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Abstract

Strategic reasoning in sequential games rests on figuring out how (co)players would react to information about past play, which in turn depends on how players update or revise their beliefs. Several notions of belief systems have been used to represent and discipline how players' beliefs change as they obtain new information. Such notions differ and can be nested according to the imposed consistency restrictions relating beliefs at different information sets. The minimal restriction requires that beliefs about others change from one information set to a following, more informative one in compliance with the chain rule, i.e., by standard updating whenever possible. On top of this, more demanding restrictions require that beliefs about co-players depend only on information about them, not on own past moves. Even stronger restrictions require that players update, or revise their beliefs *as if* they could notionally condition on any nonempty event about co-players' behavior in compliance with the chain rule. We analyze restrictions on belief change, providing characterizations and interpretations in terms of introspection and cognitive rationality. We then argue that these differences between consistency restrictions do not affect the behavioral implications of strategic reasoning in games.

KEYWORDS: Sequential games, chain rule, partial introspection, rational planning, rationalizability.

JEL Codes: C72, C73, D83.

1 Introduction

A key aspect of strategic reasoning in sequential games is to figure out how (co)players would react to either anticipated, or non-anticipated information about past play, which in turn depends on how players update or revise their beliefs. Several notions of belief systems have been used to represent and discipline how players' beliefs change as they obtain new information. Such notions differ and can be nested according to the imposed consistency restrictions relating beliefs at different information sets. The minimal restriction requires that beliefs about others change from one information set to a following, more informative one in compliance with the chain rule, i.e., by standard updating whenever possible. We call this restriction **"forward consistency,"** because it is an application of the chain rule as the play goes forward and new information about players' behavior is revealed. On top of this, more demanding restrictions require that beliefs about co-players depend only on information about them, not on own past moves. This is the **"standard"** consistency restriction implied by the application of the chain rule to systems of conditional probabilities given a collection of conditioning events corresponding to the information about others revealed by information sets (see Renyi 1955, and Battigalli & Siniscalchi 2002). Even stronger restrictions require that players update, or revise their beliefs *as if* they could notionally condition on any nonempty event about co-players' behavior, even if it is not observable, in compliance with the chain rule. We refer to such restrictions as **"complete consistency,"** because they are obtained by considering complete conditional probability systems (see Myerson 1986, and Battigalli 1996). We analyze restrictions on belief change, providing characterizations and interpretations in terms of introspection and cognitive rationality. We then argue that these differences between consistency restrictions do not affect the behavioral implications of strategic reasoning in games. Specifically, we prove that different notions of rationalizability for sequential games justified by epistemic foundations are invariant to restrictions on belief change beyond the minimal ones. The following example illustrates the main themes of this work, comparing forward consistency to standard and complete consistency (in this particular case, standard and complete consistency are equivalent).

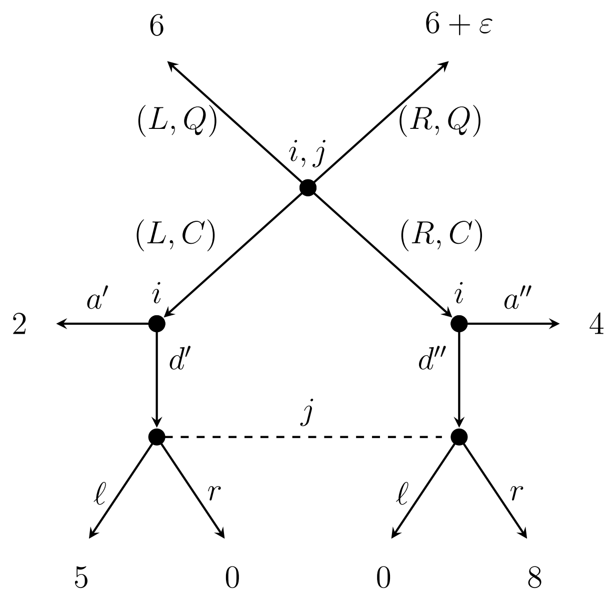

Figure 1: Γ', a common interest game between Isa and Joe.

Consider game Γ' depicted in Figure 1. At the root, Isa (i) chooses between Left and Right, and Joe (j) simultaneously chooses between Quit (which terminates the game) and Continue. If Joe Continues, Isa observes this and decides whether to go across (terminating the game) or down. If Isa goes down, Joe observes this, but not her initial move, and chooses between left and right. Assume that $0 \leq \varepsilon < 2$. Since Isa has perfect recall, she remembers her initial move and her choice may depend on it. Can her belief depend on her choice as well? According to forward consistency, this is possible. Indeed, if Isa is initially certain that Joe Quits, her initial belief does not pin down what she would believe after observing (L, C) or (R, C) and these two conditional beliefs may be different. On the other hand, if Isa initially assigns probabilities $\gamma\lambda$ and $\gamma(1-\lambda) > 0$ strategies $C.\ell$ and $C.r$, respectively, and $\gamma > 0$, then upon observing C she would assign probability λ to $C.\ell$ and $(1-\lambda)$ to $C.r$ independently of her initial move. According to standard and complete consistency, the latter restriction must hold even if Isa is initially certain that Joe will Quit ($\gamma = 0$). We will show that this difference does not matter for the analysis of the behavioral implications of rationality and strategic reasoning. This is clearer if $\varepsilon > 0$. In this case, if Isa were initially certain of Quit she would go Right and her counterfactual belief after Left would not matter. But the difference does not matter even if $\varepsilon = 0$. Intuitively, the reason is that, whatever the planned choice of Isa at the root, if she is rational she actually makes this choice and her counterfactual belief after the other does not matter. We will explain this in detail in our analysis of rational planning and rationalizable behavior. Here we only provide an informal explanation of why Isa might have different beliefs about Joe conditional on (L, C) and (R, C), given that at both nodes she has the same information about Joe. According to subjective expected utility maximization, in order to decide what to do at the root, Isa only has to her initial belief. If she is certain of Quit, then Right is the best choice (one of the best if $\varepsilon = 0$). She has no need for further planning and she come up with a partial strategy. With this, she be be only *partially introspective and unable to anticipate how she would revise her belief if surprised.* Such partial introspection may prevent her from imposing the cognitive rationality "same-information/same-belief" rule, unless such rule is in some sense "wired" into her way of thinking. Of course, if surprised, Isa would form *some* revised belief and make a choice based on it. For example, if she initially goes Right and is surprised by C, it may be the case that she is pre-disposed to believe that Joe is more likely to continue with right, and thus choose to go down. In this case, she would implement the reduced strategy (or plan of action, according to Rubinstein 1991) $R.d''$. But this does not imply that she had initially planned to go down if surprised. In other words, $R.d''$ is a *description* of Isa's behavior, but not necessarily a plan of Isa. What matters for strategic reasoning is to be able to anticipate the behavior of other. From this perspective, Joe's belief in the rationality of Isa allows for the possibility that he assigns positive probability to her behavior being described by $R.d''$. Thus, our results about the invariance of (version of) rationalizability to restrictions on belief systems beyond forward consistency may be interpreted as saying that the underlying epistemic justifications do not rely on full introspection.

2 Sequential games with perfect recall

Our analysis is restricted to *finite* sequential games without chance moves played by agents with *perfect recall*, represented in extensive form.[1] Some knowledge of the extensive and strategic-form representations of sequential games is taken for granted. Thus, for the primitive terms of the analysis, only the necessary symbols and definitions with rather terse explanations are given below. The reader interested in the details should consult, e.g., Selten (1975), or Osborne & Rubinstein (1994). We instead expand on the interpretation of some derived terms. Note also that the formalism used here is more expressive than the traditional one due to Kuhn (1953), because (i) it represents simultaneous moves directly by letting plays be sequences of action profiles, and (ii) it represents also the information of inactive players, which is potentially relevant for our analysis of belief change.

A **sequential game** (played by agents) **with perfect recall** is a structure

$$\Gamma = \langle I, \bar{X}, (A_i, H_i, u_i)_{i \in I} \rangle$$

where: I is a finite set of **players**, A_i is a nonempty finite set of potentially available **actions** for player i, and $\bar{X}$ is a *finite tree* of feasible sequences of action profiles, called **histories** or **nodes**.[2] We let: $\varnothing$ denote the **empty sequence**, that is, the root of tree $\bar{X}$, Z denote the set of **terminal** histories (or **paths**), and $X = \bar{X} \backslash Z$ denote the set of **non-terminal**

[1] Games where, according to the rules, players are always reminded of the information previously provided to them and of the actions they took are extremely rare. For this reason, we interpret the perfect recall property as resulting from the interaction between players' personal cognitive features and the rules of the game. When players mnemonic abilities are perfect, whatever the rules of the game, their information structure may be represented with information partitions satisfying perfect recall (see Battigalli & Generoso, 2021). Also, we refrain from using expressions like "extensive-form game" or "normal form game", because by "game" we mean the object being represented, not its mathematical representation. With this, the extensive and normal forms are types of representations, not types of games.

[2] An action profile of the players in nonempty subset $J \subseteq I$ is an element of the cross-product $\times_{i \in J} A_i$. A set of sequences is a **tree** if it contains every prefix of each one of its elements, including the empty sequence.

histories. We write $x \prec x'$ ($x \preceq x'$) if sequence x is a strict (weak) prefix of x', that is, node x precedes node x' in tree $\bar{X}$. For each player $i \in I$ there is a subset $X_i \subseteq X$ of nonterminal nodes where i is **alert**, i.e., she processes information; $\iota(x) = \{i \in I : x \in X_i\}$ denotes the set of players who are alert at x and we assume that $\iota(\varnothing) = I$, that is, all players are alert at the root. Alert players are **active** if they can choose between two or more alternative actions, and **inactive** otherwise, i.e., if all they can do is to "wait". This is described by a profile of nonempty-valued feasibility correspondences $\left(\bar{\mathcal{A}}_i(\cdot) : X_i \rightrightarrows A_i\right)_{i\in I}$ such that, for every $x \in X$, $(x, a_{\iota(x)}) \in \bar{X}$ if and only if $a_{\iota(x)} \in \times_{i\in\iota(x)}\bar{\mathcal{A}}_i(x)$.[3] Thus, $i \in \iota(x)$ is inactive at x if $\bar{\mathcal{A}}_i(x)$ is a singleton. In our graphical representations, such as the game tree Γ' in Figure 1, we only show the actions of active players. Information structure H_i is the collection of **information sets** of player i, where:

1. H_i is a *partition* of X_i;

2. for every $h_i \in H_i$ and $x, x' \in h_i$, $\bar{\mathcal{A}}_i(x) = \bar{\mathcal{A}}_i(x') = \mathcal{A}_i(h_i)$;

3. (*perfect recall*) for every $h_i \in H_i$ and $x, x' \in h_i$ with $x \neq x'$, we have (i) $x \not\prec x'$ and (ii) for all $(\tilde{x}, a) \preceq x$ with $\tilde{x} \in \tilde{h}_i$ for some $\tilde{h}_i \in H_i$, there exists $(\tilde{x}', a') \preceq x'$ such that $\tilde{x}' \in \tilde{h}_i$ and $a_i = a'_i$.[4]

Perfect recall implies that we can unambiguously partially order the information sets of each player i with the "prefix of" precedence relation of X: for all distinct $h_i, h'_i \in H_i$, h_i is a **predecessor** of h'_i, denoted $h_i \prec h'_i$, if and only if every history in h'_i follows some history in h_i (i.e., it has a prefix in h_i).

Finally, $u_i : Z \to \mathbb{R}$ is the **payoff function** of player i.[5] For the sake of simplicity, many of our examples feature *common interests* (CI): $u_i = u_j$ for all $i, j \in I$.

[3]This property ensures that what a player can do at a node does not depend on what co-players are simultaneously doing—otherwise actions would not be simultaneous.

[4]Property (i) says that players cannot end up twice in the same information set because they remember having moved before. Property (ii) says that if two histories are in the same information set, then player i must have been unable to distinguish the prefixes of these histories at earlier information sets and must have taken the same actions at such earlier information sets (since she recalls her past actions).

[5]This is interpreted as the composition $u_i = v_i \circ g$ of the outcome function $g : Z \to Y$ and i's utility function $v_i : Y \to \mathbb{R}$.

We consider two notable special cases: a game has **observable actions** (or perfect monitoring) if $X_i = X$ for every i (players are always alert) and all information sets are singletons, in which case we write $H_i = H = X$ for every i and we do not distinguish between histories/nodes and the singleton information sets containing them. A game with observable actions has **perfect information** if for every history $h \in H$, only one player is active. For example, game Γ'' depicted in Figure 2 has observable actions, game Γ''' depicted in Figure 3 has perfect information.

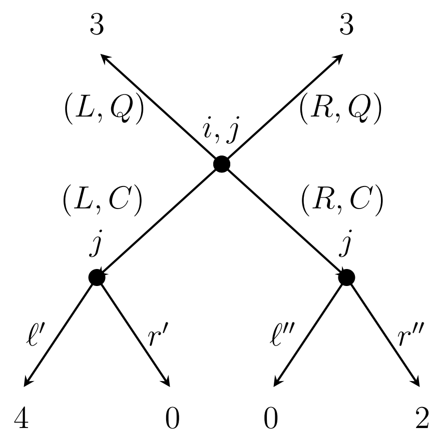

Figure 2: Γ'', a CI game with observable actions.

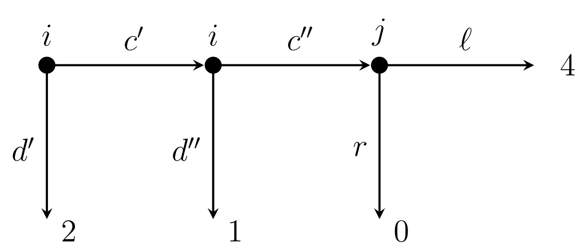

Figure 3: Γ''', a CI game with perfect information.

A **strategy** for player i is a function $s_i : H_i \to A_i$ that assigns to each information set $h_i \in H_i$ a feasible action $s_i(h_i) \in \mathcal{A}_i(h_i)$. Thus, the set of strategies of player i is the cross-product of feasible action sets, $S_i = \times_{h_i \in H_i} \mathcal{A}_i(h_i)$. We denote by $S = \times_{i \in I} S_i$ the set of strategy profiles and by $S_{-i} = \times_{j \neq i} S_j$ the set of i's co-players' strategy profiles. The implementation of a profile of strategies $s \in S$ induces a unique terminal history $\zeta(s)$, where $\zeta : S \to Z$ denotes the **path function**. Although we do not require that feasible action sets at distinct information sets where a player is active be disjoint, this condition holds in our examples. This eases notation, allowing us to write the strategies of our examples as lists of actions separated by dots, such as $R.a'.d''$ for Isa in game Γ' of Figure 1.

For each $h_i \in H_i$, the set of strategy profiles compatible with information set h_i is

$$S(h_i) = \{s \in S : \exists x \in h_i, x \prec \zeta(s)\}.$$

Let $S_i(h_i) = \text{proj}_{S_i} S(h_i)$ and $S_{-i}(h_i) = \text{proj}_{S_{-i}} S(h_i)$. Perfect recall implies that, for all $h_i, h_i' \in H_i$, we have (i) $S(h_i) = S_i(h_i) \times S_{-i}(h_i)$, (ii) if h_i' follows h_i, then $S(h_i') \subseteq S(h_i)$, hence, $S_{-i}(h_i') \subseteq S_{-i}(h_i)$, (iii) $S(h_i) \cap S(h_i') \neq \emptyset$ if and only if either $h_i \preceq h_i'$ or $h_i' \preceq h_i$.[6] Yet, it is possible that $S_{-i}(h_i) \cap S_{-i}(h_i') \neq \emptyset$ even if h_i and h_i' are not ordered. For example, in game Γ', $S_{-i}(\{(L,C)\}) = \{C.\ell, C.r\} = S_{-i}(\{(R,C)\})$.

As in most of the work on strategic reasoning, here *strategies represent both contingent plans in the minds of rational players and descriptions of information-dependent behavior.* Thus, as a player plans her strategy s_i, she assesses the likelihood of the possible "ways of behaving," or "action rules" of the others, s_{-i}.

The interpretation of strategies as plans or mere descriptions of behavior is related to an important structural equivalence relation. Let $H_i(s_i) = \{h_i \in H_i : s_i \in S_i(h_i)\}$ denote the collection of information sets that may occur if i implements strategy s_i. For example, in game Γ', $H_i(R.a'.a'') = H_i(R.a'.d'') = \{\{\varnothing\}, \{(R,C)\}\}$; in game Γ'', $H_j(Q.a'.a'') = \{\{\varnothing\}\}$ for all $a'.a'' \in \{\ell', r'\} \times \{\ell'', r''\}$, and $H_j(C.a'.a'') = \{\{\varnothing\}, \{(L,C)\}, \{(R,C)\}\}$.

Definition 1. Two strategies $s_i', s_i'' \in S_i$ are (1) **behaviorally equivalent** if $H_i(s_i') = H_i(s_i'')$ and $s_i'(h_i) = s_i''(h_i)$ for every $h_i \in H_i(s_i)$, (2) **realization-equivalent** if $\zeta(s_i', s_{-i}) =$

[6] To see this, if $s \in S(h_i)$ and $s \in S(h_i')$, then there exist $x \in h_i$ such that $x \prec \zeta(s)$ and $x' \in h_i'$ such that $x' \prec \zeta(s)$. Hence, either $x' \preceq x$ or $x \preceq x'$, which implies by perfect recall that either $h_i' \preceq h_i$ or $h_i \preceq h_i'$. The reason is that perfect recall implies that for any $h_i, h_i' \in H_i$, if $x \in h_i$, $x' \in h_i'$ and $x \prec x'$, then all nodes of h_i' are preceded by some node of h_i, that is, $h_i \prec h_i'$.

$\zeta(s_i'', s_{-i})$ for every $s_{-i} \in S_{-i}$.

Kuhn (1953) proved that these two equivalence relations coincide:

Remark 1. (Kuhn, 1953, Theorem 1) Two strategies are behaviorally equivalent if and only if they are realization-equivalent.

Let $\equiv_i$ denote this (behavioral or realization) equivalence relation. We call the elements of the quotient set $S_i| \equiv_i$ "structurally reduced strategies," abbreviated in **"reduced strategies"**; e.g., Isa has $2^3 = 8$ strategies, but only 4 reduced strategies in Γ': $L.a'$, $L.d'$, $R.a''$, and $R.d''$, with $L.a' = \{L.a'.a'', L.a'.d''\}$ etc. Often these equivalence classes are instead called "plans of action," suggesting that how s_i is defined outside of $H_i(s_i)$ is irrelevant for planning, and sometimes adding that the restriction of s_i to $H_i \backslash H_i(s_i)$ should be interpreted as an expectation of the co-players (e.g., Osborne & Rubinstein 1994, p 103). We do not adopt this terminology because we have a different perspective that will be fully spelled out in Section 6.

