Belief change, Rationality, and Strategic Reasoning in Sequential Games

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Abstract

Strategic reasoning in sequential games rests on figuring out how (co)players would react to information about past play, which in turn depends on how players update or revise their beliefs. Several notions of belief systems have been used to represent and discipline how players’ beliefs change as they obtain new information. Such notions differ and can be nested according to the imposed consistency restrictions relating beliefs at different information sets. The minimal restriction requires that beliefs about others change from one information set to a following, more informative one in compliance with the chain rule, i.e., by standard updating whenever possible. On top of this, more demanding restrictions require that beliefs about co-players depend only on information about them, not on own past moves. Even stronger restrictions require that players update, or revise their beliefs as if they could notionally condition on any nonempty event about co-players’ behavior in compliance with the chain rule. We analyze restrictions on belief change, providing characterizations and interpretations in terms of introspection and cognitive rationality. We then argue that these differences between consistency restrictions do not affect the behavioral implications of strategic reasoning in games.
1 Introduction

A key aspect of strategic reasoning in sequential games is to figure out how (co)players would react to either anticipated, or non-anticipated information about past play, which in turn depends on how players update or revise their beliefs. Several notions of belief systems have been used to represent and discipline how players’ beliefs change as they obtain new information. Such notions differ and can be nested according to the imposed consistency restrictions relating beliefs at different information sets. The minimal restriction requires that beliefs about others change from one information set to a following, more informative one in compliance with the chain rule, i.e., by standard updating whenever possible. We call this restriction “forward consistency,” because it is an application of the chain rule as the play goes forward and new information about players’ behavior is revealed. On top of this, more demanding restrictions require that beliefs about co-players depend only on information about them, not on own past moves. This is the “standard” consistency restriction implied by the application of the chain rule to systems of conditional probabilities given a collection of conditioning events corresponding to the information about others revealed by information sets (see Renyi 1955, and Battigalli & Siniscalchi 2002). Even stronger restrictions require that players update, or revise their beliefs as if they could notionally condition on any nonempty event about co-players’ behavior, even if it is not observable, in compliance with the chain rule. We refer to such restrictions as “complete consistency,” because they are obtained by considering complete conditional probability systems (see Myerson 1986, and Battigalli 1996). We analyze restrictions on belief change, providing characterizations and interpretations in terms of introspection and cognitive rationality. We then argue that these differences between consistency restrictions do not affect the behavioral implications of
strategic reasoning in games. Specifically, we prove that different notions of rationalizability for sequential games justified by epistemic foundations are invariant to restrictions on belief change beyond the minimal ones. The following example illustrates the main themes of this work, comparing forward consistency to standard and complete consistency (in this particular case, standard and complete consistency are equivalent).

Consider game $\Gamma'$ depicted in Figure 1. At the root, Isa ($i$) chooses between Left and Right, and Joe ($j$) simultaneously chooses between Quit (which terminates the game) and Continue. If Joe Continues, Isa observes this and decides whether to go across (terminating the game) or down. If Isa goes down, Joe observes this, but not her initial move, and chooses between left and right. Assume that $0 \leq \varepsilon < 2$. Since Isa has perfect recall, she remembers her initial move and her choice may depend on it. Can her belief depend on her choice as well? According to forward consistency, this is possible. Indeed, if Isa is initially certain that Joe Quits, her initial belief does not pin down what she would believe after observing $(L, C)$ or $(R, C)$ and these two conditional beliefs may be different. On the other hand, if Isa initially assigns probabilities $\gamma\lambda$ and $\gamma(1 - \lambda) > 0$ strategies $C.\ell$ and $C.r$, respectively,

