}, interpreted as retailers or downstream firms. Let \(i\) denote an arbitrary retailer and let \(-i\) denote the other retailer. The producer sells a good. An offer takes the form of a price, which is either high \((p^H \in \mathbb{R}_{++})\) or low \((p^L \in \mathbb{R}_{++})\). Let \(p_1, p_2 \in \{p^H, p^L\}\) be the prices
offered by the producer to retailers 1 and 2. The strategy set of the producer is $S_0 = \{(p^H, p^H), (p^H, p^L), (p^L, p^H), (p^L, p^L)\}$, where the first component indicates the offer made to retailer 1 and the second the offer made to retailer 2. Offers are simultaneously and privately made, so that each retailer observes the offer she receives, but not the offer the other retailer receives. Retailers choose how many units to purchase. Because we prefer a more parsimonious game, we constrain the strategy set of retailers. If the price is $p^H$, we assume that retailers can buy zero or one units; and if the price is $p^L$, zero, two or three units. Hence the strategy set for each retailer $i$ is $S_i = \{0, 1\} \times \{0, 2, 3\}$, where each strategy $(s_{iH}, s_{iL}) \in S_i$ corresponds to how much to purchase following a $p^H$ offer (first coordinate) and following a $p^L$ offer (second coordinate). A strategy profile is an element of $\{(p^H, p^L) \times (\{0, 1\} \times \{0, 2, 3\})\}^2$.

The producer incurs a transaction cost $c_0 \in \mathbb{R}_+$ for each executed trade, and each retailer incurs a transaction cost $c_r \in \mathbb{R}_+$ for each executed transaction (but they do not incur any cost if an offer is not accepted). Let $q_i \in \{0, 1, 2, 3\}$ be the quantity purchased by retailer $i$. Retailers sell their units of the good in the consumer market. The price of the good in the consumer market is a function $p : \{0, 1, 2, 3, 4, 5, 6\} \rightarrow \mathbb{R}_+$ of the aggregate supply $Q = q_1 + q_2$. We assume $p(Q) : \{0, 1, 2, 3, 4, 5, 6\} \rightarrow \mathbb{R}_+$ is strictly decreasing in $Q$.

Profits for the producer are $\sum_{i=1}^2 (p_i q_i - c_0 1[q_i > 0])$, where $1[q_i > 0]$ is an indicator function such that $1[q_i > 0] = 1$ if $q_i > 0$, and $1[q_i > 0] = 0$ otherwise. Retailer $i$ profits are $(p(Q) - p_i)q_i - c_r 1[q_i > 0]$.

Because retailer profits depend on the number of units purchased by the other retailer, contracts between the producer and one retailer exert an externality to the other retailer. Because contracts with one retailer are not observed by the other, each retailer faces uncertainty about the expected payoff of purchasing any positive number of units from the producer: willingness to purchase depends on beliefs about the trades executed between the producer and the other retailer.

The game form contains ten parameters. Let $\theta = (p^H, p^L, c_0, c_r, p(1), p(2), p(3), p(4), p(5), p(6))$
be a parameter vector. We are interested in parameter values that generate multiple (two) equilibria in which different selection criteria generate different predictions.

**Definition 2** Let Θ be the set of parameter vectors such that θ ∈ Θ if and only if

i) \(3p^L > p^H > 2p^L > c_0\), and

ii) for each retailer \(i\), the best response correspondences \(BR_i(p_i, q_{-i})\) are given by the following table:

<table>
<thead>
<tr>
<th>Quantity (q_{-i})</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(BR_i(p^H, q_{-i}))</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(BR_i(p^L, q_{-i}))</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The first condition is that for the producer, selling one unit at a high price is better than selling two at a low price, but not as good as selling three at a low price (and any transaction is better than no transaction). The second condition on retailers’ best responses leads to the existence of two equilibria: one in which both retailers purchase one unit at price \(p^H\), and another in which both retailers purchase two units at price \(p^L\). No other outcome can be sustained in a pure equilibrium.

For any parameter vector \(θ\), let \(Γ_θ\) be the game with the specified game form, and parameter values \(θ\).

In this game each equilibrium action profile is played in only one equilibrium strategy profile, and therefore each equilibrium action profile identifies an equilibrium strategy profile. It follows that \(Δ_{Ω(θ_ρ)}^s = Δ_{Ω(θ_ρ)}^o\).

**Claim 2** For any \(θ ∈ Θ\), the set of action profiles that are sustained in a pure Perfect Bayesian Equilibrium (PBE) of game \(Γ_θ\) is \{((\(p^L, p^L\), 2, 2), ((\(p^H, p^H\), 1, 1))\}.
The first equilibrium action profile is \(((p^L, p^L), 2, 2)\), that is, the producer offers a low price to both retailers, and each retailer purchases two units. We refer to this action profile as the $L$ equilibrium action profile. Producer profits are low; the producer would like to deviate offering a high price to both retailers. The strategy profile that sustains this action profile in equilibrium is such that retailers would not purchase any unit if offered the high price, because they both mutually fear that the other retailer would still purchase two units at a low price. The producer is thus stuck in a low price equilibrium, being unable to credibly deviate to offer the high price to both retailers.

In the second equilibrium action profile, denoted $H$, prices are high, quantities low. The producer could be tempted to deviate to offer a low price to a retailer, who would then benefit from purchasing three units, which would increase the producer’s profit. However, the strategy profile that sustains action profile $H$ in equilibrium is such that retailers who observe a deviation to a low price do not believe that the offer is exclusive to them: they fear the other retailer got it as well, and thus that quantities in the market will be large, which limits their willingness to purchase to only two units, thwarting the producer’s incentive to deviate in the first place. Off-path beliefs are critical to sustain one or the other equilibrium. The most common assumption on off-path beliefs is that they are passive, i.e., retailer $i$ does not update her beliefs on the offer received by retailer $-i$, when $i$ receives a price offer $p_i$ that is off equilibrium. Rather, retailer $i$ believes that $-i$ still receives the equilibrium price offer.

**Claim 3** For any $\theta \in \Theta$, the unique action profile supported by passive beliefs in a pure Perfect Bayesian equilibrium of game $\Gamma_\theta$ is \(((p^L, p^L), 2, 2)\).

In the equilibrium action profile supported by passive beliefs, retailers buy two units at a low price in either game. **Wary beliefs** are such that if the producer offers an off equilibrium price $\bar{p}_i$ to retailer $i$, retailer $i$ believes that the offer to retailer $-i$ is
an optimal one for the producer, given $\tilde{p}_i$. Because contracts are unobservable, **wary beliefs** coincide with passive beliefs (McAfee and Schwartz (28)) and action profile $L$ is also the unique pure equilibrium action profile supported by *wary beliefs*. We prove in the Appendix (Claim 7) that there is a unique action profile sustained in a **proper equilibrium** and it coincides with the action profile supported in equilibrium by wary beliefs or by passive beliefs, and also by In and Wright’s RI-equilibrium.

Alternatively, under symmetric beliefs, if retailer $i$ receives an out of equilibrium offer $p_i$, then she believes that $p_{-i} = p_i$ with probability one (the producer must have deviated to make the same offer to all retailers).

**Claim 4** For any $\theta \in \Theta$, the unique action profile supported by symmetric beliefs in a pure Perfect Bayesian equilibrium of game $\Gamma_\theta$ is $((p^H, p^H), 1, 1)$.

In the equilibrium action profile supported by symmetric beliefs, retailers purchase one unit at a high price.

Whether an equilibrium holds or not hinges on the belief that the other retailer was offered a high price $p^H$, following a deviation. If, following a deviation, the probability assigned to a high price for the other retailer is low enough, the equilibrium holds. Otherwise, it fails.