To anticipate, one may think of a player *planning forward* or *backward*. Planning forward means comparing the expected payoffs of different courses of action like Q (quit), or $C.a'.a''$ (continue, then a' if L and a'' if R) for Joe in game Γ'' (opting for $C.\ell'.r''$ if $\mu^j(L|\varnothing) > \frac{1}{2}$, and for Q if $\mu^j(L|\varnothing) < \frac{1}{2}$). When planning forward, there is no need to specify actions for contingencies that cannot occur given the plan under consideration (in Γ'', if Joe considers quitting, he does not have to plan what to do if he instead continues); thus, *reduced strategies correspond to forward plans.* Planning backward means first asking oneself what to do at any last move, use the answer as a contingent prediction about own behavior, and—given this—recursively plan what to do at earlier moves. For example, in game Γ'', without having made up his mind about the move at the root, Joe first plans to choose ℓ' after (L, C) and r'' after (R, C); next, he plans to continue (quit) at the root if $\mu^j(L|\varnothing) > 0.5$ ($\mu^j(L|\varnothing) < 0.5$). Whatever Joe plans to do at the root, *the result of* such *backward planning is a complete strategy* $s_j = a_j.\ell'.r''$ with $a_j \in \{C, Q\}$.[7]

[7]It also makes sense to plan only for moves that are deemed possible under current beliefs. For example,

The foregoing arguments suggest that, if we regard relation $\equiv_i$ as behavioral equivalence, we are led to interpret reduced strategies as "forward plans." If we instead regard $\equiv_i$ as realization equivalence, we can think of reduced strategies as *sufficient descriptions of i's behavior in the eyes of the co-players* (or an external observer). Indeed, if $s_i' \equiv_i s_i''$, independently of the co-players' behavior, it is impossible to distinguish s_i' between s_i'' by observing the realized path; furthermore, it is not necessary for i's co-players to distinguish s_i' between s_i'' in order to assess the likely consequences of taking different actions at (or implementing different continuation strategies starting from) an information set. For example, all that matters for Joe in game Γ' of Figure 1 are the probabilities of the reduced strategies $L.a'$, $L.d'$, $R.a''$, and $R.d''$, interpreted as sufficient descriptions of Isa's behavior.

3 Conditional beliefs

In this section, we introduce several representations of beliefs for sequential games. Players are uncertain about how the others would behave in the various circumstances, and form beliefs on others' behavior to assess the likely consequences of their actions. The beliefs of a player will typically change as the game progresses. At information set $h_i \in H_i$, player i learns that the co-players are behaving according to a strategy profile in $S_{-i}(h_i)$. If player i's beliefs before the realization of information set h_i assigned probability 0 to $S_{-i}(h_i)$, then the realization of h_i falsifies i's earlier beliefs, which have to be revised, rather than just updated according to the rules of conditional probability. We thus have to model what players would believe in all circumstances, and how their beliefs change as they receive new information.

We first consider an abstract representation of conditional thinking by means of "conditional probability systems" (Renyi 1955), which requires some consistency between beliefs conditional on different events. Next we move to "systems of beliefs", which specify the beliefs a player would hold at each information set. We introduce different degrees of consis-

if in game Γ'' Isa is initially certain that Joe is going to quit, then she just plans what to do at the root, which yields a partial reduced strategy. If Joe continues, thus surprising Isa, she decides on the spot what to do next. See Section 6.

tency among beliefs at different information sets and relate them to conditional probability systems.

We use the following notation: For any finite set Ω, interpreted as the space of uncertainty, let $\Delta(\Omega)$ be the set of probability measures on Ω. Whenever the underlying space of uncertainty Ω is understood, for all events $E \subseteq \Omega$ we let $\Delta(E) = \{\mu \in \Delta(\Omega) : \mu(E) = 1\}$.

3.1 Conditional probability systems

Let Ω be a finite space of uncertainty. Fix a nonempty collection of "conceivable conditioning events" $\mathcal{C} \subseteq 2^{\Omega} \setminus \{\emptyset\}$. We call the pair $(\Omega, \mathcal{C})$ a **conditional space**.[8] For example, if we consider player i in a game, we have $\Omega = S_{-i}$, and a natural collection of conditioning events is the observable events about others' behavior $\mathcal{H}_i := \{S_{-i}(h_i) \subseteq S_{-i} : h_i \in H_i\}$.

Definition 2. Fix a conditional space $(\Omega, \mathcal{C})$. An array of probability measures $\mu = (\mu(\cdot|C))_{C \in \mathcal{C}} \in \times_{C \in \mathcal{C}} \Delta(C)$ is a **conditional probability system** (**CPS**) on $(\Omega, \mathcal{C})$, written $\mu \in \Delta^{\mathcal{C}}(\Omega)$, if it satisfies the *chain rule*: for all $E \subseteq \Omega$, $C, D \in \mathcal{C}$,

$$E \subseteq D \subseteq C \Rightarrow \mu(E|C) = \mu(E|D)\,\mu(D|C).$$

The CPSs on $\left(\Omega, 2^{\Omega} \setminus \{\emptyset\}\right)$, whose set is denoted by $\Delta^*(\Omega)$, are called **complete**.

In words, whenever possible, the beliefs for distinct conditioning events must be related to each other by the standard rules of conditional probability. Indeed, the chain rule is equivalent to requiring that, for all $E \subseteq \Omega$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$,

$$\mu(D|C) > 0 \Rightarrow \mu(E|D) = \frac{\mu(E|C)}{\mu(D|C)}.$$

(Note that, since $0 \leq \mu(E|C) \leq \mu(D|C)$, $\mu(D|C) = 0$ implies that the chain-rule equality holds trivially as $0 = 0$.)

[8] If Ω is infinite, consider a triple $(\Omega, \mathcal{B}, \mathcal{C})$, where $\mathcal{B}$ is the relevant sigma-algebra and $\mathcal{C} \subseteq \mathcal{B}$.

As anticipated, a natural representation of conditional beliefs in games is to consider CPSs on $(S_{-i}, \mathcal{H}_i)$, where player i forms beliefs conditional on each observable event about the behavior of others (e.g., Battigalli & Siniscalchi 2002). Complete CPSs instead assume that players form their beliefs conditional on every possible event — for every C, the player thinks: "If I knew C, I would believe $\mu^i(\cdot|C)$." Event C need not represent information player i may obtain during the game, yet she can still ask herself this question and answer to it. Complete CPSs represent the coherent conditional beliefs a player would form if she asked herself this question for every possible event. This representation is used, e.g., in Battigalli (1996).

In general, a richer class of conditioning events gives more bite to the chain rule, because each event is related by set inclusion to more events. Specifically, consider $\mathcal{C} \subset \mathcal{D}$ and let

$$\mathrm{proj}_{\Delta^{\mathcal{C}}(\Omega)}\Delta^{\mathcal{D}}(\Omega) = \left\{\mu \in \Delta^{\mathcal{C}}(\Omega) : \exists \bar{\mu} \in \Delta^{\mathcal{D}}(\Omega), \forall C \in \mathcal{C}, \mu(\cdot|C) = \bar{\mu}(\cdot|C)\right\}$$

denote the set of CPSs on $(\Omega, \mathcal{C})$ that can be derived from some CPS on $(\Omega, \mathcal{D})$.

Remark 2. If $\mathcal{C} \subseteq \mathcal{D}$, then $\mathrm{proj}_{\Delta^{\mathcal{C}}(\Omega)}\Delta^{\mathcal{D}}(\Omega) \subseteq \Delta^{\mathcal{C}}(\Omega)$.

Thus, complete CPSs deserve special attention in that they embody the most stringent restrictions on how players form conditional beliefs.

3.2 Systems of beliefs

We now turn to the main object of our analysis, belief systems. We assume that player i would hold at each of her information sets a belief $\mu^i(\cdot|h_i)$ assigning positive probability only to strategies of co-players' that are compatible with h_i. Thus, players are endowed with **systems of beliefs** (often abbreviated in **"belief systems"**)

$$\mu^i = (\mu^i(\cdot|h_i))_{h_i \in H_i} \in \times_{h_i \in H_i}\Delta\left(S_{-i}\left(h_i\right)\right).$$

We discipline belief change with coherency properties. The first property is embodied in the definition of belief system: *knowledge implies probability-1 belief*, that is, players assign positive probability only to behavior of others consistent with their information. The second property concerns how players update or revise their beliefs across information sets; depending on these assumptions, we obtain different notions of belief systems.[9] We consider three increasingly restrictive notions of consistency in updating/revision. The weakest one requires beliefs to be linked by the chain rule of conditional probabilities *as one moves forward* on a path, that is, if h_i' follows h_i and i deems h_i' reachable from h_i, then beliefs at h_i' should be derived from beliefs at h_i by conditioning on $S_{-i}(h_i')$.

Definition 3. A system of beliefs μ^i is **forward consistent**, written $\mu^i \in \Delta_{\mathcal{F}}^{H_i}(S_{-i})$, if for all $h_i, h_i' \in H_i$ with $h_i \prec h_i'$, for all $E_{-i} \subseteq S_{-i}(h_i')$

$$\mu^i(S_{-i}(h_i')|h_i) > 0 \Rightarrow \mu^i(E_{-i}|h_i') = \frac{\mu^i(E_{-i}|h_i)}{\mu^i(S_{-i}(h_i')|h_i)}.$$

With a slight abuse of language, we often refer to the implication above as the **"forward chain rule."** As discussed in the heuristic example of the Introduction, forward consistency allows players to form different beliefs at two information sets representing the same information on others' behavior. This may happen if h_i and h_i' were both unexpected and differ only in i's own behavior at the previous information set, so that $S_{-i}(h_i) = S_{-i}(h_i')$. In this case, h_i and h_i' do not precede each other, and so forward consistency allows a player to revise beliefs differently at h_i and h_i'. This need not be viewed as a form of "irrationality." Belief systems can be interpreted as an external observer's description of the beliefs a player would hold in every circumstance. Players may not be fully introspective, and know only their current beliefs. We clarify in Section 6 that to form rational plans, players need not plan in advance

[9] There is an analogy between our systems of beliefs and those of Kreps & Wilson (1982). The latter assign conditional probabilities to nonterminal nodes, so that the probabilities of nodes in an information set sum to 1. Focusing on a single player, systems of beliefs in the sense of Kreps & Wilson can be derived from systems of beliefs in our sense. In both cases, across-information-sets restrictions are explicitly added by considering subsets of systems of beliefs.

how they would behave at unexpected contingencies, and hence need not think in advance about what they would believe in such occurrences. For some psychological reasons, players may then revise their beliefs in different ways depending on their past actions.

Yet, one may still want to assume that players do not form different beliefs about others depending on their own behavior. Our intermediate restriction on belief systems excludes such cases by deriving belief systems from CPSs on $(S_{-i}, \mathcal{H}_i)$. CPSs embody by construction the property that same knowledge on others' behavior implies same beliefs. Indeed, if $S_{-i}(h_i) = S_{-i}(h_i')$, then h_i and h_i' correspond to the same element of $\mathcal{H}_i$, and hence for a CPS $\bar{\mu}^i$ we have by construction $\bar{\mu}^i(\cdot|S_{-i}(h_i)) = \bar{\mu}^i(\cdot|S_{-i}(h_i'))$.

Definition 4. A system of beliefs μ^i is **standard**, written $\mu^i \in \Delta^{H_i}(S_{-i})$, if there exists a CPS $\bar{\mu}^i \in \Delta^{\mathcal{H}_i}(S_{-i})$ such that $\mu^i(\cdot|h_i) = \bar{\mu}^i(\cdot|S_{-i}(h_i))$ for every $h_i \in H_i$.

(We call them "standard" because CPSs on $(S_{-i}, \mathcal{H}_i)$ have been widely used in the literature on strategic reasoning in games since Battigalli & Siniscalchi 1999.)

Remark 3. A belief system μ^i is standard if and only if for all $h_i, h_i' \in H_i$ with $S_{-i}(h_i') \subseteq S_{-i}(h_i)$, for all $E_{-i} \subseteq S_{-i}(h_i')$ we have

$$\mu^i(S_{-i}(h_i')|h_i) > 0 \Rightarrow \mu^i(E_{-i}|h_i') = \frac{\mu^i(E_{-i}|h_i)}{\mu^i(S_{-i}(h_i')|h_i)}.$$

By perfect recall, $h_i \prec h_i'$ implies $S_{-i}(h_i') \subseteq S_{-i}(h_i)$; thus, all standard belief systems are forward consistent.

As already noted, H_i typically has a larger cardinality than $\mathcal{H}_i$ because information sets may also represent information about player i's behavior, not just the behavior of the co-players $-i$. For example, in game Γ' of Figure 1, $H_i = \{\{\varnothing\}, \{(L, C)\}, \{(R, C)\}\}$ has three elements, while $\mathcal{H}_i = \{S_j, \{C.\ell, C.\text{r}\}\}$ has two elements. Despite this, the set of standard belief systems $\Delta^{H_i}(S_{-i})$ is isomorphic to $\Delta^{\mathcal{H}_i}(S_{-i})$. Indeed, for every standard belief system μ^i, $S_{-i}(h_i'') = S_{-i}(h_i')$ implies $\mu^i(\cdot|h_i') = \mu^i(\cdot|h_i'')$; with this, given a standard belief system μ^i, the array $(\bar{\mu}^i(\cdot|C))_{C\in\mathcal{H}_i}$ where $\bar{\mu}^i(\cdot|S_{-i}(h_i)) = \mu^i(\cdot|h_i)$ for each $h_i \in H_i$ defines uniquely a CPS on $(S_{-i}, \mathcal{H}_i)$. So $\mu^i(\cdot|h_i) \longleftrightarrow \bar{\mu}^i(\cdot|S_{-i}(h_i))$ is a bijection between $\Delta^{H_i}(S_{-i})$ and $\Delta^{\mathcal{H}_i}(S_{-i})$. In Section 4, we characterize standard belief systems in games with observable actions.

The strongest notion of consistency requires a belief system to be induced by some complete CPS.

Definition 5. A system of beliefs μ^i is **completely consistent**, written $\mu^i \in \Delta_{\mathcal{C}}^{H_i}(S_{-i})$, if there exists a complete CPS $\bar{\mu}^i \in \Delta^*(S_{-i})$ such that $\mu^i(\cdot|h_i) = \bar{\mu}^i(\cdot|S_{-i}(h_i))$ for every $h_i \in H_i$.

By Remark 2, complete consistency is more restrictive than standard consistency. To give a sense of the strength of these restrictions, complete consistency implies that the odds ratio of any pair of strategy profiles of the co-players (if well-defined) is the same at all the information sets that are compatible with both strategy profiles.

Remark 4. If belief system μ^i is completely consistent then, for all $h_i, h_i' \in H_i$ and $s_{-i}, t_{-i} \in S_{-i}(h_i) \cap S_{-i}(h_i')$,

$$\mu^i(t_{-i}|h_i), \mu^i(t_{-i}|h_i') > 0 \Rightarrow \frac{\mu^i(s_{-i}|h_i)}{\mu^i(t_{-i}|h_i)} = \frac{\mu^i(s_{-i}|h_i')}{\mu^i(t_{-i}|h_i')}.$$

Proof: Consider a belief system μ^i derived from some complete CPS $\bar{\mu}^i$. Fix any $h_i, h_i' \in H_i$ and $s_{-i}, t_{-i} \in S_{-i}(h_i) \cap S_{-i}(h_i')$ such that $\mu^i(t_{-i}|h_i) > 0$ and $\mu^i(t_{-i}|h_i') > 0$. Since $\bar{\mu}^i$ is a complete CPS, the chain rule relates $\bar{\mu}^i(\cdot|S_{-i}(h_i))$ and $\bar{\mu}^i(\cdot|S_{-i}(h_i'))$ to $\bar{\mu}^i(\cdot|\{s_{-i}, t_{-i}\})$. Hence,

$$\mu^i(s_{-i}|h_i) = \bar{\mu}^i(s_{-i}|S_{-i}(h_i)) = \bar{\mu}^i(s_{-i}|\{s_{-i}, t_{-i}\}) \cdot \bar{\mu}^i(\{s_{-i}, t_{-i}\}|S_{-i}(h_i)),$$

$$\mu^i(s_{-i}|h_i') = \bar{\mu}^i(s_{-i}|S_{-i}(h_i')) = \bar{\mu}^i(s_{-i}|\{s_{-i}, t_{-i}\}) \cdot \bar{\mu}^i(\{s_{-i}, t_{-i}\}|S_{-i}(h_i')).$$

Since $\mu^i(t_{-i}|h_i), \mu^i(t_{-i}|h_i') > 0$ we have $\bar{\mu}^i(\{s_{-i}, t_{-i}\}|S_{-i}(h_i)) = \mu^i(\{s_{-i}, t_{-i}\}|h_i) > 0$ and so

$$\frac{\mu^i(s_{-i}|h_i)}{\mu^i(\{s_{-i}, t_{-i}\}|h_i)} = \bar{\mu}^i(s_{-i}|\{s_{-i}, t_{-i}\}) = \frac{\mu^i(s_{-i}|h_i')}{\mu^i(\{s_{-i}, t_{-i}\}|h_i')}.$$

It can be analogously verified for t_{-i} that

$$\frac{\mu^i(t_{-i}|h_i)}{\mu^i(\{s_{-i},t_{-i}\}|h_i)} = \bar{\mu}^i(t_{-i}|\{s_{-i},t_{-i}\}) = \frac{\mu^i(t_{-i}|h_i')}{\mu^i(\{s_{-i},t_{-i}\}|h_i')}.$$

The last two equations yield, dividing the former by the latter,

$$\frac{\mu^i(s_{-i}|h_i)}{\mu^i(t_{-i}|h_i)} = \frac{\mu^i(s_{-i}|h_i')}{\mu^i(t_{-i}|h_i')}.$$

■

Standard belief systems need not satisfy this property, as shown in Example 1 of Section 4. Yet, this necessary condition, although quite strong, is not sufficient for complete consistency. We provide in Section 5 an example of this fact, as well as the exact characterization of complete consistency in terms of coherency of odds ratios at different information sets. It is also worth noting that complete consistency is connected to Kreps & Wilson's (1982) consistency in the following sense:

Remark 5. A system of beliefs μ^i is completely consistent if and only if there is a sequence of strictly positive probability measures $(\nu_n)_{n\in\mathbb{N}}$ on S_{-i} such that

$$\forall h_i \in H_i, \forall s_{-i} \in S_{-i}(h_i), \; \mu^i(s_{-i}|h_i) = \lim_{n\to\infty} \frac{\nu_n(s_{-i})}{\nu_n(S_{-i}(h_i))}.$$

This is implied by Remark 10 in the Appendix. Kreps & Wilson (1982) put forward an across-players notion of consistency of assessments derived from Selten's (1975) "trembling hand" idea. Remark 5 implies that, in two-person games, complete consistency is equivalent to a single-player version of Kreps & Wilson's consistency. See the Appendix for details.

To summarize, we defined three nested subsets of belief systems (the inclusions may be strict):

$$\Delta_{\mathcal{C}}^{H_i}(S_{-i}) \subseteq \Delta^{H_i}(S_{-i}) \subseteq \Delta_{\mathcal{F}}^{H_i}(S_{-i}).$$

In the next sections, we shed further light on the interpretation of these belief systems by providing a characterization of standard consistency in games with observable actions and a characterization of complete consistency. Then we move to solution concepts, and we show that in spite of a considerable difference in the degree of consistency assumed by these types of belief systems, these differences are inconsequential for the analysis of strategic reasoning, as we obtain equivalent versions of rationalizability for sequential games.

4 Standard belief systems in games with observable actions

In this section, we provide a characterization of standard belief systems in *games with observable actions*. We show that a belief system is standard if and only if it is forward consistent and it assigns the same beliefs at histories that differ only in player i's own action at the immediate predecessor —i.e., beliefs are independent of own behavior. Then we show that in perfect-information games, forward consistency and "standard" consistency coincide.