![Figure 1: $\Gamma'$, a common interest game between Isa and Joe.](#)
and $\gamma > 0$, then upon observing $C$ she would assign probability $\lambda$ to $C.\ell$ and $(1 - \lambda)$ to $C.r$ independently of her initial move. According to standard and complete consistency, the latter restriction must hold even if Isa is initially certain that Joe will Quit ($\gamma = 0$). We will show that this difference does not matter for the analysis of the behavioral implications of rationality and strategic reasoning. This is clearer if $\varepsilon > 0$. In this case, if Isa were initially certain of Quit she would go Right and her counterfactual belief after Left would not matter. But the difference does not matter even if $\varepsilon = 0$. Intuitively, the reason is that, whatever the planned choice of Isa at the root, if she is rational she actually makes this choice and her counterfactual belief after the other does not matter. We will explain this in detail in our analysis of rational planning and rationalizable behavior. Here we only provide an informal explanation of why Isa might have different beliefs about Joe conditional on $(L, C)$ and $(R, C)$, given that at both nodes she has the same information about Joe. According to subjective expected utility maximization, in order to decide what to do at the root, Isa only has to her initial belief. If she is certain of Quit, then Right is the best choice (one of the best if $\varepsilon = 0$). She has no need for further planning and she come up with a partial strategy. With this, she be be only partially introspective and unable to anticipate how she would revise her belief if surprised. Such partial introspection may prevent her from imposing the cognitive rationality “same-information/same-belief” rule, unless such rule is in some sense “wired” into her way of thinking. Of course, if surprised, Isa would form some revised belief and make a choice based on it. For example, if she initially goes Right and is surprised by $C$, it may be the case that she is pre-disposed to believe that Joe is more likely to continue with right, and thus choose to go down. In this case, she would implement the reduced strategy (or plan of action, according to Rubinstein 1991) $R.d''$. But this does not imply that she had initially planned to go down if surprised. In other words, $R.d''$ is a description of Isa’s behavior, but not necessarily a plan of Isa. What matters for strategic reasoning is to be able to anticipate the behavior of other. From this perspective, Joe’s belief in the rationality of Isa allows for the possibility that he assigns positive probability to her behavior being described by $R.d''$. Thus, our results about the invariance of (version of) rationalizability to
restrictions on belief systems beyond forward consistency may be interpreted as saying that the underlying epistemic justifications do not rely on full introspection.

2 Sequential games with perfect recall

Our analysis is restricted to finite sequential games without chance moves played by agents with perfect recall, represented in extensive form. Some knowledge of the extensive and strategic-form representations of sequential games is taken for granted. Thus, for the primitive terms of the analysis, only the necessary symbols and definitions with rather terse explanations are given below. The reader interested in the details should consult, e.g., Selten (1975), or Osborne & Rubinstein (1994). We instead expand on the interpretation of some derived terms. Note also that the formalism used here is more expressive than the traditional one due to Kuhn (1953), because (i) it represents simultaneous moves directly by letting plays be sequences of action profiles, and (ii) it represents also the information of inactive players, which is potentially relevant for our analysis of belief change.

A sequential game (played by agents) with perfect recall is a structure

\[
\Gamma = \langle I, \bar{X}, (A_i, H_i, u_i)_{i \in I} \rangle
\]

where: \( I \) is a finite set of players, \( A_i \) is a nonempty finite set of potentially available actions for player \( i \), and \( \bar{X} \) is a finite tree of feasible sequences of action profiles, called histories or nodes. We let: \( \emptyset \) denote the empty sequence, that is, the root of tree \( \bar{X} \), \( Z \) denote the set of terminal histories (or paths), and \( X = \bar{X} \setminus Z \) denote the set of non-terminal histories. These sequences form a tree if they contain every prefix of each one of its elements, including the empty sequence.
histories. We write $x < x'$ ($x \preceq x'$) if sequence $x$ is a strict (weak) prefix of $x'$, that is, node $x$ precedes node $x'$ in tree $\bar{X}$. For each player $i \in I$ there is a subset $X_i \subseteq X$ of nonterminal nodes where $i$ is alert, i.e., she processes information; $\iota(x) = \{i \in I : x \in X_i\}$ denotes the set of players who are alert at $x$ and we assume that $\iota(\emptyset) = I$, that is, all players are alert at the root. Alert players are active if they can choose between two or more alternative actions, and inactive otherwise, i.e., if all they can do is to “wait”. This is described by a profile of nonempty-valued feasibility correspondences $(\bar{A}_i(\cdot) : X_i \rightarrow A_i)_{i \in I}$ such that, for every $x \in X$, $(x, a_{i(x)}) \in \bar{X}$ if and only if $a_{i(x)} \in \times_{i \in \iota(x)} \bar{A}_i(x)$. Thus, $i \in \iota(x)$ is inactive at $x$ if $\bar{A}_i(x)$ is a singleton. In our graphical representations, such as the game tree $\Gamma'$ in Figure 1, we only show the actions of active players. Information structure $H_i$ is the collection of

information sets of player $i$, where:

1. $H_i$ is a partition of $X_i$;

2. for every $h_i \in H_i$ and $x, x' \in h_i$, $\bar{A}_i(x) = \bar{A}_i(x') = A_i(h_i)$;

3. (perfect recall) for every $h_i \in H_i$ and $x, x' \in h_i$ with $x \neq x'$, we have (i) $x \npreceq x'$ and (ii) for all $(\tilde{x}, a) \preceq x$ with $\tilde{x} \in \tilde{h}_i$ for some $\tilde{h}_i \in H_i$, there exists $(\tilde{x}', a') \preceq x'$ such that $\tilde{x}' \in \tilde{h}_i$ and $a_i = a'_i$.