Out of equilibrium beliefs are a pair $(\omega_1(p^H|\tilde{p}_1), \omega_2(p^H|\tilde{p}_2)) \in [0, 1]^2$, where $\tilde{p}_1$ and $\tilde{p}_2$ are off-equilibrium offers observed by retailer 1 and retailer 2, and $\omega_i(p^H|\tilde{p}_i)$ is the probability assigned to $p_{-i} = p^H$ by agent $i$ with beliefs $\omega_i$ who observes $\tilde{p}_i$.

Equilibrium action profile $L$ is supported by beliefs such that, after observing offer $p_i = p^H$, retailer $i$ chooses not to purchase any units, which implies that

$$\omega_i(p^H|\tilde{p}_i)p(1) + (1 - \omega_i(p^H|\tilde{p}_i))p(3) - p^H - c_r \leq 0$$

$$\omega_i(p^H|\tilde{p}_i) \leq \frac{p^H + c_r - p(3)}{p(1) - p(3)}.$$

Equilibrium action profile $H$ is supported by beliefs such that, observing offer $p_i = p^L$, retailer $i$ chooses to purchase two units, and not three, units, which implies
that
\[
2[\omega_i(p^H|\tilde{p}_i)p(3) + (1 - \omega_i(p^H|\tilde{p}_i))p(4) - p^L] - c_r
\]
\[
\leq 3[\omega_i(p^H|\tilde{p}_i)p(4) + (1 - \omega_i(p^H|\tilde{p}_i))p(5) - p^L] - c_r
\]
\[
\omega_i(p^H|\tilde{p}_i) \leq \frac{p^L + 3p(5) - 2p(4)}{2p(3) - 5p(4) + 3p(5)}.
\]

Let
\[
\Theta_L = \left\{ \theta \in \Theta : \frac{p^H + c_r - p(3)}{p(1) - p(3)} > \frac{p^L + 3p(5) - 2p(4)}{2p(3) - 5p(4) + 3p(5)} \right\}
\]
\[
\Theta_H = \left\{ \theta \in \Theta : \frac{p^H + c_r - p(3)}{p(1) - p(3)} < \frac{p^L + 3p(5) - 2p(4)}{2p(3) - 5p(4) + 3p(5)} \right\}.
\]

The hyperplane \(\frac{p^H + c_r - p(3)}{p(1) - p(3)} = \frac{p^L + 3p(5) - 2p(4)}{2p(3) - 5p(4) + 3p(5)}\) divides the set of parameter vectors \(\Theta\) into a subset \(\Theta_L\) in which a larger set of beliefs sustains the equilibrium action profile \(L\) and a subset \(\Theta_H\) in which a larger set of beliefs sustains the equilibrium action profile \(H\). We obtain the following claim.

**Claim 5** For any \(\theta \in \Theta\), the unique action profile supported by the largest set of beliefs in a pure Perfect Bayesian equilibrium of game \(\Gamma_\theta\) is \(((p^L, p^L), 2, 2)\) if \(\theta \in \Theta_L\) and \(((p^H, p^H), 1, 1)\) if \(\theta \in \Theta_H\).

Notice that \(\Theta - (\Theta_L \cup \Theta_H)\) has measure zero; the equilibrium supported by the largest set of beliefs is generically unique over the set of parameters.

### 3.2 Experimental Design and Procedures

We use controlled laboratory experiments to evaluate the predictive power of the different selection criteria. We designed two treatments that only differ in their parameter constellations, one with parameter values \(\theta_L \in \Theta_L\) and another with parameter values \(\theta_H \in \Theta_H\). We label these Treatments \(L\) and \(H\) respectively.
The parameters related to producer payoffs are kept fixed across the two games: producer prices are $p^H = 36$, $p^L = 15$ and the producer’s transaction cost for any executed trade is $c_0 = 15$. The two treatments differ, however, in the variables that affect retailers’ payoffs. In Treatment $L$, the retailer’s transaction cost for any positive purchase is $c_r = 29$ and the vector of consumer market prices is $(72, 71, 35, 34, 27, 22)$, where coordinate $k$ denotes the price if $k$ units are sold in the market, for any number of units $k \in \{1, 2, 3, 4, 5, 6\}$. In Treatment $H$, the retailer’s transaction cost for any positive purchase is $c_r = 33$, and the vector of consumer market prices is $(120, 105, 59, 45, 27, 18)$.

In Treatment $L$ with parameter values $\theta_L$, the size of the set of beliefs that support action profile $L$ in equilibrium is 0.81, and the size of the set of beliefs that support $H$ is 0.10. In Treatment $H$ with parameter values $\theta_H$, this size is 0.16 for action profile $L$ and 0.93 for action profile $H$.

We made an affine transformation to the payoff function, which has no strategic consequences, but yields two advantages for experimental purposes: we avoid negative payoffs to subjects by adding a constant, and we equalize producer and retailers’ expected payoffs by multiplying the producer’s payoffs by $4/3$. Tables 1 and 2 respectively summarize retailers’ payoffs and supplier’s payoffs. All these payoffs are expressed in talers, the experimental currency.\textsuperscript{12}

All participants were given the role of either a supplier or a retailer, and kept that role throughout the experiment. Participants played the game for 50 rounds, being rematched after every round within matching groups of 12 subjects. After each round, subjects received full feedback about actions of all subjects in their subgroup and their payoffs for that round. To determine payment, the computer randomly selected five periods for the final payment.

Experiments were conducted at the BonnEconLab of the University of Bonn in

\textsuperscript{12}The total amount earned in these periods was transformed into euros through the conversion rate of 0.03 in Treatment $L$ and 0.045 in Treatment $H$. In total, subjects earned an average of €12.87, including a show-up fee of €4. Each experimental session lasted approximately one hour.
<table>
<thead>
<tr>
<th>Treatment L</th>
<th>Quantity bought by the other retailer</th>
<th>Treatment H</th>
<th>Quantity bought by the other retailer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Quantity bought</td>
<td>0</td>
<td>33</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>40</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>116</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>64</td>
<td>61</td>
</tr>
</tbody>
</table>

Table 1: Retailer’s payoffs.

<table>
<thead>
<tr>
<th>Price Charged by the supplier</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>0</td>
<td>28</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Low</td>
<td>0</td>
<td>—</td>
<td>20</td>
<td>40</td>
</tr>
</tbody>
</table>

Table 2: Supplier’s payoffs for each transaction.

March 2013. We ran a total of 6 sessions with 24 subjects each. No subject participated in more than one session. Therefore, we have six independent observations per treatment. Students were recruited through the online recruitment system ORSEE (Greiner 2004) and the experiment was programmed and conducted with the software z-Tree (Fischbacher 2007).

All experimental sessions were organized along the same procedure. Subjects received detailed written instructions, which an instructor read aloud (see supplementary appendix). Subjects were asked to answer a questionnaire to check their full understanding of the experimental design before beginning the experiment. At the end of the experiment, subjects completed a short questionnaire which we describe in more detail below.
3.3 Experimental Results

We organize our discussion of the experimental results by focusing, in turn, on prices, quantities and beliefs.

3.3.1 Prices

Figure 1 plots the aggregate prices offered by suppliers in both treatments, aggregated by groups of five periods. Each bar is divided into three tones of gray: the dark gray part represents the frequency of suppliers who offered both retailers the high price, the medium gray part represents the frequency of suppliers who offered a high price for one retailer and a low price for the other, and the light gray part represents the frequency of suppliers who offered the low price to both retailers.