It is first convenient to study the structure of the "strategic-form information sets" $S_{-i}(h)$ $(h \in H)$ in games with observable actions. In such games, non-terminal histories and information sets coincide; therefore, for every $h \in H$, we have $S(h) = \times_{i \in I} S_i(h)$. For each $i \in I$, $h \in H$, and $a_{-i} \in \mathcal{A}_{-i}(h)$, let

$$S_{-i}(h, a_{-i}) = \{s_{-i} \in S_{-i}(h) : s_{-i}(h) = a_{-i}\}$$

denote the set of co-players' strategy profiles consistent with h that select a_{-i} at h. For any $h', h'' \in H$, let $\pi(h', h'') \in H$ be the **last common predecessor** (longest common prefix) of h' and h''. For any $h \prec h'$, let $\alpha(h, h') = (\alpha_j(h, h'))_{j \in I} \in \mathcal{A}(h)$ be the unique action profile such that $(h, \alpha(h, h')) \preceq h'$, and let $f(h, h')$ be the **immediate follower** of h (weakly) **preceding** h'. Note, by definition, $S_{-i}(f(h, h')) = S_{-i}(h, \alpha_{-i}(h, h'))$. The following lemma is crucial to characterize standard belief systems.

Lemma 1. *In games with observable actions, if two histories $h', h'' \in H$ are not ordered by precedence and $\bar{h}$ is their longest common predecessor, then $S_{-i}(h') \subseteq S_{-i}(h'')$ implies that (a) $\alpha_{-i}(\bar{h}, h') = \alpha_{-i}(\bar{h}, h'')$ and (b) $S_{-i}(h'') = S_{-i}(f(\bar{h}, h'')) = S_{-i}(f(\bar{h}, h'))$.*

Proof: Take any two histories $h', h'' \in H$ not ordered by precedence and consider their last common predecessor $\bar{h} = \pi(h', h'')$. We first prove by contraposition that, if $S_{-i}(h') \subseteq S_{-i}(h'')$, then only player i can be active at histories h such that $\bar{h} \prec h \prec h''$. Indeed, suppose that some player $j \neq i$ is active at such h; then, j has an action $a_j^* \in \mathcal{A}_j(h) \setminus \{\alpha_j(h, h'')\}$, which implies that there exists $a_{-i}^* \in \mathcal{A}_{-i}(h)$ such that $a_{-i}^* \neq \alpha_{-i}(h, h'')$. Take any $s_{-i}^* \in S_{-i}$ such that (i) for all $\tilde{h} \prec h'$, $s_{-i}(\tilde{h}) = \alpha_{-i}(\tilde{h}, h')$, so that $s_{-i}^* \in S_{-i}(h')$, and (ii) $s_{-i}(h) = a_{-i}^*$, so that $s_{-i}^* \notin S_{-i}(h'')$ (noting that $h \not\prec h'$, this does not conflict with (i)). With this, there exists $s_{-i}^* \in S_{-i}(h') \backslash S_{-i}(h'')$, that is, $S_{-i}(h') \nsubseteq S_{-i}(h'')$.

Now suppose that $S_{-i}(h') \subseteq S_{-i}(h'')$. Then $\alpha_{-i}(\bar{h}, h') = \alpha_{-i}(\bar{h}, h'')$, otherwise $S_{-i}(h')$ and $S_{-i}(h'')$ would be disjoint. This implies $S_{-i}(f(\bar{h}, h')) = S_{-i}(f(\bar{h}, h''))$. Since only player i can be active at histories h with $\bar{h} \prec h \prec h''$, we have $S_{-i}(h'') = S_{-i}(f(\bar{h}, h''))$. ■

Using this result, we can characterize standard belief systems by means of a property of independence of beliefs from own behavior.

Definition 6. A system of beliefs μ^i satisfies **own-action independence** (**OI**) if $\mu^i(\cdot|h') = \mu^i(\cdot|h'')$ for all $h' = (h, a')$ and $h'' = (h, a'')$ with $a'_{-i} = a''_{-i}$.

Denote by $\Delta^H_{\mathcal{F},OI}(S_{-i})$ and $\Delta^H_{OI}(S_{-i})$ the sets of forward consistent and standard belief systems that satisfy OI.

Remark 6. In games with observable actions, all standard belief systems satisfy OI, i.e., $\Delta^H_{OI}(S_{-i}) = \Delta^H(S_{-i})$. Indeed, if $h' = (h, a')$ and $h'' = (h, a'')$ are such that $a'_{-i} = a''_{-i}$, then $S_{-i}(h') = S_{-i}(h'')$. Then by Remark 3 the chain rule applies to the pair (h', h''), which implies, for equal conditioning events, equal conditional measures.

Theorem 1. *In games with observable actions, a system of beliefs is standard if and only if it is forward consistent and satisfies OI, that is, $\Delta^H(S_{-i}) = \Delta^H_{\mathcal{F},OI}(S_{-i})$.*

Proof: The inclusion $\Delta^H(S_{-i}) = \Delta^H_{OI}(S_{-i}) \subseteq \Delta^H_{\mathcal{F},OI}(S_{-i})$ is obvious. We show the other inclusion. Take any $\mu^i \in \Delta^H_{\mathcal{F},OI}(S_{-i})$ and two histories $h', h'' \in H$ with $S_{-i}(h') \subseteq S_{-i}(h'')$. By Remark 3, it is enough to show that the forward chain rule relates $\mu^i(\cdot|h'')$ to $\mu^i(\cdot|h')$. If $h'' \preceq h'$ the forward chain rule relates $\mu^i(\cdot|h'')$ to $\mu^i(\cdot|h')$. If h'' and h' are not related by precedence, then let $h = \pi(h', h'')$, $a' = \alpha(h, h')$ and $a'' = \alpha(h, h'')$. Since $S_{-i}(h') \subseteq S_{-i}(h'')$, Lemma 1 implies (a) $a'_{-i} = a''_{-i}$ and (b) $S_{-i}(h'') = S_{-i}((h, a'')) = S_{-i}((h, a'))$. Since $(h, a'') \preceq h''$, the forward chain rule relates $\mu^i(\cdot|(h, a''))$ to $\mu^i(\cdot|h'')$. Since $S_{-i}(h'') = S_{-i}((h, a''))$, this implies $\mu^i(\cdot|h'') = \mu^i(\cdot|(h, a''))$. Since $a'_{-i} = a''_{-i}$ and μ^i satisfies OI, we have $\mu^i(\cdot|(h, a'')) = \mu^i(\cdot|(h, a'))$. Finally, since $(h, a') \preceq h'$, the forward chain rule relates $\mu^i(\cdot|(h, a'))$ to $\mu^i(\cdot|h')$. Thus, $\mu^i(\cdot|h'') = \mu^i(\cdot|(h, a'')) = \mu^i(\cdot|(h, a'))$ and the forward chain rule indirectly relates $\mu^i(\cdot|h'')$ to $\mu^i(\cdot|h')$. ■

This characterization is useful in constructive proofs when working with "standard" CPSs in games with observable actions: it is often easier to show that a system of beliefs satisfies the forward chain rule and OI than showing directly standard consistency.

Note, OI is key to be able to conclude that $\mu^i(\cdot|(h, a')) = \mu^i(\cdot|(h, a''))$ in the proof. Otherwise, if $\mu^i(S_{-i}(h, a'_{-i})|h) = 0$, forward consistency does not guarantee that i revises her beliefs at (h, a') and (h, a'') in the same way and hence we would not be guaranteed that the forward chain rule indirectly relate $\mu^i(\cdot|h')$ and $\mu^i(\cdot|h'')$.

On the other hand, a forward consistent belief system μ^i may violate OI at two histories $h' = (h, a')$, $h'' = (h, a'')$ with $a'_{-i} = a''_{-i}$ only if (a) $a'_i \neq a''_i$, which implies that i is active at h, and (b) $\mu^i(S_{-i}(h, a'_{-i})|h) = 0$, which implies that at least one co-player of i is active at h, since otherwise $S_{-i}(h, a'_{-i}) = S_{-i}(h)$ and thus $\mu^i(S_{-i}(h, a'_{-i})|h) = \mu^i(S_{-i}(h)|h) = 1 > 0$. So, forward consistent belief systems might feature own-action *dependence* only in games where there are simultaneous moves at some stage, which yields the following result:

Corollary 1. *In games with perfect information, a system of beliefs is standard if and only if it is forward consistent:* $\Delta^H(S_{-i}) = \Delta^H_{\mathcal{F}}(S_{-i})$.

Since these results are independent of players' interactive knowledge of the profile of payoff functions, they extend to games with incomplete information and observable actions.

As for complete consistency, we typically have that it is stronger than forward/standard consistency, even in games with perfect information, as shows the example below.

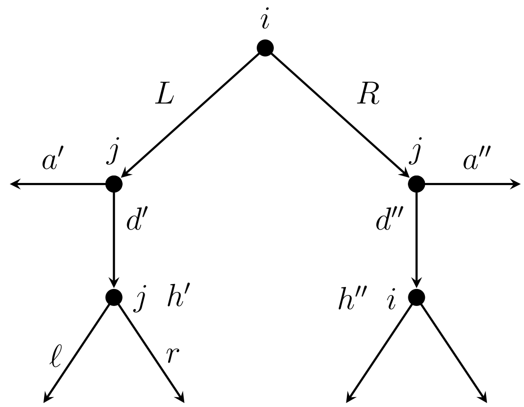

Figure 4:

Example 1. Consider histories $h' = (L, d')$ and $h'' = (R, d'')$ in the perfect-information game structure depicted in Figure 4. For strategies $s_{-i} = d'.d''.\ell$, $s'_{-i} = d'.d''.r$, and $s^*_{-i} = a'.a''.\ell$ (where $-i = j$ is Joe), we have $s_{-i}, s'_{-i} \in S_{-i}(h') \cap S_{-i}(h'')$, $s^*_{-i} \notin S_{-i}(h')$, and $s^*_{-i} \notin S_{-i}(h'')$. Take a forward consistent (hence, by Corollary 1, *standard*) belief system μ^i of Isa such that $\mu^i(s^*_{-i}|\varnothing) = 1$. Isa's beliefs after L and R coincide with her initial belief, because Joe is not active at the root. It follows that $\mu^i(S_{-i}(h')|L) = 0$ and $\mu^i(S_{-i}(h'')|R) = 0$, so that Isa has to revise her beliefs at both h' and h''. Since $S_{-i}(h')$ and $S_{-i}(h'')$ are different and not nested, we can set $\mu^i(s_{-i}|h') = 1$ and $\mu^i(s'_{-i}|h'') = 1$. But this violates the odds-ratio property of Remark 4, which would require in this specific case that $\mu^i(s_{-i}|h') = \mu^i(s_{-i}|h'')$. Thus, μ^i is a standard belief system that is not completely consistent. ▲

5 A characterization of complete consistency

In this section, we show that complete consistency is characterized by a strong coherency condition relating probability ratios across different information sets.

We already noted that complete consistency implies constant odds ratios between pair of opponents' strategy profiles at all information sets consistent with both strategy profiles. Yet, this condition is not sufficient for complete consistency, as shown by the following example.

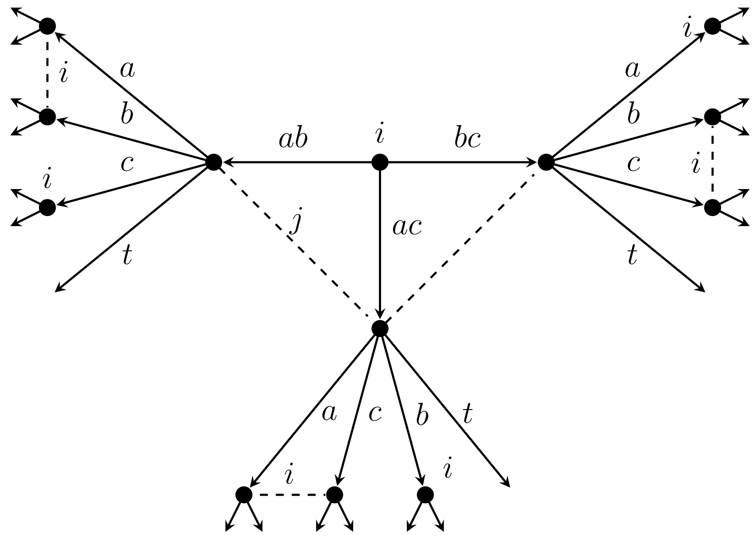

Figure 5:

Example 2. Consider the game structure depicted in Figure 5. Note that there is no pair (h_i, h_i') of third-stage information sets of Isa such that $S_j(h_i) \cap S_j(h_i')$ contains two distinct actions/strategies of Joe. Thus, if belief system μ^i is forward consistent it also satisfies the "odds-ratio" property of Remark 4, which applies only if either h_i or h_i', is the initial information set of Isa. Suppose that Isa is initially certain of termination, that is, $\mu^i(t|\varnothing) = 1$, then forward consistency and the "odds-ratios" property are trivially satisfied by μ^i. Yet, μ^i need not be completely consistent. To see this, let $[xy] = \{(x, xy), (y, xy)\}$ denote the third-stage information set of Isa after she chose xy and Joe chose either x or y (both different from t). Since $\mu^i(t|\varnothing) = 1$, forward consistency puts no constraints on beliefs at such information sets. Then we may have $\mu^i(a|[ab]) > \frac{1}{2}$, $\mu^i(b|[bc]) > \frac{1}{2}$, and $\mu^i(c|[ac]) > \frac{1}{2}$, which violates complete consistency. Indeed, any complete CPS $\bar{\mu}^i \in \Delta^*(S_j)$ defines i's belief conditional on $\{a, b, c\} \subset S_j$, and hence the chain rule relates beliefs conditional on $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ to $\bar{\mu}^i(\cdot|\{a, b, c\})$. ▲

A standard odds ratio is computed using the conditional probabilities of two elements s_{-i} and s'_{-i} at the same information set. As such, it does not allow to compare the probabilities of s_{-i} and s'_{-i} at different information sets. This comparison is made possible by the following notion of "generalized odds ratio."

Definition 7. Given a belief system $(\mu^i(\cdot|h_i))_{h_i \in H_i}$ and a pair $s_{-i}, s'_{-i} \in S_{-i}$, a (possibly infinite) **generalized odds ratio** between s_{-i} and s'_{-i} is the product of a finite concatenation of odds ratios, namely:

$$q = \frac{\mu^i(s_{-i}|h_i^1)}{\mu^i(s_{-i}^1|h_i^1)} \cdot \frac{\mu^i(s_{-i}^1|h_i^2)}{\mu^i(s_{-i}^2|h_i^2)} \cdot \ldots \cdot \frac{\mu^i(s_{-i}^{n-1}|h_i^n)}{\mu^i(s'_{-i}|h_i^n)},$$

where $s_{-i}^{k-1}, s_{-i}^k \in S_{-i}\left(h_i^k\right)$ for all $k \in \{1, ...n\}$, with $s_{-i}^0 = s_{-i}$, $s_{-i}^n = s'_{-i}$, and if one odds ratio is infinite, no odds ratio is zero.

For instance, if player i deems s_{-i} twice more likely than another strategy s'_{-i} at some information set where both s_{-i} and s'_{-i} are still possible, and deems s'_{-i} twice more likely than s''_{-i} at some other information set where s'_{-i} and s''_{-i} are still possible, then we can say that there is a generalized odds ratio of 4 between s_{-i} and s''_{-i} in player i's belief system.

A generalized odds ratio can be zero or infinite (but not indeterminate). A zero generalized odds ratio between s_{-i} and s'_{-i} can be given the following interpretation: If player i was to make an indirect assessment of the relative likelihood of s_{-i} and s'_{-i} by looking at the relative likelihoods between these and other strategies at different information sets, she must conclude that s_{-i} has zero probability whenever s'_{-i} is deemed possible. Whether she actually does so depends on the degree of consistency in her belief system. Complete consistency implies that all generalized odds ratios for any pair of co-players' strategy profiles are equal.[10]

Theorem 2. *A belief system μ^i is completely consistent if and only if for every pair $s_{-i}, s'_{-i} \in S_{-i}$, all generalized odds ratios of μ^i between s_{-i} and s'_{-i} are identical.*

[10]The (2020) working-paper version of Siniscalchi (2021) provides an alternate characterization of complete consistency closely connected to ours.

Proof. The part showing that constant generalized odds ratios (GORs) given μ^i implies complete consistency of μ^i is rather technical and is proved by Catonini (2021). We provide a direct proof that complete consistency implies constant GORs. Fix a pair (s_{-i}, s'_{-i}) and consider two GORs

$$\begin{aligned} q^1 &= \frac{\mu^i(s_{-i}|h_i^1)}{\mu^i(s_{-i}^1|h_i^1)} \cdot \frac{\mu^i(s_{-i}^1|h_i^2)}{\mu^i(s_{-i}^2|h_i^2)} \cdot ... \cdot \frac{\mu^i(s_{-i}^{n-1}|h_i^n)}{\mu^i(s'_{-i}|h_i^n)}, \\ q^2 &= \frac{\mu^i(s_{-i}|h_i^m)}{\mu^i(s_{-i}^{m-1}|h_i^m)} \cdot \frac{\mu^i(s_{-i}^{m-1}|h_i^{m-1})}{\mu^i(s_{-i}^{m-2}|h_i^{m-1})} \cdot ... \cdot \frac{\mu^i(s_{-i}^{n+1}|h_i^{n+1})}{\mu^i(s'_{-i}|h_i^{n+1})}, \end{aligned}$$

where we use a labeling convention so that $m > n$. We want to show that $q^1 = q^2$. If they are both zero or both infinite, we are done. Otherwise, suppose without loss of generality that $q^2 \geq q^1$, so that $q^1 \neq \infty$ and $q^2 > 0$. Calling $s_{-i}^n = s'_{-i}$ and $s_{-i}^0 = s_{-i}$, let

$$q := \frac{q^1}{q^2} = \frac{\mu^i(s_{-i}^0|h_i^1)}{\mu^i(s_{-i}^1|h_i^1)} \cdot ... \cdot \frac{\mu^i(s_{-i}^{n-1}|h_i^n)}{\mu^i(s_{-i}^n|h_i^n)} \cdot \frac{\mu^i(s_{-i}^n|h_i^{n+1})}{\mu^i(s_{-i}^{n+1}|h_i^{n+1})} \cdot ... \cdot \frac{\mu^i(s_{-i}^{m-1}|h_i^m)}{\mu^i(s_{-i}^0|h_i^m)}.$$

Thus, all denominators are positive. Clearly, $q^1 = q^2$ if $q = 1$. To show $q = 1$, we proceed as follows: Let $\bar{\mu}^i = (\bar{\mu}^i(\cdot|h_i))_{2^{S_{-i}}\setminus\{\emptyset\}}$ be a complete CPS that extends μ^i. Let

$$C = \cup_{k=1}^m S_{-i}(h_i^k).$$

Since $\bar{\mu}^i(C|C) = 1$, there is $k \in \{1, ..., m\}$ such that $\bar{\mu}^i(S_{-i}(h_i^k)|C) > 0$. Since $\bar{\mu}^i(s_{-i}^k|S_{-i}(h_i^k)) = \mu^i(s_{-i}^k|h_i^k) > 0$, by the chain rule $\bar{\mu}^i(s_{-i}^k|C) > 0$. By definition of the odds ratios, $s_{-i}^k \in S_{-i}(h_i^{k+1})$ (whenever $k + 1 \leq m$), hence $\bar{\mu}^i(S_{-i}(h_i^{k+1})|C) > 0$. Expanding this reasoning we obtain that $\bar{\mu}^i(S_{-i}(h_i^p)|C) > 0$ for every $p = k, ..., m$. Now, note that $s_{-i}^m = s_{-i}^0$, and that $s_{-i}^0 \in S_{-i}(h_i^1)$. So, $\bar{\mu}^i(S_{-i}(h_i^m)|C) > 0$ implies $\bar{\mu}^i(s_{-i}^0|C) > 0$ by the chain rule with $\bar{\mu}^i(\cdot|S_{-i}(h_i^m))$ and $\bar{\mu}^i(s_{-i}^0|S_{-i}(h_i^m)) > 0$. Thus, $\bar{\mu}^i(S_{-i}(h_i^1)|C) > 0$, and repeating the above argument we have $\bar{\mu}^i(S_{-i}(h_i^p)|C) > 0$ for all $p = 1, ..., m$. Therefore, by the chain rule

$$\frac{\mu^i(s_{-i}^{p-1}|h_i^p)}{\mu^i(s_{-i}^p|h_i^p)} = \frac{\bar{\mu}^i(s_{-i}^{p-1}|C)}{\bar{\mu}^i(s_{-i}^p|C)}.$$

Hence,

$$q = \frac{\bar{\mu}^i(s^0_{-i}|C)}{\bar{\mu}^i(s^1_{-i}|C)} \cdot \frac{\bar{\mu}^i(s^1_{-i}|C)}{\bar{\mu}^i(s^2_{-i}|C)} \cdot ... \cdot \frac{\bar{\mu}^i(s^{m-1}_{-i}|C)}{\bar{\mu}^i(s^0_{-i}|C)} = 1.$$

■

Thus, complete consistency assumes a very high degree of coherency concerning how players revise their beliefs at different information sets. Yet this coherency need not be viewed as a consequence of conscious calculations of a fully introspective player about his conditional beliefs. A player may still be partially introspective about how she would revise her beliefs at unexpected contingencies, yet complete consistency could be a "wired-in" property of belief formation. Catonini (2021) relates complete consistency to the possibility of being "dutch-booked", suggesting that individuals violating it may be willing to undertake detrimental courses of actions.