Perfect recall implies that we can unambiguously partially order the information sets of each player $i$ with the “prefix of” precedence relation of $X$: for all distinct $h_i, h'_i \in H_i$, $h_i$ is a predecessor of $h'_i$, denoted $h_i \prec h'_i$, if and only if every history in $h'_i$ follows some history in $h_i$ (i.e., it has a prefix in $h_i$).

Finally, $u_i : Z \rightarrow \mathbb{R}$ is the payoff function of player $i$. For the sake of simplicity, many of our examples feature common interests (CI): $u_i = u_j$ for all $i, j \in I$.

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3This property ensures that what a player can do at a node does not depend on what co-players are simultaneously doing—otherwise actions would not be simultaneous.

4Property (i) says that players cannot end up twice in the same information set because they remember having moved before. Property (ii) says that if two histories are in the same information set, then player $i$ must have been unable to distinguish the prefixes of these histories at earlier information sets and must have taken the same actions at such earlier information sets (since she recalls her past actions).

5This is interpreted as the composition $u_i = v_i \circ g$ of the outcome function $g : Z \rightarrow Y$ and $i$’s utility function $v_i : Y \rightarrow \mathbb{R}$.
We consider two notable special cases: a game has **observable actions** (or perfect monitoring) if \( X_i = X \) for every \( i \) (players are always alert) and all information sets are singletons, in which case we write \( H_i = H = X \) for every \( i \) and we do not distinguish between histories/nodes and the singleton information sets containing them. A game with observable actions has **perfect information** if for every history \( h \in H \), only one player is active. For example, game \( \Gamma'' \) depicted in Figure 2 has observable actions, game \( \Gamma''' \) depicted in Figure 3 has perfect information.

![Figure 2: \( \Gamma'' \), a CI game with observable actions.](image)

![Figure 3: \( \Gamma''' \), a CI game with perfect information.](image)

A **strategy** for player \( i \) is a function \( s_i : H_i \rightarrow A_i \) that assigns to each information set \( h_i \in H_i \) a feasible action \( s_i(h_i) \in A_i(h_i) \). Thus, the set of strategies of player \( i \) is the cross-product of feasible action sets, \( S_i = \times_{h_i \in H_i} A_i(h_i) \). We denote by \( S = \times_{i \in I} S_i \) the set of strategy profiles and by \( S_{-i} = \times_{j \neq i} S_j \) the set of \( i \)'s co-players’ strategy profiles. The
implementation of a profile of strategies \( s \in S \) induces a unique terminal history \( \zeta(s) \), where \( \zeta : S \to Z \) denotes the **path function.** Although we do not require that feasible action sets at distinct information sets where a player is active be disjoint, this condition holds in our examples. This eases notation, allowing us to write the strategies of our examples as lists of actions separated by dots, such as \( R.a'.d'' \) for Isa in game \( \Gamma' \) of Figure 1.

For each \( h_i \in H_i \), the set of strategy profiles compatible with information set \( h_i \) is

\[
S(h_i) = \{ s \in S : \exists x \in h_i, x \prec \zeta(s) \}.
\]

Let \( S_i(h_i) = \text{proj}_{S_i} S(h_i) \) and \( S_{-i}(h_i) = \text{proj}_{S_{-i}} S(h_i) \). Perfect recall implies that, for all \( h_i, h'_i \in H_i \), we have (i) \( S(h_i) = S_i(h_i) \times S_{-i}(h_i) \), (ii) if \( h'_i \) follows \( h_i \), then \( S(h'_i) \subseteq S(h_i) \), hence, \( S_{-i}(h'_i) \subseteq S_{-i}(h_i) \), (iii) \( S(h_i) \cap S(h'_i) \neq \emptyset \) if and only if either \( h_i \preceq h'_i \) or \( h'_i \preceq h_i \). Yet, it is possible that \( S_{-i}(h_i) \cap S_{-i}(h'_i) \neq \emptyset \) even if \( h_i \) and \( h'_i \) are not ordered. For example, in game \( \Gamma' \), \( S_{-i}(\{(L,C)\}) = \{C.l, C.r\} = S_{-i}(\{(R,C)\}) \).

As in most of the work on strategic reasoning, here strategies represent both contingent plans in the minds of rational players and descriptions of information-dependent behavior. Thus, as a player plans her strategy \( s_i \), she assesses the likelihood of the possible “ways of behaving,” or “action rules” of the others, \( s_{-i} \).

The interpretation of strategies as plans or mere descriptions of behavior is related to an important structural equivalence relation. Let \( H_i(s_i) = \{ h_i \in H_i : s_i \in S_i(h_i) \} \) denote the collection of information sets that may occur if \( i \) implements strategy \( s_i \). For example, in game \( \Gamma' \), \( H_i(R.a'.a'') = H_i(R.a'.d'') = \{\emptyset, \{(R,C)\}\} \); in game \( \Gamma'' \), \( H_j(Q.a'.a'') = \{\emptyset\} \) for all \( a'.a'' \in \{l', r'\} \times \{l'', r''\} \), and \( H_j(C.a'.a'') = \{\emptyset, \{(L,C)\}, \{(R,C)\}\} \).