The figure makes two points quite clearly. First, there is stark contrast between the pricing strategies used in both treatments: although in Treatment L prices are high only in 12% of the cases, in Treatment H they are high in 72.21% of the cases. This difference is clearly significant (Mann-Whitney test, $z = 2.882$, $p = 0.0039$).\footnote{In all nonparametric tests we used a matching group as an independent observation, because from period 2 onwards, individual choices were affected by observing other group members. Unless otherwise noted, we aggregated the data across all periods in a matching group.}
|                                | Coef. | Std. Err. | Odds Ratio | z     | Pr > |z| |
|--------------------------------|-------|-----------|------------|-------|------|---|
| Treatment L                   | -1.66 | 0.58      | 0.19       | -2.86 | 0.00 | |
| Period                        | 0.07  | 0.01      | 1.07       | 13.57 | 0.00 | |
| Treatment L * Period          | -0.10 | 0.01      | 0.90       | -12.58| 0.00 | |
| Cut 1                         | -0.39 | 0.40      | -0.96      | 0.34  |      | |
| Cut 2                         | 0.95  | 0.40      | 2.35       | 0.02  |      | |
| Group variance                | 0.49  | 0.38      |            |       |      | |
| Subject variance              | 1.52  | 0.44      |            |       |      | |
| Number of Observations        | 2400  |           |            |       |      | |
| Number of Groups              | 12    |           |            |       |      | |
| Number of Subjects            | 48    |           |            |       |      | |
| Log Likelihood                | -1450.58|        |            |       |      | |
| Wald $\chi^2$                 | 255.56|           |            |       |      | |
| Hausman test $\chi^2$ *       | 0.00  |           |            |       |      | |

Table 3: Three-level (overall, group, and subject) mixed effects ordered logistic regression of high prices offered by suppliers as a function of treatment, period, and treatment and period interacted. Treatment L is a dummy variable that takes value one if the treatment is L and zero otherwise. * The Hausman test is done on a linear mirror version of the model.

This difference across treatments is in line with the prediction of the largest set of beliefs but not by the competing selection criteria. Second, although this difference is evident from the first periods, it increases over time due to opposite convergence processes. In the last five periods, the percentage of high prices in Treatment $L$ is as low as 8.33%, while in Treatment $H$ is as high as 91.67%.

Table 3 formalizes these ideas by displaying the results of a three-level (overall, matching group, and subject levels) mixed-effects ordered logistic regression of the amount of high prices set by suppliers as a function of treatment and the period number, and both interacted. Table 3 indicates the strong treatment effect found above and the convergence pattern observed in Figure 1.

### 3.3.2 Quantities

Figure 2 displays the aggregate quantities bought conditional on the price in each treatment in groups of five periods. The graphs in the top (bottom) belong to Treat-
Figure 2: Quantities bought conditional on the price in each treatment.

ment $L$ ($H$). The graphs on the left (right) correspond to situations in which retailers were offered a high (low) price. Each bar is divided into different tones of gray which represent the amount bought: dark gray if the retailer bought zero units, dark/medium for one, light/medium for two and light gray for three.

Figure 2 shows two clear patterns. First, retailers’ behavior when receiving a low price is in line with the predictions of all equilibria and similar across treatments: they buy two units in around 89.96% of the time in Treatment $L$ and 91.90% in Treatment $H$. The small difference across treatments is not significant (Mann-Whitney test, $z = 0.961$, $p = 0.3367$). Second, there is a substantial difference in retailers’ behavior across treatments when receiving a high price. In Treatment $L$, retailers demand one unit 24.65% of the time. In contrast, this percentage is 93.48% in Treatment $H$. This difference in percentages is not surprisingly highly significant (Mann-Whitney test, $z = 2.882$, $p = 0.0039$). Awe we found with prices, this significant difference across treatments is in line with the predictions of behavior given by our largest set of beliefs.
Table 4: Measure of fit of the different selection criteria. "Random" displays a measure of fit of a randomly picked equilibrium. The data is restricted to the second half of the experiment.

criterion but not with the predictions from passive, symmetric or wary beliefs criteria.

### 3.3.3 Goodness of Fit

The main goal of the experimental part of this paper is to evaluate the predictive power of the different selection criteria described in the previous section. In this subsection we address this question by comparing a measure of goodness of fit. The measure of fit that we use is the percentage of observations in which the entire group behaved as predicted by a given equilibrium conditional on playing an equilibrium. Table 4 displays the results for the second half of the experiment as well as the percentage of groups that played an equilibrium action profile.\(^{14}\)

Overall, we find that our selection criterion outperforms both the common prediction of passive/wary beliefs and proper or RI equilibrium, as well as the prediction of symmetric beliefs. While our selection criterion makes the right prediction in 94.46% of cases, passive/wary beliefs/proper equilibrium and symmetric beliefs make the right prediction in 54.10% and 45.90% of cases, respectively. These differences are significant (Wilcoxon test, \(z = 1.819, p = 0.0690\) and \(z = 2.411, p = 0.0159\)).

When we disaggregate by the different games, the predictive power of our selection

\(^{14}\)We restrict the measure to the second half of the experiment due to the convergence process observed in Section 3.3.1. Qualitatively similar results are obtained when considering the whole sample.
criterion equalizes the best of the other criteria. Recall that in \( L \), the prediction of our criterion coincides with the prediction of the equilibrium under passive and wary beliefs or proper equilibrium, while in \( H \), the prediction of our criterion coincides with the prediction of the equilibrium under symmetric beliefs. Therefore, our selection criterion coincides with one of the others in each game by construction. The noticeable feature is that our criterion matches in each case the best performer of the other criteria.

3.3.4 Beliefs

Beliefs are a crucial element of equilibrium selection in the games presented. In order to assess whether game play was related to participants’ beliefs, we elicited beliefs of voters in a non-incentivized manner at the end of the experiment. Immediately after finishing the main part of the experiment, subjects completed a questionnaire. The two first questions of the questionnaire related to their beliefs. In particular, we asked the following questions: “Suppose that you play an additional period as a retailer. If the supplier offers you a low / high price, which price do you think the supplier will offer to the other retailer?” They could either answer “Low Price” or “High Price.”

Table 5 shows the joint distributions of beliefs. The matrix shows a substantial difference between treatments. As indicated in the table, according to the equilibrium with symmetric beliefs, a retailer \( i \) would expect the retailer \( j \) to be offered a low price when \( i \) is offered a low price, and a high price when \( i \) is offered a high price. In the equilibrium with passive beliefs, retailers expect the other retailer to be offered a low price regardless of the price offered to themselves.

Note that in Treatment \( L \), most retailers’ beliefs are in line with the beliefs predicted by passive beliefs although in Treatment \( H \), instead, most retailers’ beliefs are

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\(^{15}\)Because eliciting beliefs can produce significant changes to the actions played (Croson (11) and Gächter and Renner, (17)), we elicited them only at the end of the experiment, providing subjects with instructions about belief elicitation only after the contracting game experiment was over.
Other Retailer’s price if offered a high price

<table>
<thead>
<tr>
<th>Treatment $L$</th>
<th>Treatment $H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>High</td>
</tr>
<tr>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>15$^P$</td>
<td>20$^S$</td>
</tr>
<tr>
<td>30$^P$</td>
<td>7$^P$</td>
</tr>
</tbody>
</table>

Table 5: Joint distribution of beliefs (in absolute numbers). $^P$ indicates the prediction by the passive beliefs and $^S$ indicates the prediction by symmetric beliefs.

This is in line with the prediction of symmetric beliefs. These findings reinforce the results shown in the previous section: a selection criterion that imposes an invariant assumption on beliefs regardless of the nature of the game is inappropriately restrictive, and a poor fit of the data. Off-path beliefs depend on the particular game.

4 Application: Electoral Competition

The second application to test our selection criterion is a theory of electoral competition with two candidates and three voters. Each voter lives in a different district. Candidates compete by offering inefficient local public goods (pork) in each voter’s district. The candidate who gets most votes carries out her offers, and the provided local public goods are paid by general taxation.