6 Rational planning

In this section, we define standard criteria of optimality for strategies given a system of beliefs and develop a novel interpretation in terms of "partial planning", which does not require players to be fully introspective about their conditional beliefs, but only requires players to know their current beliefs. We then relate this interpretation to the degree of consistency that belief systems may feature.

Recall that $H_i(s_i)$ is the collection of information sets that may occur if i implements strategy s_i. For any s_i, $h_i \in H_i(s_i)$, and belief system μ^i, the conditional expected payoff of s_i given h_i under μ^i is

$$U_i(s_i, \mu^i(\cdot|h_i)) = \sum_{s'_{-i} \in S_{-i}(h_i)} u_i(\zeta(s_i, s'_{-i}))\mu^i(s'_{-i}|h_i).$$

Remark 7. Fix s_i, μ^i, and $h_i \in H_i(s_i)$ arbitrarily. By Remark 1, expected payoff $U_i(s_i, \mu^i(\cdot|h_i))$ is independent of how s_i is specified at information sets that cannot occur under s_i: $U_i(s_i, \mu^i(\cdot|h_i)) = U_i(s_i', \mu^i(\cdot|h_i))$ for every s_i' that selects the same actions as s_i on the sub-domain $H_i(s_i)$, that is, for every $s_i' \equiv_i s_i$.

For every $h_i \in H_i$, let $\rho_i[s_i/h_i]$ denote the "h_i-replacement" strategy that, at every $h_i' \prec h_i$, chooses the unique action $\alpha_i(h_i', h_i)$ leading from h_i' toward h_i, and coincides with s_i at all other information sets, that is,

$$\forall h_i' \in H_i,\ \rho_i[s_i/h_i](h_i') = \begin{cases} \alpha_i(h_i', h_i) & \text{if } h_i' \prec h_i, \\ s_i(h_i') & \text{if } h_i' \not\prec h_i. \end{cases}$$

By definition, $\rho_i[s_i/h_i] \in S_i(h_i)$ and $U_i(\rho_i[s_i/h_i], \mu^i(\cdot|h_i))$ is well posed for all $s_i \in S_i$ and $h_i \in H_i$. The most demanding notion of optimality of behavior consists in planning and implementing a strategy that maximizes expected payoff starting from every information set.

Definition 8. A strategy $\bar{s}_i$ is **sequentially optimal** under belief system μ^i, written $\bar{s}_i \in BR_i^*(\mu^i)$, if

$$\forall h_i \in H_i, \forall s_i' \in S_i(h_i), \quad U_i(\rho_i[\bar{s}_i/h_i], \mu^i(\cdot|h_i)) \geq U_i(s_i', \mu^i(\cdot|h_i)).$$

Note that, if belief system μ^i is not forward consistent, the set of sequentially optimal strategies under μ^i may be empty because of conflicts between expected-payoff maximization at different ordered information sets.

Example 3. In game Γ''' of Figure 3, the chain rule implies that Isa must hold the same belief about Joe at the first and second node, because at both nodes she has no information about Joe. Thus, if she initially believes $\mu^i(\ell|\varnothing) > \frac{1}{2}$ *and the chain rule holds*, strategy $c'.c''$ is sequentially optimal. If instead $\mu^i(\ell|\varnothing) > \frac{1}{2}$ and $\mu^i(\ell|c') < \frac{1}{4}$, violating the chain rule, then the maximization problem at the root is still solved by $c'.c''$, but the one at history/node (c') is solved by $c'.d''$, and no strategy is sequentially optimal under μ^i. ▲

The aforementioned issue of conflicting conditional preferences is circumvented by an "intra-personal equilibrium" property that can always be satisfied in finite games with perfect recall: one-step optimality. For any strategy $s_i \in S_i$, information set $h_i \in H_i$, and action $a_i \in \mathcal{A}_i(h_i)$, let $\lambda_i [s_i/h_i, a_i]$ denote the "local (h_i, a_i)-replacement" strategy of i that chooses a_i at h_i and coincides with $\rho_i [s_i/h_i]$ at all other information sets of i, that is,

$$\forall h_i' \in H_i,\ \lambda_i [s_i/h_i, a_i] (h_i') = \begin{cases} a_i & \text{if } h_i' = h_i, \\ \alpha_i(h_i', h_i) & \text{if } h_i' \prec h_i, \\ s_i (h_i') & \text{if } h_i' \not\preceq h_i. \end{cases}$$

Definition 9. A strategy $\bar{s}_i \in S_i$ is **one-step optimal** under belief system μ^i if

$$\forall h_i \in H_i, \forall a_i \in \mathcal{A}_i(h_i), \quad U_i(\rho_i [\bar{s}_i/h_i], \mu^i(\cdot|h_i)) \geq U_i(\lambda_i [\bar{s}_i/h_i, a_i], \mu^i(\cdot|h_i)).$$

One-step optimality can be interpreted as the result of a *folding-back planning* algorithm: A fully introspective player i who knows her belief system μ^i first considers the information sets h_i with height 0 in H_i, that is, those where she makes a last move;[11] for any such information set h_i, she plans to choose an expected-payoff-maximizing action given belief $\mu^i (\cdot|h_i)$ about the co-players (breaking ties arbitrarily). In step $\ell > 0$ of the algorithm, as she considers any information set h_i with height ℓ in H_i, that is, one that followed by at most ℓ moves of her, she plans to choose an expected-payoff-maximizing action given belief $\mu^i (\cdot|h_i)$ about the co-players and the prediction that she would behave as already planned in earlier steps of the algorithm at the following information sets of height $k < \ell$ (again, breaking ties arbitrarily). Note that this algorithm is equivalent to one-step optimality even if the chain rule does not hold.

Example 4. Go back to game Γ'''. If $\mu^i (\ell|\varnothing) > \frac{1}{2}$ and $\mu^i (\ell|c') < \frac{1}{4}$, the only one-step

[11] For the sake of this discussion, the distinction between information sets where i is active or inactive is immaterial.

optimal strategy is $d'.d''$: Isa understands how she would change her beliefs and that she would choose d'' at the second node, if reached; thus, she chooses d' at the root. In other words, d'' is just a conditional prediction of Isa about herself, not something that she initially deems optimal conditional on c'. ▲

Finiteness and perfect recall imply that a one-step optimal (pure) strategy can always be found by folding back:

Remark 8. For every belief system μ^i, at least one strategy is one-step optimal under μ^i.

When the chain rule holds, conditional preferences are dynamically consistent and—by a relatively standard dynamic programming argument—one can show that a strategy is sequentially optimal if and only if can be obtained by folding-back planning. This implies that sequential and one-step optimality are equivalent, a result known as the *one-shot-deviation principle*. Notably, Perea (2002) proves that the one-shot-deviation principle holds also for the weakest form of chain rule considered here, forward consistency.[12]

Theorem 3. *(Perea 2002) For every forward consistent belief system μ^i, a strategy is sequentially optimal under μ^i if and only if it is one-step optimal under μ^i.*

Clearly, since $\Delta_{\mathcal{C}}^{H_i}(S_{-i}) \subseteq \Delta^{H_i}(S_{-i}) \subseteq \Delta_{\mathcal{F}}^{H_i}(S_{-i})$, the one-shot deviation principle also holds for standard and completely consistent belief systems. Intuitively, the reason why the equivalence between sequential and one-step optimality holds despite possible violations of strong versions of the chain rule is the following. Under a forward consistent belief system, player i may hold conflicting beliefs at information sets h_i' and h_i'' only if they follow (a) different actions of her and (b) unexpected actions of the co-players, such as $h_i' = \{(L, C)\}$ and $h_i'' = \{(R, C)\}$ for Isa in game Γ', if she initially assigns probability 0 to action C of Joe. In this case, the decision problems of i at h_i' and h_i'' are mutually independent; furthermore,

[12]Perea shows the result for two-person games, but the extension to n-person games is straightfoward. The proof is available upon request.

from the perspective of earlier information sets (the root for Isa in Γ'), planning at h_i' and h_i'' does not affect expected payoffs, because i deems both h_i' and h_i'' unreachable.

Theorem 3 and Remark 8 imply the following:

Corollary 2. *For every forward consistent belief system μ^i, at least one strategy is sequentially optimal under μ^i, that is, $BR_i^*(\mu^i) \neq \emptyset$.*

We now turn to a seemingly less demanding notion of rationality.

Definition 10. A strategy $\bar{s}_i$ is **weakly sequentially optimal** under belief system μ^i, written $\bar{s}_i \in BR_i(\mu^i)$, if

$$\forall h_i \in H_i(\bar{s}_i), \forall s_i \in S_i(h_i), \quad U_i(\bar{s}_i, \mu^i(\cdot|h_i)) \geq U_i(s_i, \mu^i(\cdot|h_i)).$$

Like sequential optimality, also weak sequential optimality may be unsatisfiable if μ^i is not forward consistent. Focusing on forward consistent belief systems, weak sequential optimality can be given the following *"forward-planning"* interpretation. Suppose player i forms her beliefs according to $\mu^i \in \Delta_{\mathcal{F}}^{H_i}(S_{-i})$ as the play unfolds, although she may not know how she would revise her beliefs upon observing unexpected information, i.e., she may not be fully introspective. At the root, she plans to follow a (possibly partial) strategy that maximizes her expected payoff restricted to the collection of information sets $H_i(\mu^i|\varnothing)$ she deems possible under $\mu^i(\cdot|\varnothing)$. Let $\bar{s}_{i,\varnothing}$ be such a strategy. One way to compute $\bar{s}_{i,\varnothing}$ is to perform a folding-back planning algorithm on the restricted collection $H_i(\mu^i|\varnothing)$, that is, starting from information sets that are terminal within $H_i(\mu^i|\varnothing)$, and then focusing on behavior in the subcollection of information sets in $H_i(\mu^i|\varnothing)$ that are possible if such partial plan is implemented; indeed, by Remark 7, the specification of $\bar{s}_{i,\varnothing}$ at information sets in $H_i(\mu^i|\varnothing)$ that cannot occur under $\bar{s}_{i,\varnothing}$ is immaterial for the maximization problem. Since conditional beliefs within $H_i(\mu^i|\varnothing)$ are obtained by updating, player i would have no incentive to deviate from $\bar{s}_{i,\varnothing}$ at any $h_i \in H_i(\mu^i|\varnothing)$.

Example 5. Consider again game Γ''' and let μ^i be a forward consistent belief system of Isa such that $\mu^i(\ell|\varnothing) < \frac{1}{2}$, then it is optimal to go down immediately, $\bar{s}_{a,\varnothing}(\varnothing) = d'$, (c') is inconsistent with $\bar{s}_{a,\varnothing}$, and the specification of $\bar{s}_{i,\varnothing}$ at (c') does not matter. ▲

If, implementing $\bar{s}_{i,\varnothing}$, player i unexpectedly obtains information h_i, where h_i is a first follower in $H_i \backslash H_i(\mu^i|\varnothing)$ of the information sets in $H_i(\mu^i|\varnothing)$, then i comes up with a new belief $\mu^i(\cdot|h_i) \in \Delta(S_{-i}(h_i))$ and plans to follow a (possibly partial) continuation strategy $\bar{s}_{i,h_i}$ within the collection $H_i(\mu^i|h_i)$ of information sets weakly following h_i that she deems possible under $\mu^i(\cdot|h_i)$. If an hypothetical external observer knew μ^i (and how i breaks ties), she could determine $\bar{s}_i$ for all information sets consistent with $\bar{s}_i$, that is, those in $H_i(\bar{s}_i)$, and $\bar{s}_i$ would be weakly sequentially optimal under μ^i.[13]

The following result shows that the two notions of planning – weak sequential optimality and sequential optimality – are behaviorally equivalent.

Lemma 2. *A strategy is weakly sequentially optimal under a forward consistent belief system μ^i if and only if it is behaviorally equivalent to some strategy that is sequentially optimal under μ^i.*

Proof: Suppose that $\bar{s}_i$ is weakly sequentially optimal under $\mu^i \in \Delta^{H_i}_{\mathcal{F}}(S_{-i})$. By Corollary 2, there is some strategy $\hat{s}_i$ that is sequentially optimal under μ^i. Let $\bar{s}'_i$ denote the strategy coincides with $\bar{s}_i$ on $H_i(\bar{s}_i)$ and with $\hat{s}_i$ on $H_i \backslash H_i(\bar{s}_i)$. By construction, $\bar{s}'_i$ is behaviorally equivalent to $\bar{s}_i$. We must show that $\bar{s}'_i$ is sequentially optimal under μ^i.

Since $\bar{s}_i$ is weakly sequentially optimal under μ^i, Remark 7 implies that

$$\forall h_i \in H_i(\bar{s}_i), \forall s_i \in S_i(h_i), \quad U_i(\bar{s}'_i, \mu^i(\cdot|h_i)) = U_i(\bar{s}_i, \mu^i(\cdot|h_i)) \geq U_i(s_i, \mu^i(\cdot|h_i)).$$

[13] The strategies obtained by "folding-back" on the collections $H_i\left(\mu^i|h_i\right)$ of information sets deemed possible by beliefs $\mu^i(\cdot|h_i)$ ($h_i \in H_i$) form a subset of $BR_i\left(\mu^i\right)$, those weakly sequentially optimal under μ^i (and a superset of $BR^*_i\left(\mu^i\right)$). But, for each one of them, there is a structurally equivalent strategy in $BR_i\left(\mu^i\right)$. We omit the details.

Since $\hat{s}_i$ is sequentially optimal under μ^i and $\bar{s}_i'$ coincides with $\hat{s}_i$ on $H_i \backslash H_i(\bar{s}_i)$,

$$\forall h_i \in H_i \backslash H_i(\bar{s}_i), \forall s_i \in S_i(h_i), \quad U_i(\rho_i[\bar{s}_i'/h_i], \mu^i(\cdot|h_i)) = U_i(\rho_i[\hat{s}_i/h_i], \mu^i(\cdot|h_i)) \geq U_i(s_i, \mu^i(\cdot|h_i)).$$

It follows that $\bar{s}_i'$ is sequentially optimal under μ^i. ■

Clearly, the equivalence result also holds for standard and completely consistent belief systems.

Example 6. Strategy $R.a'.a''$ of Isa in game Γ' of Figure 1 is not sequentially optimal under any standard belief system, because there is no common value $\mu^i(C.\ell|(L,C)) = \mu^i(C.\ell|(R,C))$ that makes a' a best reply given (L,C) (which requires $\mu^i(C.\ell|(L,C)) \leq \frac{2}{5}$), and a'' a best reply given (R,C) (which requires $\mu^i(C.\ell|(R,C)) \geq \frac{1}{2}$). Yet, there is a continuum of standard belief systems that make the behaviorally equivalent strategy $R.d'.a''$ sequentially optimal, including all the $\mu^i \in \Delta^{H_i}(S_{-i})$ with $\mu^i(\{Q.\ell, Q.r\}) = 1$ and $\mu^i(C.\ell|(L,C)) = \mu^i(C.\ell|(R,C)) \geq \frac{1}{2}$ (there are more if $\varepsilon > 0$). ▲

To summarize, the analysis and discussion of this section clarifies two points:

1. In order to rationally decide what to do at an information set h_i, player i only has to know her current belief at h_i and what she whould believe at followers she deems possible given h_i, that is, information sets $h_i' \in H_i(\mu^i|h_i)$. The resulting plan is dynamically consistent (optimal at each $h_i' \in H_i(\mu^i|h_i)$, or at least the elements of $H_i(\mu^i|h_i)$ possible under the plan itself) as long as the forward chain rule holds. Therefore, *partial introspection and the forward chain rule embedded in forward consistent belief systems are sufficient for rational planning.*

2. *Forward planning* (iterated whenever a player is surprised, as explained above) *determines reduced rather than full strategies*, as anticipated in Section 2.

7 Belief change and strategic reasoning: equivalence results

In this section, we argue that the behavioral implications of rationality and different versions of "common belief in rationality" do not depend on coherency restrictions of conditional beliefs across information sets beyond those implied by forward consistency. The literature on the epistemic foundations of solution concepts shows that such behavioral implications are characterized by extensions of the rationalizability idea from simultaneous-move to sequential games.

7.1 Strong rationalizability

The idea of common belief in rationality has been extended from simultaneous-move games to sequential games in different ways, depending on how players are assumed to revise their beliefs after unexpected moves. In this section, we analyze Strong Rationalizability,[14] which is based on the idea that players "strongly believe" in the rationality and strategic sophistication of co-players. A player strongly believes an event if she would be always certain of that event unless observing evidence in direct contradiction with it. Strong Rationalizability captures the implications of rationality and common strong belief in rationality, a formalization of the *"best rationalization principle"*: players always ascribe to their co-players the highest degree of strategic sophistication consistent with their past behavior, even when surprised by their behavior. This form of reasoning is also often referred to as "forward-induction." See Battigalli (1996) and Battigalli & Siniscalchi (2002).

A belief system μ^i **strongly believes** event $E_{-i} \subseteq S_{-i}$ if $\mu^i(E_{-i}|h_i) = 1$ for every $h_i \in H_i$ such that $E_{-i} \cap S_{-i}(h_i) \neq \emptyset$. Similarly, a complete CPS $\mu^i \in \Delta^*(S_{-i})$ strongly believes E_{-i} if $\mu^i(E_{-i}|C_{-i}) = 1$ for every conditioning event $C_{-i} \subseteq S_{-i}$ such that $C_{-i} \cap E_{-i} \neq \emptyset$. We let

[14]Strong Rationalizability used to be called "extensive-form rationalizability" (Pearce 1984, Battigalli 1997). We avoid this terminology because there are different ways to define "rationalizability" for the extensive-form analysis of sequential games; see, e.g., *Initial Rationalizability*, which captures rationality and common *initial* belief in rationality (Ben-Porath 1997, Battigalli & Siniscalchi 1999) and *Backwards Rationalizability*, which we analyze in the next section.