**Definition 1.** Two strategies \( s'_i, s''_i \in S_i \) are (1) **behaviorally equivalent** if \( H_i(s'_i) = H_i(s''_i) \) and \( s'_i(h_i) = s''_i(h_i) \) for every \( h_i \in H_i(s_i) \), (2) **realization-equivalent** if \( \zeta(s'_i, s_{-i}) = \zeta(s''_i, s_{-i}) \).

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\(^6\)To see this, if \( s \in S(h_i) \) and \( s \in S(h'_i) \), then there exist \( x \in h_i \) such that \( x \prec \zeta(s) \) and \( x' \in h'_i \) such that \( x' \prec \zeta(s) \). Hence, either \( x' \preceq x \) or \( x \preceq x' \), which implies by perfect recall that either \( h'_i \preceq h_i \) or \( h_i \preceq h'_i \). The reason is that perfect recall implies that for any \( h_i, h'_i \in H_i \), if \( x \in h_i, x' \in h'_i \) and \( x \prec x' \), then all nodes of \( h'_i \) are preceded by some node of \( h_i \), that is, \( h_i \prec h'_i \).
\[ \zeta(s_i', s_{-i}) \] for every \( s_{-i} \in S_{-i} \).

Kuhn (1953) proved that these two equivalence relations coincide:

**Remark 1.** (Kuhn, 1953, Theorem 1) Two strategies are behaviorally equivalent if and only if they are realization-equivalent.

Let \( \equiv_i \) denote this (behavioral or realization) equivalence relation. We call the elements of the quotient set \( S_i | \equiv_i \) “structurally reduced strategies,” abbreviated in “reduced strategies”; e.g., Isa has \( 2^3 = 8 \) strategies, but only 4 reduced strategies in \( \Gamma' \): \( L.a', L.d', R.a'', \) and \( R.d'' \), with \( L.a' = \{ L.a'.a'', L.a'.d'' \} \) etc. Often these equivalence classes are instead called “plans of action,” suggesting that how \( s_i \) is defined outside of \( H_i(s_i) \) is irrelevant for planning, and sometimes adding that the restriction of \( s_i \) to \( H_i \setminus H_i(s_i) \) should be interpreted as an expectation of the co-players (e.g., Osborne & Rubinstein 1994, p 103). We do not adopt this terminology because we have a different perspective that will be fully spelled out in Section 6.

To anticipate, one may think of a player planning forward or backward. Planning forward means comparing the expected payoffs of different courses of action like \( Q \) (quit), or \( C.a'.a'' \) (continue, then \( a' \) if \( L \) and \( a'' \) if \( R \)) for Joe in game \( \Gamma'' \) (opting for \( C.\ell'.r'' \) if \( \mu^i(L|\emptyset) \geq 1/2 \), and for \( Q \) if \( \mu^i(L|\emptyset) < 1/2 \)). When planning forward, there is no need to specify actions for contingencies that cannot occur given the plan under consideration (in \( \Gamma'' \), if Joe considers quitting, he does not have to plan what to do if he instead continues); thus, *reduced strategies correspond to forward plans.* Planning backward means first asking oneself what to do at any last move, use the answer as a contingent prediction about own behavior, and—given this—recursively plan what to do at earlier moves. For example, in game \( \Gamma'' \), without having made up his mind about the move at the root, Joe first plans to choose \( \ell' \) after \( (L, C) \) and \( r'' \) after \( (R, C) \); next, he plans to continue (quit) at the root if \( \mu^i(L|\emptyset) > 0.5 \) (\( \mu^j(L|\emptyset) < 0.5 \)). Whatever Joe plans to do at the root, *the result of such backward planning is a complete strategy* \( s_j = a_j.\ell'.r'' \) with \( a_j \in \{C, Q\} \).

\[ \text{It also makes sense to plan only for moves that are deemed possible under current beliefs. For example,} \]
The foregoing arguments suggest that, if we regard relation $\equiv_i$ as behavioral equivalence, we are led to interpret reduced strategies as “forward plans.” If we instead regard $\equiv_i$ as realization equivalence, we can think of reduced strategies as *sufficient descriptions of $i$’s behavior in the eyes of the co-players* (or an external observer). Indeed, if $s'_i \equiv_i s''_i$, independently of the co-players’ behavior, it is impossible to distinguish $s'_i$ between $s''_i$ by observing the realized path; furthermore, it is not necessary for $i$’s co-players to distinguish $s'_i$ between $s''_i$ in order to assess the likely consequences of taking different actions at (or implementing different continuation strategies starting from) an information set. For example, all that matters for Joe in game $\Gamma'$ of Figure 1 are the probabilities of the reduced strategies $L.a'$, $L.d'$, $R.a''$, and $R.d''$, interpreted as sufficient descriptions of Isa’s behavior.