Each voter observes the offers of pork made to her district, but is imperfectly informed about the offers made to the other districts. This game features multiple equilibria that hinge on beliefs about the unobserved offers off the equilibrium path.

We consider two treatments: one in which pork is almost efficient (0.9 units of benefit per unit of cost), and one in which it is very inefficient (0.4). In either treatment, the efficient policy is to offer no pork, and this policy can be supported in an equilibrium with symmetric beliefs. On the other hand, in either treatment, the unique equilibrium supported by passive beliefs is such that candidates propose the
least efficient policy, which is to provide pork to every district.

Equilibrium selection by the largest set of supporting beliefs predicts the equilibrium in which candidates offer pork to every district in the treatment in which pork is almost efficient (benefit 0.9 per unit of cost), and the equilibrium in which candidates do not offer pork to any district in the treatment in which pork is very inefficient (benefit 0.4 per unit of cost).

Therefore, selection by the largest set of supporting beliefs offers an intuitive comparative static prediction across treatments: candidates offer more pork in the treatment in which pork is almost efficient, than in the treatment in which pork is very inefficient. None of the other selection criteria feature this comparative static prediction.

We conducted our experiments at the BonnEconLab of the University of Bonn in October 2012. Participants were divided into groups of 10 subjects, four of them assigned to the role of candidates, and six to the role of voters. Within each group, candidates and voters were randomly rematched in each of 40 rounds to play the local public good provision and voting game in a subgroup of two candidates and three voters.

The experimental results agree with our predicted comparative static. For the second half of the experiment (rounds 21-40), in the treatment in which pork is almost efficient and we predict that candidates offer pork to every district, 62% of groups play an equilibrium action profile, and among these groups that play an equilibrium, 94% of them play the equilibrium action profile in which candidates offer pork to every district (as we predict), while 0% of them play the equilibrium with no pork.

Whereas, in the treatment in which we predict that candidates do not offer pork to any district, only 48% of groups who play an equilibrium action profile, play the equilibrium with pork for every district, and 43% play the equilibrium with no pork (the one we predict). However, only 35% of groups play according to an equilibrium in this treatment, which makes results on equilibrium selection more difficult to
In order to satisfy the journal’s space constraints, we include a full description of the theory and predictions, the experimental design procedures, and the experimental results in an online appendix, and in the working paper version (Eguia, Llorente-Saguer, Morton and Nicolò (12)).

**Discussion**

In the absence of a theory that explains how agents form out of equilibrium beliefs, any “ad hoc” assumption on out of equilibrium beliefs is arbitrary. The out of equilibrium beliefs held by the players depend on the specific game they play: games with different characteristics lead to different patterns of beliefs. To the extent that we cannot predict with confidence which specific beliefs are salient in a given game, we recommend that we consider all out of equilibrium beliefs as equally likely.

Under a uniform prior over these out of equilibrium beliefs, any equilibrium may hold, but not all equilibrium strategy profiles and action profiles are equally likely to emerge in the equilibrium actually played: those that require very specific beliefs are less likely to be played than action profiles that hold in equilibrium for a large set of beliefs. The probability that a particular strategy profile is played in equilibrium increases in the size of the set of beliefs that support this strategy profile as an equilibrium profile. The equilibrium action profile that is more likely to be played is the one which is supported by the largest set of beliefs. This is the equilibrium action profile we select.

To compute the size of beliefs that support a strategy profile $s$ in equilibrium, we must construct a measure over sets of beliefs. In finite games, we use the standard Lebesgue measure defined over the set of all possible beliefs at each out of equilibrium information set that follows an individual deviation. For each information set $I$, we calculate the fraction of the Lebesgue measure of the set of beliefs at information set $I$
that support the strategy profile in equilibrium, over the Lebesgue measure of the set of all possible beliefs at information set $I$. We then calculate the Cartesian product of these fractions over all information sets that follow from an individual deviation from the equilibrium strategy profile by a first mover in the sequential game under consideration. This Cartesian product defines the size of supporting beliefs of the strategy profile $s$.

Our criterion selects the action profile that has the largest set of supporting beliefs. The confidence in our prediction is increasing in the ratio of the largest set of supporting beliefs, over the size of the set of beliefs that sustain other action profiles. If the largest set is little larger than others, we conjecture that the equilibrium action profile with the largest set of beliefs may not always be played, but that nevertheless the frequency of play of each equilibrium action profile that is played will be increasing in the size of beliefs that support it in equilibrium. If our reasoning is correct, we can construct an order or ranking of equilibrium action profiles from most to least likely to be played, where each action profile is ranked according to the size of the set of beliefs that support it in equilibrium. In this paper we have used the size of the set of beliefs only to identify the action profile(s) with the largest set of supporting beliefs (without seeking to construct an order of action profiles) because we wished to compare our criterion to criteria such as selection by passive, symmetric or wary beliefs, which offer only a binary partition of the set of equilibrium action profiles (selected / not selected), not a full order. Future experiments can test the order prediction, and measure how the predictive success of selection by the largest set of beliefs depends on the relative size of the largest set of supporting beliefs over the size of the smaller sets of beliefs that support other equilibrium action profiles.

The seminal experiment on equilibrium selection in vertical contracting games, Martin, Normann and Snyder (27), finds that no specific restriction on the set of beliefs can fit well with the data.\textsuperscript{16} Their finding is consistent with our theory. Beyond

\textsuperscript{16}To our knowledge, there is no previous experimental work on equilibrium selection in games of electoral competition.
this negative finding, it is difficult to conduct an equilibrium analysis with their data, because in their experiment, no group of players play according to any equilibrium. Since all their subject groups, without exception, are out of equilibrium, comparisons of equilibrium selection predictions are problematic. Martin, Normann and Snyder (27) suggest that subjects have heterogeneous beliefs and that their behavior can be explained by a (non-equilibrium) model in which a fraction of subjects hold passive beliefs and the remaining subjects hold symmetric beliefs. We agree that agents may have different beliefs (see our Table 5 and Table 10) and this explanation can ex-post rationalize their observed experimental results. However, a theory based on heterogeneous beliefs has poor predictive power as long as the source of this heterogeneity remains unexplained and the fraction of agents who have each class of beliefs is ex-ante unknown. We propose a selection criterion that offers a clear and in most cases unique prediction, determined ex-ante by the characteristics and payoff matrix of each specific game. Our selection criterion not only rationalizes the observed data ex-post, but more importantly, it also predicts ex-ante the results we expect to obtain.

Our theory can be directly generalized to larger classes of finite sequential games, including games with more than two stages or with a more complex information structure. Our intuition—select the equilibrium supported by the largest set of beliefs—also applies to continuous games. While a precise formalization poses some technical challenges to define a measure of beliefs over a continuous strategy space, a formal definition that overcomes these challenges and a solved example with a continuous strategy space drawn from Segal and Whinston (39) are available from the authors.\footnote{Relatedly, see Myerson and Reny (30) to understand the technical obstacles that make the notion of Sequential Equilibrium difficult to formalize in continuous games.}

The motivation to use our selection criterion is that we cannot predict players’ out of equilibrium beliefs, and it is therefore more appropriate to consider that any beliefs might emerge, than to make ad hoc assumptions about them. However, if previous experience in a particular game, or survey data, provide good reasons to anticipate that players are more likely to hold some beliefs than others, we can consider a
more general family of selection criteria based on weighted sizes of beliefs. For each information set $I$ with $k$ information nodes, let $\mu^I = (\mu_1^I, \mu_2^I, \ldots, \mu_k^I)$ be a vector of weights ($\sum_{i=1}^{k} \mu_i^I = 1$ and $\mu_i^I \in [0, 1]$ for each $i \in \{1, 2, \ldots, k\}$), which is determined by the prior (held by a theorist modeling the game) over the beliefs that players may hold at that information set. For each strategy profile $s$, we can then calculate the weighted size of the set of beliefs that support $s$ as an equilibrium strategy profile, where beliefs that are deemed a priori more likely carry a greater weight. This approach generates a family of selection criteria, characterized by the weights $\mu$. Selection by passive, symmetric or wary beliefs are special cases of this family, in which the prior is degenerate and assigns all weight to passive, symmetric or wary beliefs.