$\mathrm{SB}_i(E_{-i})$ denote the set of either belief systems or complete CPSs (which one will be clear from the context) that strongly believe E_{-i}.

To define Strong Rationalizability, it is convenient to use weak sequential optimality as our notion of best reply.[15] When we consider a complete CPS $\bar{\mu}^i \in \Delta^*(S_{-i})$, with a slight abuse of notation, we let $BR_i(\bar{\mu}^i)$ denote the set of strategies of i that are weakly sequentially optimal under the (completely consistent) belief system μ^i derived from $\bar{\mu}^i$ (that is, with $\mu^i(\cdot|h_i) = \bar{\mu}^i(\cdot|S_{-i}(h_i))$ for all $h_i \in H_i$). Depending on which kind of beliefs we consider for the definition, we let Δ_i denote either forward consistent, or standard, or completely consistent belief systems, or complete CPSs, which yields four distinct definitions of strong rationalizability.[16]

Definition 11. Fix $\Delta = (\Delta_i)_{i\in I} \in \left\{ \left(\Delta_{\mathcal{F}}^{H_i}(S_{-i})\right)_{i\in I}, \left(\Delta^{H_i}(S_{-i})\right)_{i\in I}, \left(\Delta_{\mathcal{C}}^{H_i}(S_{-i})\right)_{i\in I}, \left(\Delta^*(S_{-i})\right)_{i\in I} \right\}$. Let $(S_i^{\Delta,0})_{i\in I} = (S_i)_{i\in I}$ and define inductively for each $m \in \mathbb{N}_0$ and i:

$$S_i^{\Delta,m+1} = \{s_i \in S_i : \exists \mu^i \in \Delta_i \cap (\cap_{k=0}^m \mathrm{SB}_i(S_{-i}^{\Delta,k})), s_i \in BR_i(\mu^i)\}.$$

The set of **strongly rationalizable** strategies for i (given Δ) is $S_i^{\Delta,\infty} = \cap_{k=0}^\infty S_i^{\Delta,k}$.

The sequence $(S_i^{\Delta,m})_{m=0}^\infty$ is by definition a weakly decreasing sequence of events. Notice that at step $m+1$, we require strong belief in each $S_{-i}^{\Delta,k}$ with $k \le m$, not just $S_{-i}^{\Delta,m}$. The reason for this demanding requirement is that the strong belief operator is not monotone: suppose that $E_{-i} \subset F_{-i}$; although $\mu^i(E_{-i}|h_i) = 1$ implies $\mu^i(F_{-i}|h_i) = 1$ for any given h_i, strong belief in E_{-i} does not imply strong belief in F_{-i}, because E_{-i} might be consistent with a strictly smaller collection of information sets than F_{-i}; in this case, strong belief in F_{-i} implies

[15] Battigalli & De Vito (2021) define strong rationalizability and provide epistemic justifications using the notion of sequential best reply to a (standard) belief system. They discuss the equivalence between the two approaches.

[16] One may wonder why we consider separate definitions for completely consistent belief systems and complete CPSs. Indeed, (i) completely consistent belief systems are the "projection" of complete CPSs, and (ii) for any complete CPS $\bar{\mu}^i \in \Delta^*(S_{-i})$, the set of (weakly) sequentially optimal strategies under $\bar{\mu}^i$ depends only on the conditional beliefs at information sets $\left(\bar{\mu}^i(\cdot|S_{-i}(h_i))\right)_{h_i\in H_i}$. Yet, the strong-belief requirement is more demanding for complete CPSs than for belief systems, which might, in principle, cause a difference between the two solution concepts.

restrictions on conditional beliefs not implied by strong belief in E_{-i}. The best rationalization principle is captured by strong belief in the whole sequence $(S_{-i}^{\Delta,m})_{m=0}^{\infty}$: once a degree m of strategic rationality is contradicted at information set h_i (i.e., $S_{-i}(h_i) \cap S_{-i}^{\Delta,m} = \emptyset$), player i interprets her information on co-players' past behavior by assuming they are implementing plans of the highest degree of strategic rationality compatible with such information, i.e., she infers that $s_{-i} \in S_{-i}^{\Delta,k^*}$ for the highest k^* such that $S_{-i}(h_i) \cap S_{-i}^{\Delta,k^*} \neq \emptyset$. This form of strategic sophistication shapes how players revise their beliefs upon reaching unexpected histories.

The main result of this section is that these definitions obtained with different notions of belief change are all equivalent.

Theorem 4. *The strong rationalizability procedures of Definition 11 coincide.*

Before turning to the proof, we introduce some preliminary results.

Lemma 3. *Fix a strategy s_i, a subset $E_{-i} \subseteq S_{-i}$, and a forward consistent belief system μ^i that strongly believes E_{-i}. Then there exists a complete CPS $\bar{\mu}^i$ such that (i) $\mu^i(\cdot|h_i) = \bar{\mu}^i(\cdot|S_{-i}(h_i))$ for all $h_i \in H_i(s_i)$ and (ii) $\bar{\mu}^i$ strongly believes E_{-i}.*

Proof: See the Appendix.

This first lemma establishes that we can map a forward consistent belief system that strongly believes in an event into a complete CPS that (i) coincides with the original belief system at all information sets consistent with a strategy s_i and (ii) strongly believes in the event. Property (i) implies that weak sequential optimality of s_i under μ^i is preserved by the transformation.

Lemma 3, however, can only be used for strong belief in one event, while the strong rationalizability procedures require strong belief in a sequence of events. Yet, the strong rationalizability procedures with complete CPSs and FBSs can be characterized by procedures that require strong belief only in one event at each step:[17]

[17]As a corollary of Theorem 4, we show in the Appendix that all our procedures are characterized by the one-memory procedure.

Definition 12. Given $\Delta = (\Delta_i)_{i \in I}$ as in Definition 11, let $(\bar{S}_i^{\Delta,0})_{i \in I} = (S_i)_{i \in I}$ and define inductively for each $m \in \mathbb{N}_0$ and i:

$$\bar{S}_i^{\Delta,m+1} = \{s_i \in \bar{S}_i^{\Delta,m} : \exists \mu^i \in \Delta_i \cap \mathrm{SB}_i(\bar{S}_{-i}^{\Delta,m}), s_i \in BR_i(\mu^i)\}.$$

Let $\bar{S}_i^{\Delta,\infty} = \cap_{k=0}^{\infty} \bar{S}_i^{\Delta,k}$.

Lemma 4. *Let* $\Delta = \left(\Delta_{\mathcal{F}}^{H_i}(S_{-i})\right)_{i \in I}$ *or* $\Delta = (\Delta^*(S_{-i}))_{i \in I}$. *Then* $S_i^{\Delta,m} = \bar{S}_i^{\Delta,m}$ *for every* i *and* $m \in \mathbb{N}$.

Proof: See the Appendix.

This result allows us to exploit Lemma 3 for the proof of Theorem 4. To see why Lemma 4 is true, note that a key difference between the definitions of $S_i^{\Delta,m+1}$ and $\bar{S}_i^{\Delta,m+1}$ is that at step $m+1$, the latter requires $s_i \in \bar{S}_i^{\Delta,m}$. But then, recursively, $s_i \in \bar{S}_i^{\Delta,k}$ for all $k \leq m$, and thus for each $k \leq m$ we can find a belief system $\mu^{i,k}$ that strongly believes $\bar{S}_{-i}^{\Delta,k}$ and justifies s_i as a weak sequential best reply. Then, we show that we can compose these belief systems to construct one that strongly believes the whole sequence $(\bar{S}_{-i}^{\Delta,k})_{k=1}^{m}$ and justifies s_i as a weak sequential best reply. We are now in a position to prove the theorem.

Proof of Theorem 4: We begin to show that the procedures obtained with complete CPSs and forward consistent belief systems (FBSs), denoted respectively $(S^{\mathcal{C},k})_{k=0}^{\infty}$ and $(S^{\mathcal{F},k})_{k=0}^{\infty}$, coincide at each step. Then we conclude by showing that for any notion of belief system in between forward consistency and complete consistency, the induced procedure is equal to the one of complete CPSs and FBSs.

Claim 1: $S^{\mathcal{C},m} = S^{\mathcal{F},m}$ *for all* $m \in \mathbb{N}_0$.

Proof: The basis step for $m = 0$ holds by definition: $S_i^{\mathcal{C},0} = S_i = S_i^{\mathcal{F},0}$ for every i. Suppose by way of induction that for some $m \geq 0$, $S_i^{\mathcal{C},m} = S_i^{\mathcal{F},m}$ for every i. We show that $S_i^{\mathcal{C},m+1} = S_i^{\mathcal{F},m+1}$, or equivalently, by Lemma 4, that $\bar{S}_i^{\mathcal{C},m+1} = \bar{S}_i^{\mathcal{F},m+1}$. Note that by the inductive hypothesis and Lemma 4, $\bar{S}_i^{\mathcal{C},m} = \bar{S}_i^{\mathcal{F},m}$ for every i. Fix $i \in I$. We show both inclusions.

Take any $s_i \in \bar{S}_i^{\mathcal{C},m+1}$. Then $s_i \in \bar{S}_i^{\mathcal{C},m}$, and thus $s_i \in \bar{S}_i^{\mathcal{F},m}$ since $\bar{S}_i^{\mathcal{F},m} = \bar{S}_i^{\mathcal{C},m}$. Moreover, there exists a complete CPS μ^i that strongly believes $\bar{S}_{-i}^{\mathcal{C},m}$, hence $\bar{S}_{-i}^{\mathcal{F},m}$, such that $s_i \in BR_i(\mu^i)$. The belief system $\bar{\mu}^i$ induced by μ^i strongly believes $\bar{S}_{-i}^{\mathcal{F},m}$, is forward consistent, and we have $s_i \in BR_i(\bar{\mu}^i)$. This shows that $s_i \in \bar{S}_i^{\mathcal{F},m+1}$.

Conversely, take any $s_i \in \bar{S}_i^{\mathcal{F},m+1}$. Then $s_i \in \bar{S}_i^{\mathcal{F},m}$, hence $s_i \in \bar{S}_i^{\mathcal{C},m}$ since $\bar{S}_i^{\mathcal{F},m} = \bar{S}_i^{\mathcal{C},m}$. Moreover, there exists an FBS μ^i that strongly believes $\bar{S}_{-i}^{\mathcal{F},m}$, hence strongly believes $\bar{S}_{-i}^{\mathcal{C},m}$, such that $s_i \in BR_i(\mu^i)$. It follows by Lemma 3 that there exists a complete CPS $\bar{\mu}^i$ that strongly believes $\bar{S}_{-i}^{\mathcal{C},m}$ such that $\bar{\mu}^i(\cdot|S_{-i}(h_i)) = \mu^i(\cdot|h_i)$ for every $h_i \in H_i(s_i)$, which implies, since $s_i \in BR_i(\mu^i)$, that $s_i \in BR_i(\bar{\mu}^i)$. Therefore $s_i \in \bar{S}_i^{\mathcal{C},m+1}$. □

Claim 2: *Let $\Delta = (\Delta_i)_{i\in I}$ be such that for every $i \in I$, $\Delta_{\mathcal{C}}^{H_i}(S_{-i}) \subseteq \Delta_i \subseteq \Delta_{\mathcal{F}}^{H_i}(S_{-i})$. Then $S^{\Delta,m} = S^{\mathcal{C},m}$ for every m.*

Proof: The proof is by induction on $m \in \mathbb{N}_0$. The basis step is immediate: $S_i^{\mathcal{C},0} = S_i^{\Delta,0} = S_i^{\mathcal{F},0}$ for every i. Suppose by way of induction that for some $m \in \mathbb{N}_0$, we have for every $k \leq m$ and $i \in I$ that $S_i^{\mathcal{C},k} = S_i^{\Delta,k} = S_i^{\mathcal{F},k}$. Fix $i \in I$. By inspection of the definitions and the inductive hypothesis, because $\Delta_i \subseteq \Delta_{\mathcal{F}}^{H_i}(S_{-i})$ it is immediate that $S_i^{\Delta,m+1} \subseteq S_i^{\mathcal{F},m+1}$. Moreover, it is easy to see that $S_i^{\mathcal{C},m+1} \subseteq S_i^{\Delta,m+1}$. Suppose $s_i \in S_i^{\mathcal{C},m+1}$. Then there is a complete CPS μ^i that strongly believes $(S_{-i}^{\mathcal{C},k})_{k=0}^m = (S_{-i}^{\Delta,k})_{k=0}^m$ such that $s_i \in BR_i(\mu^i)$. But then the belief system $\bar{\mu}^i$ induced by μ^i is completely consistent (hence $\mu^i \in \Delta_i$ since $\Delta_{\mathcal{C}}^{H_i}(S_{-i}) \subseteq \Delta_i$), strongly believes $(S_{-i}^{\Delta,k})_{k=0}^m$, and $s_i \in BR_i(\bar{\mu}^i)$. Hence $s_i \in S_i^{\Delta,m+1}$. To conclude, we have that $S_i^{\mathcal{C},m+1} \subseteq S_i^{\Delta,m+1} \subseteq S_i^{\mathcal{F},m+1}$. Since by Claim 1 we have $S_i^{\mathcal{F},m+1} = S_i^{\mathcal{C},m+1}$, it follows that $S_i^{\mathcal{C},m+1} = S_i^{\Delta,m+1} = S_i^{\mathcal{F},m+1}$. □

Since Δ defined as standard belief systems or completely consistent belief systems satisfy the assumption of Claim 2, the result follows. ■

Theorem 4 implies, in particular, that the set of strategies that can be justified as weak sequential best replies to at least one belief system is invariant to consistency restrictions beyond the forward chain rule.

7.2 Backwards rationalizability

Backwards Rationalizability characterizes the behavioral implications of rationality and common belief in continuation rationality starting from every information set (cf. Perea 2014, Battigalli & De Vito 2021). In generic games with perfect information, backwards rationalizability coincides with subgame perfect equilibrium and can therefore be computed by backward induction. Catonini & Penta (2021) show that in more general games backwards rationalizability can be computed by means of a convenient Backwards Procedure.[18] A result by Chen & Micali (2013) on the order-independence of iterated dominance procedures in sequential games implies that the set of strongly rationalizable paths is (weakly) contained in the set of backwards rationalizable path. Therefore, backwards rationalizability (possibly computed with the backward procedure) can help find the strongly rationalizable paths.

In this section, we focus on games with **observable actions** and consider the definition of backwards rationalizability given by Battigalli & De Vito (2021),[19] which is behaviorally equivalent to the one of Perea (2014) for this class of games, as it yields the same *reduced* strategies.[20] To justify this solution concept, Battigalli & De Vito (2021) start from the following epistemic assumptions: Each player is fully introspective about her belief system and plans rationally by folding back, thus obtaining a strategy that is sequentially optimal under her belief system. The player is rational if her behavior complies with her strategy. Players initially believe that co-players are rational. The key assumption is that when a player observes unexpected behavior she keeps believing that the co-players planned rationally and that, even if in the past they deviated from their plans, they will nonetheless comply with their

[18] In a related vein, Perea (2014) introduces the Backward Dominance procedure, a generalization of the backward induction algorithm which is more permissive than Backwards Rationalizability, also in terms of induced paths.

[19] Catonini & Penta (2021) define Backwards Rationalizability for games with incomplete information. Their definition coincides with the definition of Battigalli & De Vito (2021) in games with complete information.

[20] Perea (2014) only considers reduced strategies. Battigalli & De Vito (2021) consider full strategies.

plans in the subgame (according to strong rationalizability, instead, surprised players change their beliefs about the plans that their supposedly rational co-players are implementing). The assumptions concerning higher levels of mutual belief in rationality have a similar flavor.

With this, we want to express the idea that a player in a subgame believes that her co-players are going behave according to plans having some properties, even if such properties were violated in the past. For example, a nonterminal history $h \in H$ may be unreachable if the co-players' are rational, and yet player i may believe that in the subgame with root h the co-players will implement *continuation strategies* consistent with their rationality *in the subgame*. With this in mind, given a subset of j's strategies $E_j \subseteq S_j$ and a history $h \in H$, let

$$\chi_j^h(E_j) = \left\{s_j \in S_j\left(h\right) : \exists s_j' \in E_j, \forall \bar{h} \in H, h \preceq \bar{h} \Rightarrow s_j'\left(\bar{h}\right) = s_j\left(\bar{h}\right)\right\}$$

denote the set of strategies s_j consistent with h that behave like some strategy in E_j in the subgame with root h. As we consider all the co-players, for each profile of strategy subsets $(E_j)_{j \neq i}$, we obtain $\chi_{-i}^h(E_{-i}) = \times_{j \neq i}\chi_j^h(E_j)$, where $E_{-i} = \times_{j \neq i}E_j$. With this, we define the set of belief systems μ^i that "persistently believe" in the projection of $E_{-i} = \times_{j \neq i}E_j$ in each subgame:

$$\mathrm{PB}_i\left(E_{-i}\right) = \{\mu^i \in \times_{h \in H}\Delta(S_{-i}\left(h\right)) : \forall h \in H, \mu^i(\chi_{-i}^h(E_{-i})|h) = 1\}.$$

Note that, unlike the strong belief operator $\mathrm{SB}_i\left(\cdot\right)$, operator $\mathrm{PB}_i\left(\cdot\right)$ is monotone. Depending on which kind of belief systems we use for the definition of backwards rationalizability, we let Δ_i denote the set of either forward consistent, or standard, or completely consistent belief systems. The main result of this section is that the notions of backwards rationalizability obtained by considering these different kinds of belief system are behaviorally equivalent.

Recall that $BR_i^*(\mu^i)$ is the set of strategies of i that are sequentially optimal under belief system μ^i; given the observability of actions, it is the set of strategies whose continuations maximize i's expected payoff in each subgame with root $h \in H$ given belief $\mu^i\left(\cdot|h\right)$.

Definition 13. Let $\Delta_i = \Delta_{\mathcal{F}}^{H_i}(S_{-i})$, or $\Delta_i = \Delta^{H_i}(S_{-i})$, or $\Delta_i = \Delta_{\mathcal{C}}^{H_i}(S_{-i})$ for each i, and let $\Delta = (\Delta_i)_{i\in I}$. With this, let $\left(\hat{S}_i^{\Delta,0}\right)_{i\in I} = (S_i)_{i\in I}$; suppose $\left(\hat{S}_i^{\Delta,m}\right)_{i\in I}$ has been defined; then, for each i, let

$$\hat{S}_i^{\Delta,m+1} = \{s_i \in S_i : \exists \mu^i \in \Delta_i \cap \mathrm{PB}_i(\hat{S}_{-i}^{\Delta,m}), s_i \in BR_i^*(\mu^i)\}.$$

Strategies in $\hat{S}_i^{\Delta,\infty} = \cap_{k=0}^{\infty}\hat{S}_i^{\Delta,k}$ are **backwards rationalizable** for player i.

Theorem 5. *The backwards rationalizability procedures of Definition 13 coincide.*

Proof: See the Appendix. ■

7.3 Beliefs systems defined on active information sets

In the foundational works on the theory of games of von Neumann & Morgenstern (1944) and Kuhn (1953), and in most of the following game theoretic work, the information of players is specified only at information sets where they are active. Indeed, it was either argued or taken as self-evident that the information of inactive players is irrelevant for the strategic analysis of games. The results of this section confirm to some extent this intuition.

Consider systems of beliefs where players form their beliefs only at the information sets where they are active, i.e., with domain $\hat{H}_i = \{h_i \in H_i : |\mathcal{A}_i(h_i)| \geq 2\} \cup \{\varnothing\}$. (We assume $\varnothing \in \hat{H}_i$ even if i is not active at the initial history.)