3 Conditional beliefs

In this section, we introduce several representations of beliefs for sequential games. Players are uncertain about how the others would behave in the various circumstances, and form beliefs on others’ behavior to assess the likely consequences of their actions. The beliefs of a player will typically change as the game progresses. At information set $h_i \in H_i$, player $i$ learns that the co-players are behaving according to a strategy profile in $S_{-i}(h_i)$. If player $i$’s beliefs before the realization of information set $h_i$ assigned probability 0 to $S_{-i}(h_i)$, then the realization of $h_i$ falsifies $i$’s earlier beliefs, which have to be revised, rather than just updated according to the rules of conditional probability. We thus have to model what players would believe in all circumstances, and how their beliefs change as they receive new information.

We first consider an abstract representation of conditional thinking by means of “conditional probability systems” (Renyi 1955), which requires some consistency between beliefs conditional on different events. Next we move to “systems of beliefs”, which specify the beliefs a player would hold at each information set. We introduce different degrees of consis-

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if in game $\Gamma''$ Isa is initially certain that Joe is going to quit, then she just plans what to do at the root, which yields a partial reduced strategy. If Joe continues, thus surprising Isa, she decides on the spot what to do next. See Section 6.
tency among beliefs at different information sets and relate them to conditional probability systems.

We use the following notation: For any finite set \(\Omega\), interpreted as the space of uncertainty, let \(\Delta(\Omega)\) be the set of probability measures on \(\Omega\). Whenever the underlying space of uncertainty \(\Omega\) is understood, for all events \(E \subseteq \Omega\) we let \(\Delta(E) = \{\mu \in \Delta(\Omega) : \mu(E) = 1\}\).

### 3.1 Conditional probability systems

Let \(\Omega\) be a finite space of uncertainty. Fix a nonempty collection of “conceivable conditioning events” \(\mathcal{C} \subseteq 2^{\Omega}\setminus\{\emptyset\}\). We call the pair \((\Omega, \mathcal{C})\) a conditional space.\(^8\) For example, if we consider player \(i\) in a game, we have \(\Omega = S_{\neg i}\), and a natural collection of conditioning events is the observable events about others’ behavior \(\mathcal{H}_i := \{S_{\neg i}(h_i) \subseteq S_{\neg i} : h_i \in H_i\}\).

**Definition 2.** Fix a conditional space \((\Omega, \mathcal{C})\). An array of probability measures \(\mu = (\mu(\cdot|\mathcal{C}))_{\mathcal{C} \in \mathcal{C} \in \times} \in \mathcal{C} \in \Delta(\mathcal{C})\) is a conditional probability system (CPS) on \((\Omega, \mathcal{C})\), written \(\mu \in \Delta^*(\Omega)\), if it satisfies the chain rule: for all \(E \subseteq \Omega\), \(C, D \in \mathcal{C}\),

\[
E \subseteq D \subseteq C \Rightarrow \mu(E|C) = \mu(E|D) \mu(D|C).
\]

The CPSs on \((\Omega, 2^\Omega\setminus\{\emptyset\})\), whose set is denoted by \(\Delta^*(\Omega)\), are called complete.

In words, whenever possible, the beliefs for distinct conditioning events must be related to each other by the standard rules of conditional probability. Indeed, the chain rule is equivalent to requiring that, for all \(E \subseteq \Omega\) and \(C, D \in \mathcal{C}\) with \(E \subseteq D \subseteq C\),

\[
\mu(D|C) > 0 \Rightarrow \mu(E|D) = \mu(E|C) \mu(D|C).
\]

(Note that, since \(0 \leq \mu(E|C) \leq \mu(D|C)\), \(\mu(D|C) = 0\) implies that the chain-rule equality holds trivially as \(0 = 0\).)

\(^8\)If \(\Omega\) is infinite, consider a triple \((\Omega, \mathcal{B}, \mathcal{C})\), where \(\mathcal{B}\) is the relevant sigma-algebra and \(\mathcal{C} \subseteq \mathcal{B}\).
As anticipated, a natural representation of conditional beliefs in games is to consider CPSs on \((S_{-i}, H_i)\), where player \(i\) forms beliefs conditional on each observable event about the behavior of others (e.g., Battigalli & Siniscalchi 2002). Complete CPSs instead assume that players form their beliefs conditional on every possible event — for every \(C\), the player thinks: “If I knew \(C\), I would believe \(\mu^i(\cdot|C)\)” Event \(C\) need not represent information player \(i\) may obtain during the game, yet she can still ask herself this question and answer to it. Complete CPSs represent the coherent conditional beliefs a player would form if she asked herself this question for every possible event. This representation is used, e.g., in Battigalli (1996).