We interpret our selection criterion as the benchmark member of this family with a uniform prior, which we see as the default to be used in any game in which we lack solid grounds to form good conjectures on out of equilibrium beliefs. Our experiments show that vertical contracting is one application in which using passive, symmetric or wary beliefs is unwarranted: the predictive power of equilibrium selection by any of these criteria is not better than selecting equilibria at random. In contrast, selection by the largest set of beliefs predicts the right equilibrium in 94% of cases in which an equilibrium is played. While using a weighted measure could in theory (if we knew the correct prior and weights) lead to better predictions, we have shown that in our vertical contracting games, selecting the equilibrium strategy profile that can be supported by the unweighted largest set of beliefs performs remarkably well.

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Appendix

Proof of Proposition 1

Proof. Given a finite game, there are only finite many action profiles, and thus, only finitely many of them are equilibrium action profiles. Let $k$ be the number of different action profiles that can be supported in equilibrium. For an arbitrary action profile $a$ that can be supported in equilibrium, $L(\Delta^{a}_{Q(a_P)})$ and $L(\Delta^{a}_{O(a_P)})$ are well defined, and thus, there exists a maximum among them.

In Section 2 we claim that selection by the largest set of supporting beliefs generically yields a unique prediction if no more than one equilibrium action profile is supported by every off-path belief. We provide a formal statement and proof of this claim.

Let $G$ be a game form, which indicates a set of agents $N$ with $n = n_P + n_R$ players ($n_P$ proposers and a $n_R$ receivers), a set of feasible strategy profiles $S = S_1 \times \ldots \times S_n$ of size $|S|$, and the timing and information structure of a collection of games in $G$, without specifying the payoffs. Let $U \in \mathbb{R}^{|A| \times n}$ define the payoff for each agent, for each possible action profile. Then $\Gamma = (G, U) \in G$ defines a specific game. For any $\varepsilon > 0$, let $\mathcal{P}_\varepsilon(U)$ be the set of possible perturbations of $U$ such that the payoff of each proposer stays unaltered, and the payoff of each receiver is altered by no more than $\varepsilon$. Formally, let $U_a \in \mathbb{R}^n$ denote the payoff vector for any action profile $a \in A$, and let $U_{ak}$ denote the $k$-th component of vector $U_a$. Then, for any $\varepsilon \geq 0$, $\mathcal{P}_\varepsilon(U) = \{U' \in \mathbb{R}^{|A| \times n} : U_{ak} = U'_{ak}$ for any $k \in \{1, 2, \ldots n_P\}$, and $|U_{ak} - U'_{ak}| \leq \varepsilon$ for any $k \in \{n_P + 1, n_P + 2, \ldots, n_P + n_R\}$, for any $a \in A\}$. Let $\mathcal{P}_\varepsilon(\Gamma) = (G, \mathcal{P}_\varepsilon(U))$ be the collection of games with the same set of pure strategy equilibria as game $\Gamma$ that can be generated by perturbing game $\Gamma$ according to perturbations in $\mathcal{P}_\varepsilon(U)$.

Proposition 6 Assume action profiles $a$ and $a'$ can be supported by a largest set of beliefs in game $\Gamma = (G, U) \in G$ and $SSB(a) = SSB(a') \in (0, 1)$. Then there exists
\[ \varepsilon > 0 \] such that the equilibrium action profile with a largest set of supporting beliefs is generically unique over the class of games \( P_\varepsilon(\Gamma) \).

**Proof.** Assume \( SSB(a) = SSB(a') \in (0,1) \) in game \( \Gamma = (G,U) \). That is, in game \( \Gamma \), \( \frac{L(D_{\partial (a,p)}^a)}{L(D_{\partial (a,p)}^0)} \in (0,1) \). Then there exists \( \varepsilon > 0 \) such that for any \( \varepsilon \in (0,\varepsilon] \) and any \( \bar{U} \in P_\varepsilon(U) \), in game \( \bar{\Gamma} = (G,\bar{U}) \) we also find that \( \frac{L(D_{\partial (a,p)}^a)}{L(D_{\partial (a,p)}^0)} \in (0,1) \) and thus \( SSB(a) \in (0,1) \). By an analogous reasoning, in game \( \bar{\Gamma} = (G,\bar{U}) \), \( SSB(a') \in (0,1) \). Consider any perturbation \( \bar{U} \in P_\varepsilon(U) \) and let \( \bar{U}^\lambda \) be the family of perturbations such that for any \( \lambda, \bar{U}_{a''} = \bar{U}^\lambda_{a''} \) for any agent \( k \) and any action \( a'' \notin \{a,a\} \); \( \bar{U}_{ak}^\lambda = \bar{U}_{ak} - \lambda \); and \( \bar{U}_{a'k}^\lambda = \bar{U}_{a'k} + \lambda \). Then \( \Delta_{\partial (a,p)}^a \) strictly decreases with \( \lambda \), and \( \Delta_{\partial (s')_{\partial (a,p)}}^{a'} \) strictly increases with \( \lambda \), while \( \Delta_{\partial (a,p)}^{a} \) and \( \Delta_{\partial (s')_{\partial (a,p)}}^{a'} \) do not depend on \( \lambda \). Thus, there is at most one value of \( \lambda \) for which \( \frac{L(D_{\partial (a,p)}^a)}{L(D_{\partial (a,p)}^0)} = \frac{L(D_{\partial (s')_{\partial (a,p)}}^{a'})}{L(D_{\partial (s')_{\partial (a,p)}}^0)} \) and hence \( SSB(a) = SSB(a') \) is a non-generic event in the family of games perturbed by \( \bar{U}^\lambda \). ■

**Proof of Claim 2**

**Proof.** Consider an arbitrary game \( \Gamma_\varepsilon \); we first prove that there are two pure strategy profiles that are supported in a Perfect Bayesian Equilibrium (PBE) of the game. Consider first strategy profile \( ((L,L),(0,2),(0,2)) \). For each retailer to buy two units is the unique best response when the price offered is low and the other retailer is buying two units. If the producer deviates proposing a high price to some retailer, then she decreases her payoff because the retailer buys zero unit and the deviation is not observed by the other retailer. When the price offered is high, to buy zero unit is a best response for a retailer who assigns probability one that the producer is proposing a low price to the other retailer.

Consider the strategy profile \( ((H,H),(1,2),(1,2)) \). For each retailer to buy one unit is the unique best response when the price is high and the other retailer is buying one unit. If the producer deviates proposing a low price to any retailer, she decreases her payoff because the retailer buys two units and by assumption \( p^H > 2p^L \). To buy two units when the price offered is low is a best response for a retailer who assigns
probability one that the producer is proposing a low price to the other retailer, too. Consider now any other strategy profile. First, notice that if the producer proposes a high price to retailer \(i = 1, 2\), retailer \(i\) buys a positive amount if and only if retailer \(j \neq i\) buys at most one unit. If retailer \(i\) buys at most one unit (as he forced to do if the price offered to him is high), retailer \(j\)'s best response is to buy two or three units when the price offered to him is low. It follows that if the retailer in equilibrium offers a high price to retailer \(i\) and a low price to retailer \(j\), then retailer \(i\) buys zero unit, and retailer \(j\) buys two units. However, if the producer deviates and offers a low price to retailer \(i\), then retailer \(i\)'s best response is to buy a positive amount, irrespective of his beliefs about the price offered by the producer to retailer \(j\). Therefore the deviation is profitable for the producer. It follows that there are not asymmetric equilibria such that the producer offers a high price to a retailer and a low price to the other one. If the producer offers a high price to both retailers, to buy one unit is the best response for each retailer irrespective of the amount that the other retailer buys. If the producer offers a low price to both retailers, if retailer \(i\) buys three units, retailer \(j\)'s best response is to buy zero unit. However, to buy three units for retailer \(i\) is not the best response when retailer \(j\) buys zero unit. Hence, there are no equilibria such that the producer offers a low price to both retailers and some retailer buys a quantity different than two. ■

**Proof of Claim 3 and Claim 4**

These claims follow immediately from the proof of Claim 2 and the definitions of passive beliefs, and symmetric beliefs.