A unifying definition of weak sequential optimality for both $\hat{H}_i$-based and H_i-based belief systems consists in requiring maximization of conditional expected utility only at active information sets. Next we show that strong rationalizability coincides with both models of beliefs defined only on active information sets and those defined on all information sets.

We show that we can always extend a forward consistent belief system on active information sets to a forward consistent belief system on all information sets whose restriction to active information sets is equal to the original belief system. Then it is immediate that the strong rationalizability procedures with both kind of belief systems are equal.

Lemma 5. *Fix a set E_{-i} and an $\hat{H}_i$-based FBS μ^i that strongly believes E_{-i}. There exists an H_i-based FBS $\bar{\mu}^i$ that strongly believes E_{-i} such that $\mu^i(\cdot|h_i) = \bar{\mu}^i(\cdot|h_i)$ for all $h_i \in \hat{H}_i$.*

Proof: Fix an $\hat{H}_i$-based FBS $\bar{\mu}$ that strongly believes E_{-i} and an H_i-based FBS $\widetilde{\mu}$ that strongly believes E_{-i}.

For each $h_i \in \hat{H}_i$, denote by $\mathscr{H}_+(h_i)$ the set of all information sets h_i' that weakly follow h_i such that $\mu(S_{-i}(h_i')|h_i) > 0$ and by $\mathscr{H}_0(h_i)$ the set of all h_i' that weakly follow h_i such that $\bar{\mu}(S_{-i}(h_i')|h_i) = 0$. Furthermore, let $\mathscr{H}_0^*(h_i)$ be the collection of *earliest* h_i^* in $\mathscr{H}_0(h_i)$. For every $h_i \in H_i$, let $H_i(h_i)$ be the information sets that weakly follow h_i, and $\hat{H}_i^*(h_i)$ the *earliest* active information sets that follow h_i, and by $[h_i, \hat{H}_i^*(h_i))$ the information sets that weakly follow h_i and precede some element of $\hat{H}_i^*(h_i)$. Also, let $\hat{H}_i^*(h_i, E_{-i})$ be the information sets in $\hat{H}_i^*(h_i)$ consistent with E_{-i}. Denote by $H_i^-(h_i, E_{-i})$ the *earliest* information sets in $[h_i, \hat{H}_i^*(h_i))$ that are not in $[h_i, \hat{H}_i^*(h_i, E_{-i}))$

Now we construct a partition $\{\mathscr{H}_1, \mathscr{H}_2, \ldots\} \cup \{\widetilde{\mathscr{H}}\}$ of H_i as follows: $\mathscr{H}_1^* = \{\varnothing\}$ and $\mathscr{H}_1 = \mathscr{H}_+(\varnothing)$, and for $m \in \mathbb{N}$, let

$$\mathscr{H}_{m+1}^* = \cup_{h_i \in \hat{\mathscr{H}}_m} \mathscr{H}_0^*(h_i)$$

and from this, let $\mathscr{H}_{m+1}^- = \cup_{h_i \in \mathscr{H}_{m+1}^*} [h_i, \hat{H}_i^*(h_i))$ and

$$\mathscr{H}_{m+1}^+ = \cup_{h_i^* \in \hat{H}_i^*(\mathscr{H}_{m+1}^*)} \mathscr{H}_+(h_i^*).$$

and we set $\mathscr{H}_{m+1} = \mathscr{H}_{m+1}^- \cup \mathscr{H}_{m+1}^+$.

Let $\mathscr{H}^* = \cup_{m \in \mathbb{N}} \mathscr{H}_m^*$ and $\mathscr{H} = \cup_{m \in \mathbb{N}} \mathscr{H}_m$. Finally, let $\widetilde{\mathscr{H}} = H_i \setminus \mathscr{H}$. Naturally $H_i = \mathscr{H} \cup \widetilde{\mathscr{H}}$ and the sets in $\{\mathscr{H}_1, \mathscr{H}_2, \ldots\} \cup \widetilde{\mathscr{H}}$ are pairwise disjoint, so it is indeed a partition of H_i. Note that $\widetilde{\mathscr{H}} \cap \hat{H}_i = \emptyset$.

We use the arrays of measures $\bar{\mu}$ and $\widetilde{\mu}$ to construct an H_i-based FBS μ that strongly believes E_{-i} such that $\mu(\cdot|h_i) = \bar{\mu}(\cdot|h_i)$ for all $h_i \in \hat{H}_i$.

Define μ as follows: For each m and $h_i^* \in \mathscr{H}_m^*$, let $\mu(\cdot|h_i^*)$ be the equiprobable convex combination of the measures $(\mu(\cdot|h_i))_{h_i \in \hat{H}_i^*(h_i^*, \overline{S}_{-i})}$ (if any), and for all $h_i \in H_i(h_i^*)$ such that $\mu(S_{-i}(h_i)|h_i^*) > 0$, derive $\mu(\cdot|h_i)$ by conditioning from $\mu(\cdot|h_i^*)$. This derivation preserves strong belief in E_{-i} and ensures that $\mu(\cdot|h_i) = \bar{\mu}(\cdot|h_i)$ for every $h_i \in \hat{H}_i^*(h_i^*, E_{-i})$. Derive all measures at the remaining information sets of $\mathscr{H}_m$ with the same operation as we did for h_i^* starting from each element of $\bar{h}_i \in H_i^-(h_i^*, E_{-i})$. We covered all $\mathscr{H}_m$ in a way that satisfies the forward chain rule, and we have by construction $\mu(\cdot|h_i) = \bar{\mu}(\cdot|h_i)$ for every $h_i \in \mathscr{H}_m \cap \hat{H}_i$. Finally, set $\mu(\cdot|h_i) = \tilde{\mu}(\cdot|h_i)$ for every $h_i \in \widetilde{\mathscr{H}}$, which preserves strong belief of μ in E_{-i}. It is easy to see that our μ satisfies all the desired properties. ■

Note, $\hat{H}_i$-based standard belief systems, whose set is denoted by $\Delta^{\hat{H}_i}(S_{-i})$, is obtained from CPSs on $(S_{-i}, \hat{\mathcal{H}}_i)$, where $\hat{\mathcal{H}}_i = \{S_{-i}(h_i) \subseteq S_{-i} : h_i \in \hat{H}_i\}$. Extendibility for such belief systems is not always possible because when $\hat{\mathcal{H}}_i \subset \mathcal{H}_i$, these belief systems may embody less coherency restrictions across information sets not ordered by precedence than the standard ones. Yet, our results show that this is irrelevant for the predictions. Indeed, notice that:

Lemma 6. *Fix an $\hat{H}_i$-based FBS μ^i that strongly believes E_{-i} and a strategy $s_i \in BR_i(\mu^i)$. Then there is a complete CPS $\bar{\mu}^i$ such that $\bar{\mu}^i$ strongly believes E_{-i} and $s_i \in BR_i(\bar{\mu}^i)$.*

Proof: Fix a $\hat{H}_i$-based FBS μ that strongly believes E_{-i} and $s_i \in BR_i(\mu)$. By Lemma 5 there is an H_i-based FBS $\tilde{\mu}$ that strongly believes E_{-i} with $\tilde{\mu}(\cdot|h_i) = \mu(\cdot|h_i)$ for all $h_i \in \hat{H}_i$, so $s_i \in BR_i(\tilde{\mu})$. Lemma 3 in turn implies that there is a complete CPS $\bar{\mu}$ that strongly believes E_{-i} with $\bar{\mu}(\cdot|S_{-i}(h_i)) = \tilde{\mu}(\cdot|h_i)$ for all $h_i \in H_i(s_i)$, so that $s_i \in BR_i(\mu)$. ■

Since $\Delta_{\mathcal{C}}^{\hat{H}_i}(S_{-i}) \subseteq \Delta^{\hat{H}_i}(S_{-i}) \subseteq \Delta_{\mathcal{F}}^{\hat{H}_i}(S_{-i})$, the equivalence between strong rationalizability with $\hat{H}_i$-based belief systems and with complete CPSs is shown with the same sandwitch argument used in the proof of Theorem 4, which by Theorem 4 further implies equivalence with the other procedures.

Theorem 6. *The strong rationalizability procedures with our notions of belief systems defined on all or only active information sets coincide.*

8 Relation to previous work

Most of the literature on rationalizability in sequential games defined the algorithms using forward consistent belief systems, such as Battigalli & Siniscalchi (2003) on strong Δ-rationalizability, or Perea (2014) on backwards rationalizability, among others. A large body of the literature on strong rationalizability, such as Pearce (1984), Battigalli (1997), Shimoji & Watson (1998), and Shimoji (2004) use a notion of "consistent updating systems" that is equivalent to forward consistent belief systems in terms of sequential optimality. Complete CPSs were used by Battigalli (1996) to define strong rationalizability.

On the other hand, most works on the epistemic foundations of those same solution concepts use "standard" CPSs, Battigalli & Siniscalchi (1999, 2002), Battigalli & De Vito (2021). From a technical point of view, our results establish the bridge between the predictions of solution concepts that were defined with forward consistent belief systems, and the epistemic foundations that have been established for procedures defined with "standard" CPSs.

Finally, our results also connect to known characterizations of rationalizability procedures in sequential games. Shimoji & Watson's (1998) characterization of strong rationalizability in terms of iterated conditional dominance was proved with forward consistent systems of beliefs. Theorem 4 implies that the characterization holds for all our three kinds of belief systems, and similarly for Shimoji's (2004) result of generic equivalence between strong rationalizability and iterated admissibility.

9 Conclusion

In dynamic games, players must update or completely change their beliefs about the behavior of co-players upon the arrival of new information regarding their past moves. Various restrictions on how players revise their beliefs have been used in the literature, but the very meaning of these requirements and their implications for optimal planning and strategic reasoning remain unclear. We put forward and analyze three ordered notions of coherency among beliefs at different information sets, and we offer interpretations in terms of players' level of introspection.

The weakest notion, forward consistency, only requires belief updating by conditioning, as long as possible, along every path of play. Forward consistency takes the viewpoint of a player who, at each information set, only asks herself how the co-players will behave in the contingencies that she currently deems possible. When surprised by moves that she previously deemed impossible, such a player might form different beliefs depending on her own past moves. Our intermediate notion of belief system, standard belief system, rules this out by requiring that the beliefs at different information sets satisfy the chain rule of probability; hence, same information (about others) implies same belief. Standard belief systems are isomorphic to conditional probability systems a la Renyi (1955), defined over the collection of conditional events that correspond to some information set. Our strongest notion of coherency, complete consistency, requires instead that the beliefs at the information sets are consistent with a conditional probability system over the power set of the space of uncertainty, which also contains conditional events that cannot be observed at any point in the game. Such virtual events can be interpreted as "mental exercises" of a fully introspective player, who asks herself "what would I believe if I were to observe this event", and derives her beliefs at the actual information sets accordingly. Complete consistency implies that the relative probabilities of pairs of co-players' strategy profiles are mutually consistent across all information sets where such profiles are possible, a property that standard belief systems may violate, as we show by example.

Although complete consistency is much more restrictive than forward consistency for players' belief systems, the main result of the paper shows that weak sequential optimality and strong rationalizability are invariant to the choice of any of the three notions of belief system. Perhaps even more surprisingly, also backwards rationalizability, an algorithm that makes use of (non-weak) sequential rationality, is invariant to the chosen notion of belief system. The key observation for these results is that forward consistency is sufficient to identify a player's courses of action that are optimal also under a completely consistent belief system, along every path.

Appendix A: Proofs of Section 7.1

In this section, we provide the omitted proofs of the section on strong rationalizability. In order to show that strong rationalizability with FBSs and complete CPSs is characterized by the one-memory procedure, we proceed with a preliminary lemma.

Lemma 7. *Fix a chain of events $E^n_{-i} \subseteq \ldots \subseteq E^1_{-i} \subseteq E^0_{-i}$. Take any strategy $s_i \in S_i$ and two complete CPSs (or FBSs) μ and μ^0 such that μ strongly believes $E^n_{-i}, \ldots, E^1_{-i}$, μ^0 strongly believes E^0_{-i}, $s_i \in BR_i(\mu)$ and $s_i \in BR_i(\mu^0)$. Then the array μ^* defined for all $C \in \mathscr{S}_{-i}$ as $\mu^*(\cdot|C) = \mu(\cdot|C)$ if $E^1_{-i} \cap C \neq \emptyset$ and $\mu^*(\cdot|C) = \mu^0(\cdot|C)$ if $E^1_{-i} \cap C = \emptyset$ is a complete CPS that strongly believes $E^n_{-i}, \ldots, E^0_{-i}$ such that $s_i \in BR_i(\mu^*)$. The same is true for FBSs for the array μ^* defined for all $h_i \in H_i$ as $\mu^*(\cdot|h_i) = \mu(\cdot|h_i)$ if $E^1_{-i} \cap S_{-i}(h_i) \neq \emptyset$ and $\mu^*(\cdot|h_i) = \mu^0(\cdot|h_i)$ if $E^1_{-i} \cap S_{-i}(h_i) = \emptyset$.*

Proof: We show the result for complete CPSs, the proof for FBSs is analogous. Take μ^* as described in the statement of the lemma.

Claim 1: *μ^* is a complete CPS.* Take any $C_{-i}, D_{-i} \in \mathscr{S}_{-i}$ such that $D_{-i} \subseteq C_{-i}$. If $D_{-i} \cap E^1_{-i} \neq \emptyset$, then also $C_{-i} \cap E^1_{-i} \neq \emptyset$ and so we have $\mu^*(\cdot|C_{-i}) = \mu(\cdot|C_{-i})$ and $\mu^*(\cdot|D_{-i}) = \mu(\cdot|D_{-i})$. Hence the chain rule relates $\mu^*(\cdot|C_{-i})$ to $\mu^*(\cdot|D_{-i})$ since μ is a complete CPS. If $D_{-i} \cap E^1_{-i} = \emptyset$ and $C_{-i} \cap E^1_{-i} \neq \emptyset$, then $\mu^*(\cdot|C_{-i}) = \mu(\cdot|C_{-i})$ and strong belief of μ in E^1_{-i} implies $\mu(E^1_{-i}|C_{-i}) = 1$. Then, because $D_{-i} \cap E^1_{-i} = \emptyset$ we have $\mu(D_{-i}|C_{-i}) = 0$. So, the chain rule relates $\mu^*(\cdot|C_{-i})$ to $\mu^*(\cdot|D_{-i})$. If $C_{-i} \cap E^1_{-i} = \emptyset$, then $D_{-i} \cap E^1_{-i} = \emptyset$, so that $\mu^*(\cdot|C_{-i}) = \mu^0(\cdot|C_{-i})$ and $\mu^*(\cdot|D_{-i}) = \mu^0(\cdot|D_{-i})$ and the chain rule relates $\mu^*(\cdot|C_{-i})$ to $\mu^*(\cdot|D_{-i})$ since μ^0 is a complete CPS. So, μ^* is a complete CPS.

Claim 2: *μ^* strongly believes $E^n_{-i}, \ldots, E^0_{-i}$.* Fix $C \in \mathscr{S}_{-i}$ such that $C \cap E^k_{-i} \neq \emptyset$ for some $k > 0$. Then $C \cap E^1_{-i} \neq \emptyset$ since $E^k_{-i} \subseteq E^1_{-i}$, and so $\mu^*(\cdot|C) = \mu(\cdot|C)$. Since μ strongly believes E^k_{-i}, we have $\mu^*(E^k_{-i}|C) = \mu(E^k_{-i}|C) = 1$. Now fix C such that $C \cap E^0_{-i} \neq \emptyset$. If $C \cap E^1_{-i} \neq \emptyset$, then $\mu^*(\cdot|C) = \mu(\cdot|C)$, and since μ strongly believes E^1_{-i}, we have $\mu^*(E^0_{-i}|C) \geq \mu^*(E^1_{-i}|C) = \mu(E^1_{-i}|C) = 1$. If $C \cap E^1_{-i} = \emptyset$, then $\mu^*(\cdot|C) = \mu^0(\cdot|C)$, and since μ^0 strongly believes E^0_{-i}, we have $\mu^*(E^0_{-i}|C) = \mu^0(E^0_{-i}|C) = 1$.

Claim 3: *s_i is a weak sequential best reply to μ^*.* Take any $h \in H_i(s_i)$. We have either $\mu^*(\cdot|S_{-i}(h)) = \mu(\cdot|S_{-i}(h))$ or $\mu^*(\cdot|S_{-i}(h)) = \mu^0(\cdot|S_{-i}(h))$. In both cases s_i^h maximizes player i's expected payoff at h by weak sequential optimality of s_i under both μ and μ^0. ■

Proof of Lemma 4: Let Δ denote FBSs or complete CPSs. We prove that $S_i^{\Delta,m} = \bar{S}_i^{\Delta,m}$ for every i and $m \in \mathbb{N}$. The basis step is immediate, as $S_i^{\Delta,1} = \bar{S}_i^{\Delta,1}$ for every i by inspection of the definitions. Suppose by way of induction that $S_i^{\Delta,k} = \bar{S}_i^{\Delta,k}$ for every i and $k \leq m$. The inductive hypothesis implies, by inspection of the definitions, that $S_i^{\Delta,m+1} \subseteq \bar{S}_i^{\Delta,m+1}$ for every i. We show the other inclusion. Take any $s_i \in \bar{S}_i^{\Delta,m+1}$. Then because $(\bar{S}_i^{\Delta,k})_{k\in\mathbb{N}}$ is a weakly decreasing sequence of sets, we have $s_i \in \bar{S}_i^{\Delta,k}$ for every $k \leq m+1$. This implies that for each $k \leq m$, there is $\mu^k \in \Delta_i$ such that μ^k strongly believes $\bar{S}^k_{-i}$ and $s_i \in BR_i(\mu^k)$. We construct μ^* that strongly believes the chain $(\bar{S}^m_{-i}, \ldots, \bar{S}^1_{-i})$ such that $s_i \in BR_i(\mu^*)$ by recursively applying Lemma 7. Take first $\mu = \mu^m$ and $\mu^0 = \mu^{m-1}$ to obtain $\mu^{*,m-1}$ that strongly believes $(\bar{S}^m_{-i}, \bar{S}^{m-1}_{-i})$ such that $s_i \in BR_i(\mu^{*,m-1})$; for all $k = m, \ldots, 1$, recursively define $\mu^{*,k-1}$ with the same operation with $\mu = \mu^{*,k}$ and $\mu^0 = \mu^{k-1}$. Then letting $\mu^* := \mu^{*,0}$ we have that $\mu^* \in \Delta_i$ strongly believes $(\bar{S}^m_{-i}, \ldots, \bar{S}^1_{-i}) = (S^m_{-i}, \ldots, S^1_{-i})$, where the equality holds by the inductive hypothesis, and $s_i \in BR_i(\mu^*)$. Thus, $s_i \in S_i^{\Delta,m+1}$. ■

Proof of Lemma 3: Fix a strategy s_i, a forward consistent belief system $\mu = (\mu(\cdot|h_i))_{h_i\in H_i}$ that strongly believes E_{-i}, and some probability measure $\hat{\nu} \in \Delta(S_{-i})$ with $\operatorname{Supp}\hat{\nu} = E_{-i}$.