In general, a richer class of conditioning events gives more bite to the chain rule, because each event is related by set inclusion to more events. Specifically, consider \(C \subseteq D\) and let

\[
\text{proj}_{\Delta_\Omega} \Delta^D_\Omega = \left\{ \mu \in \Delta^C(\Omega) : \exists \bar{\mu} \in \Delta^D(\Omega), \forall C \in C, \mu(\cdot|C) = \bar{\mu} (\cdot|C) \right\}
\]

denote the set of CPSs on \((\Omega, C)\) that can be derived from some CPS on \((\Omega, D)\).

**Remark 2.** If \(C \subseteq D\), then \(\text{proj}_{\Delta_\Omega} \Delta^D_\Omega \subseteq \Delta^C_\Omega\).

Thus, complete CPSs deserve special attention in that they embody the most stringent restrictions on how players form conditional beliefs.

### 3.2 Systems of beliefs

We now turn to the main object of our analysis, belief systems. We assume that player \(i\) would hold at each of her information sets a belief \(\mu^i(\cdot|h_i)\) assigning positive probability only to strategies of co-players’ that are compatible with \(h_i\). Thus, players are endowed with systems of beliefs (often abbreviated in “belief systems”)

\[
\mu^i = (\mu^i(\cdot|h_i))_{h_i \in H_i} \in \times_{h_i \in H_i} \Delta(S_{-i}(h_i)).
\]
We discipline belief change with coherency properties. The first property is embodied in the definition of belief system: *knowledge implies probability-1 belief*, that is, players assign positive probability only to behavior of others consistent with their information. The second property concerns how players update or revise their beliefs across information sets; depending on these assumptions, we obtain different notions of belief systems.\textsuperscript{9} We consider three increasingly restrictive notions of consistency in updating/revision. The weakest one requires beliefs to be linked by the chain rule of conditional probabilities as one moves forward on a path, that is, if \( h'_i \) follows \( h_i \) and \( i \) deems \( h'_i \) reachable from \( h_i \), then beliefs at \( h'_i \) should be derived from beliefs at \( h_i \) by conditioning on \( S_{-i}(h'_i) \).

**Definition 3.** A system of beliefs \( \mu^i \) is **forward consistent**, written \( \mu^i \in \Delta^H_i(S_{-i}) \), if for all \( h_i, h'_i \in H_i \) with \( h_i \prec h'_i \), for all \( E_{-i} \subseteq S_{-i}(h'_i) \)

\[
\mu^i(S_{-i}(h'_i)|h_i) > 0 \Rightarrow \mu^i(E_{-i}|h'_i) = \frac{\mu^i(E_{-i}|h_i)}{\mu^i(S_{-i}(h'_i)|h_i)}.
\]

With a slight abuse of language, we often refer to the implication above as the **“forward chain rule.”** As discussed in the heuristic example of the Introduction, forward consistency allows players to form different beliefs at two information sets representing the same information on others’ behavior. This may happen if \( h_i \) and \( h'_i \) were both unexpected and differ only in \( i \)’s own behavior at the previous information set, so that \( S_{-i}(h_i) = S_{-i}(h'_i) \). In this case, \( h_i \) and \( h'_i \) do not precede each other, and so forward consistency allows a player to revise beliefs differently at \( h_i \) and \( h'_i \). This need not be viewed as a form of “irrationality.” Belief systems can be interpreted as an external observer’s description of the beliefs a player would hold in every circumstance. Players may not be fully introspective, and know only their current beliefs. We clarify in Section 6 that to form rational plans, players need not plan in advance.

\textsuperscript{9}There is an analogy between our systems of beliefs and those of Kreps & Wilson (1982). The latter assign conditional probabilities to nonterminal nodes, so that the probabilities of nodes in an information set sum to 1. Focusing on a single player, systems of beliefs in the sense of Kreps & Wilson can be derived from systems of beliefs in our sense. In both cases, across-information-sets restrictions are explicitly added by considering subsets of systems of beliefs.
how they would behave at unexpected contingencies, and hence need not think in advance about what they would believe in such occurrences. For some psychological reasons, players may then revise their beliefs in different ways depending on their past actions.