**Claim 7** For any \(\theta \in \Theta\), the unique proper equilibrium of the vertical contracting game with parameter vector \(\theta\) is such that agents play strategy profile \(((p^L, p^L), (0, 2), (0, 2)).

**Proof.** For any \(\theta \in \Theta\), the set of pure strategy profiles that are sustained in a Perfect Bayesian Equilibrium (PBE) of game \(\Gamma_\theta\) is \(\{(p^L, p^L), (0, 2), (0, 2)), ((p^H, p^H), (1, 2), (1, 2))\}. Consider first the equilibrium \(((p^H, p^H), (1, 2), (1, 2)). A PBE in which \((q_1, q_2) = (1, 1)\)
is supported by strategy profile $((p^H, p^H), (1, 2), (1, 2))$ and beliefs such that retailer $i$ observing $p_i = p^L$ believes with sufficiently high probability that $p_{-i} = p^L$ and thus expects $q_{-i} = 2$. However the deviation (consistent with $p_i = p^L$) that provides the highest payoff to the producer is such that the producer is still playing the strategy $p_{-i} = p^H$ and therefore in any proper equilibrium, the retail should assign a probability which gives a higher weight (of the order of $\frac{1}{\varepsilon}$ for $\varepsilon > 0$ infinitesimally small) to strategies according to which the deviating proposer plays $p_{-i} = p^H$. However, the best response according to these beliefs are such that when $i$ observes $p_i = p^L$ should buy three units and not two units as prescribed by the equilibrium, making the deviation profitable for the producer. Therefore this equilibrium strategy profile cannot be supported as a proper equilibrium.

Consider the equilibrium $((p^L, p^L), (0, 2), (0, 2))$. A PBE in which $(q_1, q_2) = (2, 2)$ is supported by strategy profile $((p^L, p^L), (0, 2), (0, 2))$ and beliefs such that retailer $i$ observing $p_i = p^H$ believes with sufficiently high probability that $p_{-i} = p^L$ and thus expects $q_{-i} = 2$. Since the deviation (consistent with $p_i = p^H$) that provides the highest payoff to the producer is such that the producer is still playing the strategy $p_{-i} = p^L$, then this strategy profile is supported by beliefs that satisfy the $\varepsilon$-proper equilibrium refinement. ■

Claim 5 is proved directly in the text preceding the claim.
k-Proper Equilibria

by Jon X. Eguia and Antonio Nicolò

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We consider a class of equilibrium refinements for finite games in strategic form. The refinements in this family are indexed from least to most restrictive. Proper equilibrium is obtained as a special case within the class; all other concepts are stronger than trembling-hand perfection and weaker than proper equilibrium, so they provide a collection of intermediate refinements. We argue theoretically and illustrate by examples that in some applications, the intermediate refinement concepts are preferable to either trembling-hand or proper equilibrium.

Consider a symmetric two player game in normal form with payoffs given by the following matrix:

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<tr>
<th></th>
<th>Player 2</th>
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<tr>
<td></td>
<td>$s^1$</td>
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<tr>
<td>Player 1</td>
<td>$s^1$</td>
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<td>$s^2$</td>
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<td>$s^3$</td>
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<td>$s^4$</td>
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This game has two pure Nash equilibria: $(s^1, s^1)$ and $(s^4, s^4)$. Both of these are also trembling-hand perfect equilibria (Selten 1975). However, once we consider trembles, the two equilibria do not appear equally plausible. In equilibrium $(s^1, s^1)$, the best tremble, which causes no utility loss to the deviating player, is to tremble to $s^2$, but if players occasionally tremble to $s^2$, the equilibrium strategy $s^1$ is still a best response against a mix of the equilibrium strategy $s^1$ and the best tremble $s^2$. By contrast, $(s^4, s^4)$ is more fragile: the best tremble (the least costly in terms of payoff) is to tremble to $s^3$, and as soon as a player trembles to $s^3$ with any probability, $s^4$ is not a best response given this tremble, the other player prefers to actually deviate to $s^3$ all the time, and the equilibrium $(s^4, s^4)$ breaks down.

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Proper equilibrium (Myerson 1978) was conceived to account for precisely this intuition that trembling agents are more likely to tremble toward less costly deviations from equilibrium play, than toward costlier deviations. However, the proper equilibrium refinement turns out to be too restrictive to help us in our example: neither \((s^1, s^1)\) and \((s^4, s^4)\) satisfy its requirements, and there is no pure proper equilibrium in this game. In fact, even van Damme’s (1991) concept of a weakly proper equilibrium is too strong: in this example, there is no pure weakly proper equilibrium.

We would like to identify a refinement of trembling-hand perfection that satisfies the following:

a) it selects all proper equilibria, so it inherits the good existence properties of the proper refinement.

b) it is easier, or at least not harder to compute, than existing concepts.

c) it recognizes that equilibria that do well against the most plausible trembles (such as \((s^1, s^1)\) in our motivating example) are qualitatively different from equilibria that perform poorly against the most plausible trembles (such as \((s^4, s^4)\) in our example) and should be selected.

We describe a family of concepts satisfying these desiderata.

**Definitions**

Consider a finite game, in which the set of players is \(N\) of size \(n\), the set of feasible pure strategies for each player \(i \in N\) is \(S_i\) and \(\max\{|S_1|, ..., |S_n|\} = K\), so that each player has at most \(K\) pure strategies to choose from.

Let \(S = \prod_{i=1}^{n} S_i\) be the set of all feasible pure strategy profiles. Let \(\Delta(S)\) be the set of all possible mixed strategy profiles. Let \(\sigma \in \Delta(S)\) be a mixed strategy profile. For each \(i \in N\), let \(\sigma_i \in \Delta(S_i)\) be a mixed strategy for player \(i\). Let \(\sigma\) also be represented as \(\sigma = (\sigma_i, \sigma_{-i})\) so that \(\sigma_{-i} \in S_{-i} = \prod_{j \in N \setminus \{i\}} S_j\) is a list with a mixed strategy for every other player except \(i\).

For each \(i \in N\), for any \(s_i \in S_i\) and for any \(\sigma_{-i} \in \Delta(S_{-i})\), let \(u_i(s_i, \sigma_{-i})\) be the expected utility for player \(i\) given that \(i\) plays \(s_i\) and the other players play \(\sigma_{-i}\).

For each \(i \in N\), and for any \(\sigma_{-i} \in \Delta(S_{-i})\), let \(BR^1_i(\sigma_{-i}) \subseteq S_i\) be the set of first best
responses by player \(i\) to \(\sigma_{-i}\); that is, \(BR_i^1(\sigma_{-i}) \equiv \{s_i \in S_i : u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \forall s'_i \in S_i\}\).

We define recursively second best responses as strategies that are best responses among those that are not a first best response; third best responses as strategies that are a best response when restricted to using those that are neither first nor second best responses; and more generally \(k\)-th best responses as the best responses among the remaining strategies that weren’t best responses at any previous first to \((k-1)\)-th level.