In this proof, we consider a set of $\#H_i(s_i) + 1$ indexed measures obtained from $\hat{\nu}$ and μ restricted to $H_i(s_i)$. Specifically, each measure $\mu(\cdot|h_i)$ with $h_i \in H_i(s_i)$ is indexed by the conditioning information set h_i, and $\hat{\nu}$ is indexed by an arbitrary symbol $\hat{h}_i$ (where $\hat{h}_i \notin H_i$). Formally, we consider the set of $\#H_i(s_i) + 1$ distinct pairs

$$\{(\mu(\cdot|h_i), h_i)\}_{h_i\in H_i(s_i)} \cup \{(\hat{\nu}, \hat{h}_i)\},$$

so that, by definition, $h_i' \neq h_i''$ implies $(\mu(\cdot|h_i'), h_i') \neq (\mu(\cdot|h_i''), h_i'')$ even if $\mu(\cdot|h_i') = \mu(\cdot|h_i'')$, and $(\mu(\cdot|h_i), h_i) \neq (\hat{\nu}, \hat{h}_i)$ even if $\mu(\cdot|h_i) = \hat{\nu}$. Yet, to ease notation, we just write $\mu(\cdot|h_i)$ instead of pair $(\bar{\mu}(\cdot|h_i), h_i)$, and $\hat{\nu}$ instead of pair $(\hat{\nu}, \hat{h}_i)$, since our measures are already labeled differently. We put a linear order $<$ on this indexed set in such a way that for all $h_i, h_i' \in H_i(s_i)$, (C1) if $h_i \prec h_i'$, then $\mu(\cdot|h_i) < \mu(\cdot|h_i')$; (C2) if $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$, then $\mu(\cdot|h_i) < \hat{\nu}$; (C3) if $S_{-i}(h_i) \cap E_{-i} = \emptyset$, then $\hat{\nu} < \mu(\cdot|h_i)$.

Claim A: *There is an order $<$ on $\{(\mu(\cdot|h_i), h_i)\}_{h_i \in H_i(s_i)} \cup \{(\hat{\nu}, \hat{h}_i)\}$ that satisfies (C1)-(C3).*

Proof: Construct an order $<$ as follows: Let $Z(E_{-i}) = \{z \in Z : \exists s_{-i} \in E_{-i}, \exists s_i \in S_i, \zeta(s_i, s_{-i}) = z\}$ denote the set of terminal histories that can realize if the co-players implement strategy profiles in E_{-i}. Order Z as $z_1 < \ldots < z_{|Z|}$ such that each $z_k \in Z(E_{-i})$ precedes all $z_\ell \notin Z(E_{-i})$. Begin to order all $\mu(\cdot|h_i)$ with h_i such that $S_{-i}(z_1) \subseteq S_{-i}(h_i)$ according to information-set precedence, that is, if $h_i \prec h_i'$ then $\mu(\cdot|h_i) < \mu(\cdot|h_i')$. Notice that this covers all cases since for every different h_i, h_i' with $S_{-i}(z_1) \subseteq S_{-i}(h_i), S_{-i}(h_i')$ either $h_i \prec h_i'$ or $h_i' \prec h_i$.[21] For each $k = 2, \ldots, |Z|$, do the same with z_k for the indexed measures not already ordered in the previous steps $j < k$. Indexed measures ordered at previous steps come before those ordered at subsequent steps. Place $\hat{\nu}$ directly after the last $\mu(\cdot|h_i)$ with h_i consistent with E_{-i}, that is, with $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$.

Fix $h_i, h_i' \in H_i(\bar{s}_i)$. Suppose $\mu(\cdot|h_i)$ is ordered at step k and $\mu(\cdot|h_i')$ at step ℓ. The fact that the resulting order satisfies (C1)-(C3) is implied by the following two claims.

Claim A.1: *If $h_i \prec h_i'$, then $\bar{\mu}(\cdot|h_i) < \bar{\mu}(\cdot|h_i')$.* If $k = \ell$ or $k < \ell$, the claim follows by construction of the order. Since $\mu(\cdot|h_i')$ is ordered at step ℓ, we have $S_{-i}(z_\ell) \subseteq S_{-i}(h_i')$, and since $h_i \prec h_i'$, by perfect recall $S_{-i}(h_i') \subseteq S_{-i}(h_i)$. Thus, $S_{-i}(z_\ell) \subseteq S_{-i}(h_i)$, and so $\mu(\cdot|h_i)$ is ordered at most at step ℓ, that is to say $k \leq \ell$.

Claim A.2: *If $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$ and $S_{-i}(h_i') \cap E_{-i} = \emptyset$, then $\mu(\cdot|h_i) < \mu(\cdot|h_i')$.* If $k < \ell$ the result follows. First, note that $z_\ell \notin Z(E_{-i})$.[22] Given the ordering on Z, it is then enough

[21] If both h_i' and h_i'' precede z, then there is $x' \in h_i'$ such that $x' \prec z$, and $x'' \in h_i''$ such that $x'' \prec z$. But then either $x' \preceq x''$ or $x'' \preceq x'$, which by perfect recall implies either $h_i' \preceq h_i''$ or $h_i'' \preceq h_i'$.

[22] To see this, suppose $z_\ell \in Z(E_{-i})$. Then there exists $s_{-i}^* \in E_{-i}$ such that $z_\ell = \zeta(s_i, s_{-i}^*)$ for some

to show that $z_k \in Z(E_{-i})$. Since $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$, for some s_i and $s^*_{-i} \in S_{-i}(h_i) \cap E_{-i}$ we have $z^* = \zeta(s_i, s^*_{-i}) \in Z(E_{-i})$. Since z^* is reachable from h_i, there exists $x^* \in h_i$ such that $x^* \prec z^*$, so that $S_{-i}(z^*) \subseteq S_{-i}(x^*) \subseteq S_{-i}(h_i)$. Thus, $z_k \leq z^*$ and so $z_k \in Z(E_{-i})$. $\square$

Now, fix a complete CPS $\widetilde{\mu}$ and construct an array $(\bar{\mu}(\cdot|C_{-i}))_{C_{-i}\in\mathscr{S}_{-i}} \in \times_{C_{-i}\in\mathcal{C}_{-i}}\Delta(C_{-i})$ as follows: for each $C_{-i} \in \mathscr{S}_{-i}$, derive $\bar{\mu}(\cdot|C_{-i})$ by conditioning the first indexed measure in the ordering, if any, that gives positive probability to C_{-i}; otherwise, set $\bar{\mu}(\cdot|C_{-i}) = \widetilde{\mu}(\cdot|C_{-i})$. We show that this array is a complete CPS satisfying the required properties:

Claim 1: *$\bar{\mu}$ is a complete CPS.* Fix $C_{-i}, D_{-i} \in \mathscr{S}_{-i}$ such that $D_{-i} \subseteq C_{-i}$ and $\bar{\mu}(D_{-i}|C_{-i}) > 0$. If no measure in the ordering gives positive probability to C_{-i}, the same is true by monotonicity for $D_{-i} \subseteq C_{-i}$. So, we have $\bar{\mu}(\cdot|C_{-i}) = \widetilde{\mu}(\cdot|C_{-i})$ and $\bar{\mu}(\cdot|D_{-i}) = \widetilde{\mu}(\cdot|D_{-i})$. Since $\widetilde{\mu}$ is a complete CPS and $D_{-i} \subseteq C_{-i}$, the chain rule relates $\widetilde{\mu}(\cdot|C_{-i}) = \bar{\mu}(\cdot|C_{-i})$ to $\widetilde{\mu}(\cdot|D_{-i}) = \bar{\mu}(\cdot|D_{-i})$. If $\bar{\mu}(\cdot|C_{-i})$ is derived from some ν in the ordered set, every measure $\widetilde{\nu}$ that precedes ν must assign probability 0 to C_{-i}, hence also to $D_{-i} \subseteq C_{-i}$. Moreover, since $\bar{\mu}(\cdot|C_{-i})$ is derived from ν by conditioning on C_{-i}, we have $\nu(D_{-i}) = \bar{\mu}(D_{-i}|C_{-i})\nu(C_{-i}) > 0$. Hence, $\bar{\mu}(\cdot|D_{-i})$ is derived from ν by conditioning on D_{-i}. Since $\bar{\mu}(\cdot|C_{-i})$ and $\bar{\mu}(\cdot|D_{-i})$ are derived from the same measure and $D_{-i} \subseteq C_{-i}$, the chain rule relates $\bar{\mu}(\cdot|C_{-i})$ to $\bar{\mu}(\cdot|D_{-i})$.

Claim 2: *$\bar{\mu}(\cdot|S_{-i}(h_i)) = \mu(\cdot|h_i)$ for all $h_i \in H_i(s_i)$.* Fix any $h_i \in H_i(s_i)$. Note that $\bar{\mu}(\cdot|S_{-i}(h_i))$ is derived from some indexed measure ν in the ordered set that weakly precedes $\mu(\cdot|h_i)$ since $\mu(S_{-i}(h_i)|h_i) = 1 > 0$. We show that $\nu = \mu(\cdot|\bar{h}_i)$ for some $\bar{h}_i \in H_i(s_i)$ such that $\bar{h}_i \preceq h_i$. If $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$, then by (C2) $\mu(\cdot|h_i) < \hat{\nu}$, and since $\nu \leq \mu(\cdot|h_i)$ we have $\nu \neq \hat{\nu}$. If $S_{-i}(h_i) \cap E_{-i} = \emptyset$, since $\hat{\nu}(E_{-i}) = 1$, we have $\hat{\nu}(S_{-i}(h_i)) = 0$, and since $\nu(S_{-i}(h_i)) > 0$, it must be $\nu \neq \hat{\nu}$. Thus, we have $\nu = \mu(\cdot|\bar{h}_i)$ for some $\bar{h}_i \in H_i(s_i)$. Now we show that $\bar{h}_i \preceq h_i$. Since $\bar{\mu}(\cdot|S_{-i}(h_i))$ is derived from $\mu(\cdot|\bar{h}_i)$, there must exist $s^*_{-i} \in S_{-i}(h_i) \cap S_{-i}(\bar{h}_i)$. Since $h_i, \bar{h}_i \in H_i(s_i)$, we have $s_i \in S_i(h_i)$ and $s_i \in S_i(\bar{h}_i)$. Then, by perfect recall $(s_i, s^*_{-i}) \in S(h_i)$ and $(s_i, s^*_{-i}) \in S(\bar{h}_i)$, and $S(h_i) \cap S(\bar{h}_i) \neq \emptyset$ implies that either $\bar{h}_i \preceq h_i$ or $h_i \preceq \bar{h}_i$. Since $\mu(\cdot|\bar{h}_i) \leq \mu(\cdot|h_i)$, by (C1) it must be $\bar{h}_i \preceq h_i$. Since $\bar{h}_i \preceq h_i$ and $\mu(S_{-i}(h_i)|\bar{h}_i) > 0$, by the

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s_i. Notice that $s^*_{-i} \in S_{-i}(z_\ell)$. Since by construction of the ordering $S_{-i}(z_\ell) \subseteq S_{-i}(h'_i)$, we have $s^*_{-i} \in S_{-i}(h'_i) \cap E_{-i}$, and thus $S_{-i}(h'_i) \cap E_{-i} \neq \emptyset$, a contradiction.

forward chain rule of μ we have that $\mu(\cdot|h_i)$ is derived from $\mu(\cdot|\bar{h}_i)$ by conditioning, which implies $\mu(\cdot|h_i) = \bar{\mu}(\cdot|S_{-i}(h_i))$.

CLAIM 3: *$\bar{\mu}$ strongly believes E_{-i}.* Take any $C_{-i} \in \mathscr{S}_{-i}$ such that $C_{-i} \cap E_{-i} \neq \emptyset$. Then $\bar{\mu}(\cdot|C_{-i})$ is derived from some $\nu \leq \hat{\nu}$, as $\hat{\nu}(C_{-i}) = \hat{\nu}(C_{-i} \cap E_{-i}) > 0$ since $\operatorname{Supp} \hat{\nu} = E_{-i}$. If $\nu = \hat{\nu}$, then $\bar{\mu}(E_{-i}|C_{-i}) = 1$, because $\hat{\nu}(E_{-i}) = 1$. If ν strictly precedes $\hat{\nu}$, then it must be that $\nu = \mu(\cdot|\bar{h}_i)$ for some $\bar{h}_i \in H_i(s_i)$ such that $S_{-i}(\bar{h}_i) \cap E_{-i} \neq \emptyset$, otherwise ν would follow $\hat{\nu}$ by (C3). Since μ strongly believes E_{-i}, $\mu(E_{-i}|\bar{h}_i) = 1$, hence, $\bar{\mu}(E_{-i}|C_{-i}) = 1$, as $\bar{\mu}(\cdot|C_{-i})$ is derived from $\mu(\cdot|\bar{h}_i)$. ■

As an almost immediate corollary of Theorem 4 we have that all our procedures are characterized by the one-step memory procedure.

Lemma 8. *Let $\Delta = (\Delta_i)_{i \in I}$ be such that for every $i \in I$, $\Delta_{\mathcal{C}}^{H_i}(S_{-i}) \subseteq \Delta_i \subseteq \Delta_{\mathcal{F}}^{H_i}(S_{-i})$. Then $S^{\Delta,m} = \bar{S}^{\Delta,m}$ for every m.*

Proof: First, we show by induction that for every m, $\bar{S}^{\mathcal{C},m} = \bar{S}^{\Delta,m} = \bar{S}^{\mathcal{F},m}$. The basis for $m = 0$ is immediate; using the inductive hypothesis we have by inspection of the definitions that $\bar{S}^{\mathcal{C},m+1} \subseteq \bar{S}^{\Delta,m+1} \subseteq \bar{S}^{\mathcal{F},m+1}$. Since by Theorem 4 and Lemma 4 we have $\bar{S}^{\mathcal{F},m+1} = \bar{S}^{\mathcal{C},m+1}$, it follows that $\bar{S}^{\mathcal{C},m+1} = \bar{S}^{\Delta,m+1} = \bar{S}^{\mathcal{F},m+1}$.

Finally, by Claim 2 of the proof of Theorem 4 we also have $S^{\mathcal{C},m} = S^{\Delta,m} = S^{\mathcal{F},m}$ for every m. Since $S^{\mathcal{F},m} = \bar{S}^{\mathcal{F},m} = \bar{S}^{\Delta,m}$, it follows that $S^{\Delta,m} = \bar{S}^{\Delta,m}$ for every m. ■

Appendix B: Proofs of Section 7.2

In this section, we show the behavioral equivalence result for backwards rationalizability. We proceed analogously to what we did with strong rationalizability, by showing two preliminary lemmas.

It is first convenient to introduce a stronger notion of "counterfactually persistent belief" in E_{-i}, that is, the belief at each h that the co-players will follow, or would have followed some strategy profile in E_{-i} starting from every $h' \in H$ that does not precede h, including the histories h' that are counterfactual given h. Let

$$\hat{\chi}_j^h(E_j) = \{s_j \in S_j(h) : \exists \overline{s}_j \in E_j, \forall h' \in H, h' \not\prec h \Rightarrow s_j(h') = \overline{s}_j(h')\}$$

and

$$\widehat{\mathrm{PB}}_i\,(E_{-i}) = \{\mu^i \in \times_{h\in H}\Delta(S_{-i}\,(h)) : \forall h \in H, \mu^i(\hat{\chi}_{-i}^h(E_{-i})|h) = 1\},$$

where $E_{-i} = \times_{j\neq i}E_j$ and $\hat{\chi}_{-i}^h(E_{-i}) = \times_{j\neq i}\hat{\chi}_j^h(E_j)$.

We proceed by establishing some preliminary lemmas.

Lemma 9. *Fix a strategy $\overline{s}_i$, a subset $E_{-i} \subseteq S_{-i}$, and a forward consistent belief system $\overline{\mu}^i$ in $\widehat{\mathrm{PB}}_i\,(E_{-i})$. Then there exists a fully consistent μ^i such that (i) $\mu^i(\cdot|S_{-i}(h)) = \overline{\mu}^i(\cdot|h)$ for all $h \in H_i(\overline{s}_i)$ and (ii) $\mu^i \in \mathrm{PB}_i\,(E_{-i})$.*

Proof: To economize on constructions, we take the same Myerson CPS μ^i as the one of Lemma 3, but assume that $\overline{\mu}^i$ is in $\widehat{\mathrm{PB}}_i(E_{-i})$ instead of $\mathrm{SB}_i(E_{-i})$.

Claim: $\mu^i \in \mathrm{PB}_i(E_{-i})$. Fix any $h \in H$. If $\mu^i(\cdot|S_{-i}(h)) = \widetilde{\mu}^i(\cdot|S_{-i}(h))$ or $\mu^i(\cdot|S_{-i}(h))$ was derived from ν, we have $\mu^i(\chi_{-i}^h(E_{-i})|S_{-i}(h)) = 1$. Suppose now that $\mu^i(\cdot|S_{-i}(h))$ is derived from some $\overline{\mu}^i(\cdot|\overline{h})$ in the ordering; by (C1), it cannot be that $h \prec \overline{h}$, so we have $h \not\prec \overline{h}$. Now, note that, since $\mu^i(\cdot|S_{-i}(h))$ is derived from $\overline{\mu}^i(\cdot|\overline{h})$ and $\overline{\mu}^i \in \mathrm{PB}_i(E_{-i})$, we have $\mathrm{Supp}\,\mu^i(\cdot|S_{-i}(h)) \subseteq \hat{\chi}_{-i}^{\overline{h}}(E_{-i})$. Moreover, we claim that $h \not\prec \overline{h}$ implies $\hat{\chi}_{-i}^{\overline{h}}(E_{-i}) \subseteq \chi_{-i}^h(E_{-i})$. Suppose $h \not\prec \overline{h}$ and fix $s_{-i}^* \in \hat{\chi}_{-i}^{\overline{h}}(E_{-i})$. Then there exists $s_{-i} \in E_{-i}$ such that $s_{-i}(h') = s_{-i}^*(h')$ for all $h' \not\prec \overline{h}$. Since $h \not\prec \overline{h}$, every $h' \succeq h$ satisfies $h' \not\prec \overline{h}$, and so, $s_{-i}(h') = s_{-i}^*(h')$ for all $h' \succeq h$. Hence we have $s_{-i}^*|h = s_{-i}|h$ for some $s_{-i} \in E_{-i}$, that is to say, $s_{-i}^* \in \chi_{-i}^h(E_{-i})$. So, $\mathrm{Supp}\,\mu^i(\cdot|S_{-i}(h)) \subseteq \chi_{-i}^h(E_{-i})$ for all $h \in H$, i.e., $\mu^i \in \mathrm{PB}_i(E_{-i})$. ■

Given two belief systems $\mu^i, \overline{\mu}^i$ and a nonterminal history $h \in H$, we write $\mu^i(\cdot|h) \simeq \overline{\mu}^i(\cdot|h)$ if $\mu^i(S_{-i}(z)|h) = \overline{\mu}^i(S_{-i}(z)|h)$ for every $z \succ h$. We write $\mu^i \simeq \overline{\mu}^i$ if $\mu^i(\cdot|h) \simeq \overline{\mu}^i(\cdot|h)$ for all $h \in H$.