Yet, one may still want to assume that players do not form different beliefs about others depending on their own behavior. Our intermediate restriction on belief systems excludes such cases by deriving belief systems from CPSs on \((S_{-i}, \mathcal{H}_i)\). CPSs embody by construction the property that same knowledge on others’ behavior implies same beliefs. Indeed, if \(S_{-i}(h_i) = S_{-i}(h'_i)\), then \(h_i\) and \(h'_i\) correspond to the same element of \(\mathcal{H}_i\), and hence for a CPS \(\bar{\mu}^i\) we have by construction \(\bar{\mu}^i(\cdot | S_{-i}(h_i)) = \bar{\mu}^i(\cdot | S_{-i}(h'_i))\).

**Definition 4.** A system of beliefs \(\mu^i\) is standard, written \(\mu^i \in \Delta^{H_i} (S_{-i})\), if there exists a CPS \(\bar{\mu}^i \in \Delta^{\mathcal{H}_i} (S_{-i})\) such that \(\mu^i(\cdot | h_i) = \bar{\mu}^i(\cdot | S_{-i}(h_i))\) for every \(h_i \in H_i\).

(We call them “standard” because CPSs on \((S_{-i}, \mathcal{H}_i)\) have been widely used in the literature on strategic reasoning in games since Battigalli & Siniscalchi 1999.)

**Remark 3.** A belief system \(\mu^i\) is standard if and only if for all \(h_i, h'_i \in H_i\) with \(S_{-i}(h'_i) \subseteq S_{-i}(h_i)\), for all \(E_{-i} \subseteq S_{-i}(h'_i)\) we have

\[
\mu^i(S_{-i}(h'_i)|h_i) > 0 \Rightarrow \mu^i(E_{-i}|h'_i) = \frac{\mu^i(E_{-i}|h_i)}{\mu^i(S_{-i}(h'_i)|h_i)}.
\]

By perfect recall, \(h_i \prec h'_i\) implies \(S_{-i}(h'_i) \subseteq S_{-i}(h_i)\); thus, all standard belief systems are forward consistent.

As already noted, \(H_i\) typically has a larger cardinality than \(\mathcal{H}_i\) because information sets may also represent information about player \(i\)’s behavior, not just the behavior of the co-players \(-i\). For example, in game \(\Gamma'\) of Figure 1, \(H_i = \{\emptyset\}, \{(L, C)\}, \{(R, C)\}\) has three elements, while \(\mathcal{H}_i = \{S_j, \{C.\ell, C.r\}\}\) has two elements. Despite this, the set of standard belief systems \(\Delta^{H_i} (S_{-i})\) is isomorphic to \(\Delta^{\mathcal{H}_i} (S_{-i})\). Indeed, for every standard belief system \(\mu^i, S_{-i}(h''_i) = S_{-i}(h'_i)\) implies \(\mu^i(\cdot | h'_i) = \mu^i(\cdot | h''_i)\); with this, given a standard belief system \(\mu^i\), the array \((\bar{\mu}^i(\cdot | C))_{C \in \mathcal{H}_i}\) where \(\bar{\mu}^i(\cdot | S_{-i}(h_i)) = \mu^i(\cdot | h_i)\) for each \(h_i \in H_i\) defines uniquely a CPS.
Since $\mu_s (\cdot | h_i) \longleftrightarrow \tilde{\mu}_s (\cdot | S_{-i} (h_i))$ is a bijection between $\Delta^H_i (S_{-i})$ and $\Delta^{H_i} (S_{-i})$. In Section 4, we characterize standard belief systems in games with observable actions.

The strongest notion of consistency requires a belief system to be induced by some complete CPS.

**Definition 5.** A system of beliefs $\mu^i$ is **completely consistent**, written $\mu^i \in \Delta^H_i (S_{-i})$, if there exists a complete CPS $\tilde{\mu}^i \in \Delta^* (S_{-i})$ such that $\mu^i (\cdot | h_i) = \tilde{\mu}^i (\cdot | S_{-i} (h_i))$ for every $h_i \in H_i$.

By Remark 2, complete consistency is more restrictive than standard consistency. To give a sense of the strength of these restrictions, complete consistency implies that the odds ratio of any pair of strategy profiles of the co-players (if well-defined) is the same at all the information sets that are compatible with both strategy profiles.

**Remark 4.** If belief system $\mu^i$ is completely consistent then, for all $h_i, h'_i \in H_i$ and $s_{-i}, t_{-i} \in S_{-i} (h_i) \cap S_{-i} (h'_i)$,

$$\mu^i (t_{-i} | h_i), \mu^i (t_{-i} | h'_i) > 0 \Rightarrow \frac{\mu^i (s_{-i} | h_i)}{\mu^i (t_{-i} | h_i)} = \frac{\mu^i (s_{-i} | h'_i)}{\mu^i (t_{-i} | h'_i)}.$$