**Definition 1** For any integer \(k \in \{2, \ldots, K\}\), for each \(i \in N\), and for any \(\sigma_{-i} \in \Delta(S_{-i})\), let \(BR_i^k(\sigma_{-i}) \subset S_i\) be the set of \(k\)-th best responses to \(\sigma_{-i}\), defined by

\[
BR_i^k(\sigma_{-i}) \equiv \{s_i \in S_i : u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \forall s'_i \in S_i \setminus \bigcup_{h=1}^{k-1} BR_i^h(\sigma_{-i})\}.
\]

Informally, the set of \(k\) th best responses is composed of the strategies that are a best response among those that were not an \((h-k)\) th best response for any \(h < k\).

Trembling-hand perfection (Selten 1975) and properness (Myerson 1978) both require that an equilibrium strategy profile \(\sigma \in \Delta(S)\) be approximated by a sequence of totally mixed strategy profiles \(\{\sigma^n\}_{n=1}^\infty\) with \(\lim_{n \to \infty} \sigma^n = \sigma\), and they both impose some restrictions on this sequence. Trembling-hand requires that along the sequence, any strategy that is not a best response to \(\sigma^n_{-i}\) be played with probability less than \(\varepsilon^n\) by player \(i\), where \(\{\varepsilon^n\}_{n=1}^\infty\) converges to zero. Properness requires that for any \(s_i, s'_i \in S_i\), if \(u_i(s_i, \sigma^n_{-i}) < u_i(s'_i, \sigma^n_{-i})\), then \(\sigma^n_i(s_i) \leq \varepsilon^n \sigma^n_i(s'_i)\). In our notation, properness requires that for any \(s_i, s'_i \in S_i\) such that \(s_i \in BR_i^k(\sigma^n_{-i})\) and \(s'_i \in BR_i^{k'}(\sigma^n_{-i})\), if \(k' < k\), then \(\sigma^n_i(s_i) \leq \varepsilon^n \sigma^n_i(s'_i)\). An indexed family of intermediate concepts suggests itself.

**Definition 2** For any integer \(k \in \{1, \ldots, K-1\}\), a mixed strategy profile \(\sigma \in \Delta(S)\) is an \(\varepsilon-(k\text{-proper})\) equilibrium if: i) \(\sigma\) is totally mixed, and ii) for any integers \(l, m, h \in \{1, \ldots, K\}\) such that \(l < m \leq k < h\) and for any player \(i\) and any \(s_i, s'_i, s''_i, s'''_i \in S_i\) s.t. \(s_i \in BR_i^l(\sigma_{-i})\), \(s'_i \in BR_i^m(\sigma_{-i})\), \(s''_i \in BR_i^k(\sigma_{-i})\) and \(s'''_i \in BR_i^h(\sigma_{-i})\); it follows that \(\sigma_i(s'_i) \leq \varepsilon \sigma_i(s_i)\) and \(\sigma_i(s'''_i) \leq \varepsilon \sigma_i(s_i)\).

As in Myerson’s (1978) proper equilibrium, the weights assigned by the totally mixed strategies should be \(\varepsilon\) times smaller for any strategy that yields a lower payoff than another,
but this nuanced relative order only applies to those strategies that are among the first to $k$-th best responses; all other strategies can receive any weights as long as these weights are positive but less than $\varepsilon$ times the weight of any $k$-th best response. Since any strategy except the worst one must be a $k$-th best response for some $k \in \{1, \ldots, K-1\}$, the extreme case $k = K - 1$ takes us back to Myerson’s (1978) definition of an $\varepsilon$-proper equilibrium.

**Definition 3** For any integer $k \in \{1, \ldots, K - 1\}$, a mixed strategy profile $\sigma \in \Delta(S)$ is a $k$-proper equilibrium if it is a limit of a sequence of $\varepsilon$-($k$-proper) equilibria with $\varepsilon$ converging to zero along the sequence.

A $k$-proper equilibrium requires that players assign arbitrarily greater weight to strategies that are better over those that are not as good, among the top $k$ tiers of strategies. In this it coincides with proper. It coincides with trembling hand perfection in letting the relative weights of all other strategies, which are not among the $k$-th best responses, be arbitrarily assigned as long as they are small. It follows that the concept of a $(K-1)$-proper equilibrium coincides with the concept of proper equilibrium (Myerson 1978), whereas, at the other end, the concept of trembling hand perfection is slightly weaker than 1-proper since trembling hand perfection does not require other first best responses (aside from the one sustaining the equilibrium) to be assigned greater weight than non-best responses, whereas 1-proper adds this restriction.

Observe that each level $k$ introduces a new restriction in the admissible sequences, and thus tightens the solution concept. For each integer $k \in \{1, \ldots, K\}$, let $E^k \subseteq \Delta(S)$ be the set of strategy profiles that are $k$-proper equilibrium; let $E^0$ denote the trembling-hand equilibria and note that $E^{K-1}$ is the set of proper equilibria.

**Remark 1** $E^{K-1} \subseteq E^{K-2} \subseteq \ldots \subseteq E^1 \subseteq E^0$.

Since the set of $k$-proper equilibria is contained by the set of trembling-hand perfect equilibria, and it contains the set of proper equilibria, it follows from the existence of proper equilibria in finite games that a $k$-proper equilibrium exists as well for any $k \in \{1, \ldots, K-1\}$.

It also follows that we can compare $k$-proper equilibria to extensive form solution concepts: Proposition 1 in Reny (1992) together with Remark 1 imply that given any game in
extensive form, and given any \textbf{k-proper} equilibrium \( \sigma \in \Delta(S) \) of the normal form associated with such extensive form game, there exists a system of beliefs such that \( \sigma \) and this system of beliefs satisfy weak sequential rationality; while Proposition 1 in Mailath, Samuelson and Swinkels (1997) together with Remark 1 imply that any strategy profile that is quasiperfect (van Damme 1984) in the extensive form is \textbf{k-proper} in the associated normal form game.


discussion and examples

The appeal of this family of concepts is grounded on at least three reasons, one axiomatic, and two practical for applied work.

1. \textbf{Minimize Restrictions.} Given two solution concepts that make the same sharp prediction for a given application, the weaker concept is preferred. Govindan and Wilson (2008) express this scientific norm thus: “\textit{The prevailing practice in the social sciences is to invoke the weakest refinement that suffices for the game being studied. This reflects a conservative attitude about using unnecessarily restrictive refinements. If, say, there is a unique sequential equilibrium that uses only admissible strategies, then one refrains from imposing stronger criteria.}” According to this norm, in any application in which there are multiple trembling-hand perfect equilibria so we need a refinement to make a sharper solution, and in which there is a unique \textbf{k-proper} equilibrium for some \( k < K - 1 \), the \textbf{k-proper} equilibrium is a more appropriate concept to make a unique prediction than proper equilibrium.

2. \textbf{Existence and uniqueness in pure strategies.} In many applications, researchers restrict their equilibrium analysis to pure strategies. In cases in which there exist multiple trembling-hand perfect pure equilibria, no pure proper equilibrium, and a unique \textbf{k-proper} equilibrium for some \( k < K - 1 \), using the \textbf{k-proper} equilibrium concept allows the researcher to make a unique prediction in pure strategies, which was impossible if we only consider trembling-hand perfection (too weak) or proper equilibrium (too restrictive).

3. \textbf{Ease of computation.} The proper equilibrium refinement has been used only seldom in practice, because constructing the sequence of trembles and of \( \varepsilon - \text{proper} \) equilibria that converges to the proper equilibrium (or proving that no such sequence exists) is cumbersome. There are many constraints on the trembles, and checking whether they are all satisfied is
difficult. By loosening restrictions, \textit{k-proper} equilibria with $k < K - 1$ become easier to check. The difficulty of computation increases in $k$, so incentives align with those dictated by the normative prescription of minimizing restrictions: more restrictive and harder to compute solution concepts (\textit{k-proper} equilibria with a higher $k$) should be used only if less restrictive and easier to compute concepts do not yield a sufficiently sharp prediction. Proper equilibria (the highest of the \textit{k-proper}) should then be used only as a last resort, within this family of restrictions.