Lemma 10. *Fix a subset $E_{-i} \subseteq S_{-i}$, a forward consistent belief system $\overline{\mu}^i \in \mathrm{PB}_i(E_{-i})$,* *and a strategy $\bar{s}_i$. There exists a completely consistent $\mu^i \in \mathrm{PB}_i(E_{-i})$ such that $\mu^i(\cdot|h) \simeq \overline{\mu}^i(\cdot|h)$ for every $h \in H_i(E_i)$.*

Proof: Fix a forward consistent belief system $\overline{\mu}^i \in \mathrm{PB}_i(E_{-i})$. Note that, for each h, $\hat{\chi}^h_{-i}(\overline{S}_{-i}) \subseteq \chi^h_{-i}(E_{-i})$. With this, for each $h \in H$, define a map $\varsigma^h_{-i} : \chi^h_{-i}(E_{-i}) \to \hat{\chi}^h_{-i}(E_{-i})$, which is the identity on $\hat{\chi}^h_{-i}(E_{-i})$ and is such that for each $s_{-i} \in \chi^h_{-i}(E_{-i})$, $\varsigma^h_{-i}(s_{-i})|h = s_{-i}|h$. Note that for all $h' \succ h$ and $s_{-i} \in \chi^h_{-i}(E_{-i})$, $s_{-i} \in S_{-i}(h')$ if and only if $\varsigma^h_{-i}(s_{-i}) \in S_{-i}(h')$. It is convenient to impose the following consistency property: for all $h' \succ h$ and $s_{-i} \in S_{-i}(h') \subseteq S_{-i}(h)$, set $\varsigma^{h'}_{-i}(s_{-i}) = \varsigma^h_{-i}(s_{-i})$.

Take the belief system $(\widetilde{\mu}^i(\cdot|h))_{h\in H}$ defined as follows: For every $h \in H$ and $\hat{s}_{-i} \in \hat{\chi}^h_{-i}(\overline{S}_{-i})$, let

$$\widetilde{\mu}^i(\hat{s}_{-i}|h) = \overline{\mu}^i(\{s_{-i} : \varsigma^h_{-i}(s_{-i}) = \hat{s}_{-i}\}|h) = \overline{\mu}^i(\varsigma^{h,-1}_{-i}(\hat{s}_{-i})|h).$$

Notice that for every $h \in H$, $\widetilde{\mu}^i(S_{-i}(h')|h) = \overline{\mu}^i(S_{-i}(h')|h)$ for all $h' \succ h$ ($h' \in \bar{H}$), which implies $\widetilde{\mu}^i \simeq \overline{\mu}^i$. Also, since $\overline{\mu}^i \in \mathrm{PB}_i(\overline{S}_{-i})$, we have

$$\widetilde{\mu}^i(\hat{\chi}^h_{-i}(E_{-i})|h) = \overline{\mu}^i(\varsigma^{h,-1}_{-i}(\hat{\chi}^h_{-i}(E_{-i}))|h) = \overline{\mu}^i(\chi^h_{-i}(E_{-i})|h) = 1$$

for all $h \in H$. Thus, $\widetilde{\mu}^i \in \mathrm{PB}_i(E_{-i})$. We show that $\widetilde{\mu}^i$ is a forward consistent belief system. Fix $h \prec h'$ and $E_{-i} \subseteq S_{-i}(h')$ such that $\widetilde{\mu}^i(E_{-i}|h) > 0$. Then, note first that $\varsigma^{h,-1}_{-i}(E_{-i}) \subseteq S_{-i}(h')$ and that $\overline{\mu}^i(\varsigma^{h,-1}_{-i}(E_{-i})|h) = \widetilde{\mu}^i(E_{-i}|h) > 0$. Thus, the forward chain rule of $\overline{\mu}^i$ applies and implies:

$$\widetilde{\mu}^i(E_{-i}|h') = \overline{\mu}^i(\varsigma^{h',-1}_{-i}(E_{-i})|h') = \overline{\mu}^i(\varsigma^{h,-1}_{-i}(E_{-i})|h') = \frac{\overline{\mu}^i(\varsigma^{h,-1}_{-i}(E_{-i})|h)}{\overline{\mu}^i(S_{-i}(h')|h)} = \frac{\widetilde{\mu}^i(E_{-i}|h)}{\widetilde{\mu}^i(S_{-i}(h')|h)},$$

where the second equality holds by the consistency condition of the maps $(\varsigma^h_{-i})_{h\in H}$; the third equality by the forward chain rule of $\overline{\mu}^i$; and the fourth equality by definition of $\widetilde{\mu}^i$ and $\widetilde{\mu}^i(S_{-i}(h')|h) = \overline{\mu}^i(S_{-i}(h')|h)$.

To summarize, we have a forward consistent belief system $\widetilde{\mu}^i \in \mathrm{PB}_i(E_{-i})$ such that $\widetilde{\mu}^i(\cdot|h) \simeq \overline{\mu}^i(\cdot|h)$ for every $h \in H$. Then, by Lemma 9, there exists a completely consistent μ^i in $\mathrm{PB}_i(E_{-i})$ such that $\mu^i(\cdot|h) = \widetilde{\mu}^i(\cdot|h)$ for all $h \in H_i(\overline{s}_i)$. Hence, $\mu^i(\cdot|h) \simeq \overline{\mu}^i(\cdot|h)$ for all $h \in H_i(\overline{s}_i)$. ■

Proof of Theorem 5: Denote by $(\widetilde{S}^n)_{n\in\mathbb{N}_0}$ and $(\hat{S}^n)_{n\in\mathbb{N}_0}$ the backwards rationalizability algorithms with, respectively, forward consistent and completely consistent belief systems.

We show the following inductive hypothesis: for each i, (i) $\hat{S}^n_i \subseteq \widetilde{S}^n_i$, and (ii) for all $h \in H$ there is a map $\varsigma^{n,h}_i : \chi^h_i(\widetilde{S}^n_i) \to \chi^h_i(\hat{S}^n_i)$ such that, for each $s_i \in \chi^h_i(\widetilde{S}^n_i)$, $\varsigma^{n,h}_i(s_i)(h') = s_i(h')$ for all $h' \in H_i(s_i)$. The basis step is trivial, we show the statement holds at step $n+1$.

Fix any $i \in I$. First, note that for any consistent CPS μ^i in $\mathrm{PB}_i(\hat{S}^n_{-i})$, by (i) of the inductive hypothesis and monotonicity of the $\mathrm{PB}_i(\cdot)$ operator, we have $\mu^i \in \mathrm{PB}_i(\widetilde{S}^n_{-i})$, and since a consistent CPS is a forward consistent belief system, we have (i) $\hat{S}^{n+1}_i \subseteq \widetilde{S}^{n+1}_i$.

Now, take any $\bar{h} \in H$ and $s_i \in \chi^{\bar{h}}_i(\widetilde{S}^{n+1}_i)$. Then by definition there is a forward consistent belief system $\mu^i \in \mathrm{PB}_i(\widetilde{S}^n_{-i})$ such that $s'_i|\bar{h} = s_i|\bar{h}$ for some $s'_i \in BR^*_i(\mu^i)$.

Define a belief system $(\hat{\mu}^i(\cdot|h))_{h\in H} \in \times_{h\in H}\Delta(S_{-i}(h))$ as follows: For $h = \varnothing$ and all $h \neq \varnothing$ such that $\mu^i(S_{-i}(h)|\widetilde{h}) = 0$ for all $\widetilde{h} \prec h$, set:

$$\forall \hat{s}_{-i} \in \chi^h_{-i}(\hat{S}^n_{-i}), \quad \hat{\mu}^i(\hat{s}_{-i}|h) = \mu^i(\varsigma^{n,h,-1}_{-i}(\hat{s}_{-i})|h).$$

We have the following properties: since for every $h' \succ h$ and $s_{-i} \in \chi^h_{-i}(\widetilde{S}^n_{-i})$, $s_{-i} \in S_{-i}(h')$ if and only if $\varsigma^{n,h}_{-i}(s_{-i}) \in S_{-i}(h')$, we have (a) $\hat{\mu}^i(S_{-i}(h')|h) = 0$ if and only if $\mu^i(S_{-i}(h')|h) = 0$ and (b) $\mu^i \simeq \hat{\mu}^i$. Moreover, since $\mu^i \in \mathrm{PB}_i(\widetilde{S}^n_{-i})$,

$$\hat{\mu}^i(\chi^h_{-i}(\hat{S}^n_{-i})|h) = \mu^i(\varsigma^{n,h,-1}_{-i}(\chi^h_{-i}(\hat{S}^n_{-i}))|h) = \mu^i(\chi^h_{-i}(\widetilde{S}^n_{-i})|h) = 1.$$

Thus, deriving all other probability measures by forward conditioning, we indeed obtain a forward consistent belief system $\hat{\mu}^i$. It can be checked using a forward conditioning argument that $\hat{\mu}^i \simeq \mu^i$, so that $BR^*_i(\mu^i) = BR^*_i(\hat{\mu}^i)$. Moreover, $\hat{\mu}^i$ belongs to $\mathrm{PB}_i(\hat{S}^n_{-i})$. To see this, take any $h' \succ h$. This implies $\chi^h_{-i}(\hat{S}^n_{-i}) \cap S_{-i}(h') \subseteq \chi^{h'}_{-i}(\hat{S}^n_{-i})$,[23] and so, if $\hat{\mu}^i(\cdot|h')$ is derived by conditioning from $\hat{\mu}^i(\cdot|h)$, because $\operatorname{Supp}\hat{\mu}^i(\cdot|h) \subseteq \chi^h_{-i}(\hat{S}^n_{-i})$, we have:

$$\operatorname{Supp}\hat{\mu}^i(\cdot|h') \subseteq \chi^h_{-i}(\hat{S}^n_{-i}) \cap S_{-i}(h') \subseteq \chi^{h'}_{-i}(\hat{S}^n_{-i}).$$

To summarize, we have a forward consistent belief system $\hat{\mu}^i$ such that $\hat{\mu}^i \in \mathrm{PB}_i(\hat{S}^n_{-i})$ and $\hat{\mu}^i \simeq \mu^i$. Then by Lemma 10 there exists a consistent CPS $\widetilde{\mu}^i \in \mathrm{PB}_i(\hat{S}^n_{-i})$ with $\widetilde{\mu}^i(\cdot|h) \simeq \mu^i(\cdot|h)$ for all $h \in H_i(s'_i)$. This in turn implies that there is a strategy $\widetilde{s}_i \in BR^*_i(\widetilde{\mu}^i)$ such that, for each $h \in H_i(s'_i)$ with $h \succeq \bar{h}$, $\widetilde{s}_i(h) = s'_i(h) = s_i(h)$. Thus, $\widetilde{s}_i \in \hat{S}^{n+1}_i$ and any $\overline{s}_i \in S_i(\bar{h})$ with $\overline{s}_i|\bar{h} = \widetilde{s}_i|\bar{h}$ belongs to $\chi^h_i(\hat{S}^{n+1}_i)$, so that we can let $\varsigma^{n+1,\bar{h}}_i(s_i) = \overline{s}_i$. ■

Appendix C: Comparison with Kreps & Wilson

9.1 Complete consistency and sequences

We let $\Delta^\circ(\Omega) \subseteq \Delta(\Omega)$ denote the set of strictly positive probability measures on Ω.

For any complete CPS $\mu \in \Delta^*(\Omega)$, say that x is μ-infinitely more likely than y, written $x \gg^\mu y$, if $\mu(x|\{x,y\}) = 1$ (i.e., $\mu(y|\{x,y\}) = 0$). The chain rule implies that the (typically incomplete) binary relation $\gg^\mu$ is transitive. Therefore, μ yields an ordered partition $\left(\Omega^\mu_0, ..., \Omega^\mu_{L(\mu)}\right)$, where—for $0 \le k < \ell \le L(\mu)$—each element of Ω^μ_k is μ-infinitely more likely of each element of Ω^μ_ℓ. With this, $\mu(\cdot|\Omega^\mu_0) = \mu(\cdot|\Omega)$ is the unconditional belief. More generally, let $k(\mu, C)$ denote the lowest index in $\{0, ..., L(\mu)\}$ such that $C \cap \Omega_k \neq \emptyset$; then $\mu\left(C \cap \Omega_{k(\mu,C)}|C\right) = 1$ and

$$\mu\left(\cdot|C \cap \Omega_{k(\mu,C)}\right) = \mu(\cdot|C),$$

which implies the following:

Remark 9. A complete CPS $\mu \in \Delta^*(\Omega)$ is determined by the binary conditional proba-

[23] See Lemma 4.(i) in Appendix A of Battigalli & De Vito (2021).

bility values $(\mu(\cdot|\{x,y\}))_{x,y\in\Omega}$. Indeed, if two complete CPSs μ and ν are such that

$$(\mu(\cdot|\{x,y\}))_{x,y\in\Omega} = (\bar{\mu}(\cdot|\{x,y\}))_{x,y\in\Omega}$$

then $\left(\Omega_0^\mu, ..., \Omega_{L(\mu)}^\mu\right) = \left(\Omega_0^\nu, ..., \Omega_{L(\nu)}^\nu\right)$, which implies $\mu = \nu$.

Ordered partitions can also be used to relate complete CPSs to converging sequences of strictly positive probability measures. First note that if a sequence $(\nu_n)_{n\in\mathbb{N}} \in \Delta^\circ(\Omega)^{\mathbb{N}}$ converges and yields an array $(\mu(\cdot|C))_{\emptyset\neq C\subseteq\Omega} \in \times_{\emptyset\neq C\subseteq\Omega}\Delta(C)$ in the limit, that is,

$$\forall C, \forall\omega \in C,\ \mu(\omega|C) = \lim_{n\to\infty} \frac{\nu_n(\omega)}{\nu_n(C)}, \tag{1}$$

then $(\mu(\cdot|C))_{\emptyset\neq C\subseteq\Omega}$ must be a complete CPS. Next note that, for every complete CPS μ, the sequence

$$(\nu_n)_{n\in\mathbb{N}} = \left(K_n \sum_{k=0}^{L(\mu)} n^{-k}\mu\left(\cdot|\Omega_k^\mu\right)\right)_{n\in\mathbb{N}} \in \Delta^\circ(\Omega)^{\mathbb{N}}$$

(where $K_n = \left(\sum_{k=0}^{L(\mu)} n^{-k}\right)^{-1}$ is a normalizing constant) converges and yields μ in the limit. Therefore:

Remark 10. An array of probability measures $\mu = (\mu(\cdot|C))_{\emptyset\neq C\subseteq\Omega} \in \times_{\emptyset\neq C\subseteq\Omega}\Delta(C)$ is a complete CPS if and only if there is a convergent sequence of strictly positive probability measures $(\nu_n)_{n\in\mathbb{N}}$ that yields μ in the limit as in eq. (1).

9.2 Kreps & Wilson systems of beliefs

An *assessment* à la Kreps & Wilson (1982) is a pair (ξ,β), where $\xi \in \times_{i\in I}\left(\times_{h_i\in H_i}\Delta(h_i)\right)$ is a system of beliefs over non-terminal nodes conditional the information sets containing them, and β is a profile of behavior (i.e., locally randomized) strategies. Kreps & Wilson's notion of consistency (used to define sequential equilibrium, a refinement of the Nash equilibrium concept) requires that the beliefs of different players "agree" and that they satisfy what amounts to a strong across-players independence property. Such restrictions are not germane to our (correlated) rationalizability analysis. Thus, we make them mute by considering **personal** assessments in **two-player** games.

A *personal assessment* of player i in a two-player game is a pair

$$\left(\xi^i, \beta_{-i}\right) \in \left(\times_{h_i \in H_i} \Delta\left(h_i\right)\right) \times \left(\times_{h_{-i} \in H_{-i}} \Delta\left(\mathcal{A}_{-i}(h_{-i})\right)\right),$$

where ξ^i is a system of beliefs over nodes/histories in the information sets of i and β_{-i} is a behavior strategy of the co-player $-i$, which we may interpret as a probabilistic conjecture of i about $-i$. A personal assessment (ξ^i, β_{-i}) is consistent in the sense of Kreps & Wilson if there exists a converging sequence of strictly positive behavior strategies $\left(\beta_{-i,n}\right)_{n\in\mathbb{N}} \in \left(\times_{h_{-i}\in H_{-i}} \Delta^\circ\left(\mathcal{A}_{-i}(h_{-i})\right)\right)^{\mathbb{N}}$ such that $\beta_{-i} = \lim_{n\to\infty} \beta_{-i,n}$ and

$$\xi^i\left(x|h_i\right) = \lim_{n\to\infty} \frac{\mathbb{P}_{\beta_i{}^\circ,\beta_{-i,n}}\left(x\right)}{\mathbb{P}_{\beta_i{}^\circ,\beta_{-i,n}}\left(h_i\right)},$$

for all $h_i \in H_i$ and $x \in h_i$, where $\mathbb{P}_\beta$ denotes the probability of histories induced by randomized strategy profile (pair) β, and $\beta_i{}^\circ$ is an arbitrary strictly positive randomized strategy of i.[24] By Kuhn Theorem (Kuhn 1953, Theorem 4), the latter condition is equivalent to the existence of a converging sequence of strictly positive mixed strategies $\left(\nu_{-i,n}\right)_{n\in\mathbb{N}} \in \Delta^\circ\left(S_{-i}\right)^{\mathbb{N}}$ such that $\nu_{\beta_{-i}} = \lim_{n\to\infty} \nu_{-i,n}$ and

$$\xi^i\left(x|h_i\right) = \lim_{n\to\infty} \frac{\mathbb{P}_{\nu_i{}^\circ,\nu_{-i,n}}\left(x\right)}{\mathbb{P}_{\nu_i{}^\circ,\nu_{-i,n}}\left(h_i\right)},$$

where $\nu_{\beta_{-i}}$ is the mixed strategy induced by β_{-i} according to Kuhn's transformation and $\nu_i{}^\circ$ is an arbitrary strictly positive randomized strategy.[25]

A personal assessment (ξ^i, β_{-i}) yields a conditional probability measure $\mathbb{P}_{\xi^i,\beta_{-i},\beta_i}\left(\cdot|h_i\right)$

[24]By perfect recall, it does not matter which one, because the factors involving $\beta_i{}^\circ$ in the numerator and denominator cancel out.

[25]As above, by perfect recall, it does not matter which one, because the factors involving $\nu_i{}^\circ$ in the numerator and denominator cancel out.

over paths for every behavior strategy β_i of i conditional on every information set h_i of i: if $z = \left(x, a^{\ell(x)+1}, ...a^{\ell(z)}\right)$ with $x \in h_i$ of length $\ell(x)$,

$$\mathbb{P}_{\xi^i,\beta_{-i},\beta_i}(z|h_i) = \left(\prod_{k=1}^{\ell(z)-\ell(x)} \prod_{j\in\iota(x)} \beta_j\left(a_j^{\ell(x)+k}|h_j^{\ell(x)+k}\right)\right)\xi^i(x|h_i).$$

Similarly, a system of beliefs μ^i about the co-player's strategies yields a conditional probability measure $\mathbb{P}_{\mu^i,s_i}(\cdot|h_i)$ over paths for every strategy s_i of i conditional on every information set h_i of i: let $Z(h_i)$ denote the set of terminal nodes/paths following some node in h_i, then

$$\mathbb{P}_{\mu^i,s_i}(z|h_i) = \begin{cases} 0 & \text{if } s_i \notin S_i(z) \text{ or } z \notin Z(h_i), \\ \mu^i(S_{-i}(z)) & \text{if } s_i \in S_i(z) \text{ and } z \in Z(h_i). \end{cases}$$

Note that pure strategies are special cases of degenerate behavior strategies. With a slight abuse of notation we use the same symbol s_i for a pure strategy of i and the corresponding degenerate behavior strategy of i. With this, by Remark 5 we obtain:

Remark 11. A belief system μ^i about the strategies of $-i$ is completely consistent as per Definition 5 if and only if there exists a consistent personal assessment (ξ^i, β_{-i}) such that

$$\mathbb{P}_{\mu^i,s_i}(z|h_i) = \mathbb{P}_{\xi^i,\beta_{-i},s_i}(z|h_i)$$

for all $h_i \in H_i$, $z \in Z(h_i)$, and $s_i \in S_{-i}(h_i)$.

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