We illustrate the advantages of this family of refinements with two examples in two symmetric two player games. First we return to our initial motivating example.

\textbf{Example 1} Assume $N = \{1, 2\}$, $S_1 = S_2 = \{s^1, s^2, s^3, s^4\}$ and payoffs are given by the following matrix, where player 1 is the row player and player 2 is the column player.

\begin{center}
\begin{tabular}{c|cccc}
 & $s^1$ & $s^2$ & $s^3$ & $s^4$ \\
\hline
$s^1$ & 20,20 & 0,20 & 20,0 & 0,1 \\
$s^2$ & 20,0 & 0,0 & 0,20 & 1,0 \\
$s^3$ & 0,20 & 20,0 & 18,18 & 20,0 \\
$s^4$ & 1,0 & 0,1 & 0,20 & 20,20 \\
\end{tabular}
\end{center}

The set of pure Nash equilibria is $\{(s^1, s^1), (s^4, s^4)\}$. The set of pure trembling-hand perfect Nash equilibria coincides with the set of Nash equilibria. There is no pure proper equilibria. Strategy profile $(s^1, s^1)$ is not a proper equilibrium because $u_i(s^4, \sigma_{-i}) > u_i(s^3, \sigma_{-i})$ for any $\sigma_{-i}$ sufficiently close to $(1,0,0,0)$; thus, being a proper equilibrium would require that for some sequence of totally mixed strategies $\{\sigma^n_{-i}\}_{n=1}^\infty$ that converges to the pure strategy $s^1$, if $n$ is sufficiently large, $\sigma^n_i(s^3) \leq \varepsilon \sigma^n_i(s^4)$, in which case $u_i(s^2, \sigma^n_{-i}) > u_i(s^1, \sigma^n_{-i})$ and then it must be $\sigma^n_i(s^1) \leq \varepsilon \sigma^n_i(s^2)$, which is a contradiction. A similar argument shows that $(s^4, s^4)$ also fails to satisfy the proper equilibrium refinement.

Since $u_i(s^4, s^1) > u_i(s^3, s^1)$, in order for $(s^1, s^1)$ to be a weakly proper equilibrium (van Damme 1991), there must be some sequence of totally mixed strategies $\{\sigma^n_{-i}\}_{n=1}^\infty$ that converges to the pure strategy $s^1$ such that $\sigma^n_i(s^3) \leq \varepsilon \sigma^n_i(s^4)$, in which case $u_i(s^2, \sigma^n_{-i}) > u_i(s^1, \sigma^n_{-i})$ so $s^1$ is not a best response to $\sigma^n_{-i}$ and thus $(s^1, s^1)$ is not a weakly proper equi-
librium. A similar argument shows that \((s^4, s^4)\) also fails to satisfy the weakly proper equilibrium refinement. We are thus unable to make a unique prediction in pure strategies using trembling-hand perfection, weakly proper equilibrium or proper equilibrium.

We show that \((s^1, s^1)\) is the unique 2-proper equilibrium in pure strategies. Let \(\sigma_i^n = (1 - \varepsilon^n \frac{3}{2}, \varepsilon^n \frac{3}{2}, \varepsilon^n \frac{3}{2})\) for each \(i \in \{1, 2\}\) and some small \(\varepsilon > 0\). Then \(\{\sigma_i^n\}_{n=1}^{\infty}\) converges to \(s^1\) as \(n \to \infty\), \(BR_1^1(\sigma_i^n) = s^1\), and \(BR_2^2(\sigma_i^n) = s^2\) for any \(n \in \mathbb{N}\) and thus the weights of \(\sigma_i^n\) respect this partial order \((s^1\) is played at least \(\frac{1}{\varepsilon}\) more often than \(s^2\), which is played \(\frac{1}{\varepsilon}\) more often than anything else). So \((s^1, s^1)\) is a 2-proper equilibrium. Finally, note that \((s^4, s^4)\) is not a 2-proper equilibrium. For any \(\sigma_i^n\) sufficiently close to \((0, 0, 0, 1)\), \(u_i(s^3, \sigma_i^n) > \max\{u_i(s^1, \sigma_i^n), u_i(s^2, \sigma_i^n)\}\) and \(BR_1^1(\sigma_i^n) \cup BR_2^2(\sigma_i^n) = \{s^3, s^4\}\) so \(\sigma_i^n(s^1) \leq \varepsilon \sigma_i^n(s^3)\), but then \(BR_1^1(\sigma_i^n) = s^3\) so \(\sigma_i^n(s^4) \leq \varepsilon \sigma_i^n(s^3)\) a contradiction.

Therefore, the concept of 2-proper equilibrium obtains a unique prediction in pure strategies, which was unattainable using trembling-hand perfection or proper equilibrium.

**Example 2** Assume \(N = \{1, 2\}\), \(S_1 = S_2 = \{s^1, s^2, s^3, s^4\}\) and payoffs are given by the following matrix, where player 1 is the row player and player 2 is the column player.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(s^1)</td>
</tr>
<tr>
<td>Player 1</td>
<td>(s^1)</td>
</tr>
<tr>
<td></td>
<td>(s^2)</td>
</tr>
<tr>
<td></td>
<td>(s^3)</td>
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<tr>
<td></td>
<td>(s^4)</td>
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</tbody>
</table>

The set of pure Nash equilibria is \(\{(s^1, s^1), (s^4, s^4)\}\). The set of pure trembling-hand perfect Nash equilibria coincides with the set of Nash equilibria. We show that \((s^4, s^4)\) is the unique 1-proper equilibrium in pure strategies.

Since for each player \(i\) and for any \(\sigma_{-i} \in \Delta(S_i)\), \(u_i(s^1, \sigma_{-i}) = u_i(s^2, \sigma_{-i})\), it follows that \(s^1 \in BR_i^1(\sigma_{-i}) \iff s^2 \in BR_i^1(\sigma_{-i})\). For any \(\sigma_{-i}^n\) sufficiently close to \((1, 0, 0, 0)\) and such that \(s^1 \in BR_i^1(\sigma_{-i}^n), u_i(s^1, \sigma_{-i}) > u_i(s^3, \sigma_{-i})\) and thus \(s^3 \notin BR_i^1(\sigma_{-i}^n)\), but then we require \(\sigma_i^n(s^3) \leq \varepsilon \sigma_i^n(s^2),\) which implies \(u_i(s^4, \sigma_{-i}) > u_i(s^1, \sigma_{-i})\), a contradiction. So we cannot construct a sequence that approximates \((s^1, s^1)\) satisfying the 1-proper restrictions. Let
\( \sigma_i^n = (\frac{\varepsilon_i^n}{3}, \frac{\varepsilon_i^n}{3}, \frac{\varepsilon_i^n}{3}, 1 - \varepsilon_i^n) \) for each \( i \in \{1, 2\} \) and some small \( \varepsilon > 0 \). Then \( \{\sigma_i^n\}_{n=1}^\infty \) converges to \( s^4 \) as \( n \to \infty \), and \( BR_1^i(\sigma_i^n) = s^4 \) and thus the weights of \( \sigma_i^n \) respect the 1-proper restriction. So \( (s^4, s^4) \) is a 1-proper equilibrium.

In summary, we consider a family of equilibrium refinements that take increasingly more restrictive steps from trembling-hand perfection to proper equilibrium, and we show how in some applications, these intermediate concepts provide solutions that are more satisfying and easier to compute that if we only considered the two extreme concepts of trembling-hand perfection and proper equilibrium.

References