**Proof:** Consider a belief system $\mu^i$ derived from some complete CPS $\tilde{\mu}^i$. Fix any $h_i, h'_i \in H_i$ and $s_{-i}, t_{-i} \in S_{-i} (h_i) \cap S_{-i} (h'_i)$ such that $\mu^i (t_{-i} | h_i) > 0$ and $\mu^i (t_{-i} | h'_i) > 0$. Since $\tilde{\mu}^i$ is a complete CPS, the chain rule relates $\tilde{\mu}^i (\cdot | S_{-i} (h_i))$ and $\tilde{\mu}^i (\cdot | S_{-i} (h'_i))$ to $\tilde{\mu}^i (\cdot | \{s_{-i}, t_{-i}\})$. Hence,

$$\mu^i (s_{-i} | h_i) = \tilde{\mu}^i (s_{-i} | S_{-i} (h_i)) = \tilde{\mu}^i (s_{-i} | \{s_{-i}, t_{-i}\}) \cdot \tilde{\mu}^i (\{s_{-i}, t_{-i}\} | S_{-i} (h_i)),$$

$$\mu^i (s_{-i} | h'_i) = \tilde{\mu}^i (s_{-i} | S_{-i} (h'_i)) = \tilde{\mu}^i (s_{-i} | \{s_{-i}, t_{-i}\}) \cdot \tilde{\mu}^i (\{s_{-i}, t_{-i}\} | S_{-i} (h'_i)).$$

Since $\mu^i (t_{-i} | h_i), \mu^i (t_{-i} | h'_i) > 0$ we have $\tilde{\mu}^i (\{s_{-i}, t_{-i}\} | S_{-i} (h_i)) = \mu^i (\{s_{-i}, t_{-i}\} | h_i) > 0$ and so

$$\frac{\mu^i (s_{-i} | h_i)}{\mu^i (\{s_{-i}, t_{-i}\} | h_i)} = \tilde{\mu}^i (s_{-i} | \{s_{-i}, t_{-i}\}) = \frac{\mu^i (s_{-i} | h'_i)}{\mu^i (\{s_{-i}, t_{-i}\} | h'_i)}.$$
It can be analogously verified for $t_{-i}$ that

$$\frac{\mu^i(t_{-i}|h_i)}{\mu^i(s_{-i}, t_{-i})|h_i)} = \frac{\mu^i(s_{-i}|h'_i)}{\mu^i(s_{-i}, t_{-i})|h'_i)}.$$

The last two equations yield, dividing the former by the latter,

$$\frac{\mu^i(s_{-i}|h_i)}{\mu^i(t_{-i}|h_i)} = \frac{\mu^i(s_{-i}|h'_i)}{\mu^i(t_{-i}|h'_i)}.$$

Standard belief systems need not satisfy this property, as shown in Example 1 of Section 4. Yet, this necessary condition, although quite strong, is not sufficient for complete consistency. We provide in Section 5 an example of this fact, as well as the exact characterization of complete consistency in terms of coherency of odds ratios at different information sets. It is also worth noting that complete consistency is connected to Kreps & Wilson’s (1982) consistency in the following sense:

**Remark 5.** A system of beliefs $\mu^i$ is completely consistent if and only if there is a sequence of strictly positive probability measures $(\nu_n)_{n \in \mathbb{N}}$ on $S_{-i}$ such that

$$\forall h_i \in H_i, \forall s_{-i} \in S_{-i}(h_i), \mu^i(s_{-i}|h_i) = \lim_{n \to \infty} \frac{\nu_n(s_{-i})}{\nu_n(S_{-i}(h_i))}.$$

This is implied by Remark 10 in the Appendix. Kreps & Wilson (1982) put forward an across-players notion of consistency of assessments derived from Selten’s (1975) “trembling hand” idea. Remark 5 implies that, in two-person games, complete consistency is equivalent to a single-player version of Kreps & Wilson’s consistency. See the Appendix for details.

To summarize, we defined three nested subsets of belief systems (the inclusions may be strict):

$$\Delta^H_C(S_{-i}) \subseteq \Delta^H(S_{-i}) \subseteq \Delta^H_F(S_{-i}).$$
In the next sections, we shed further light on the interpretation of these belief systems by providing a characterization of standard consistency in games with observable actions and a characterization of complete consistency. Then we move to solution concepts, and we show that in spite of a considerable difference in the degree of consistency assumed by these types of belief systems, these differences are inconsequential for the analysis of strategic reasoning, as we obtain equivalent versions of rationalizability for sequential games.

4 Standard belief systems in games with observable actions

In this section, we provide a characterization of standard belief systems in games with observable actions. We show that a belief system is standard if and only if it is forward consistent and it assigns the same beliefs at histories that differ only in player $i$’s own action at the immediate predecessor —i.e., beliefs are independent of own behavior. Then we show that in perfect-information games, forward consistency and “standard” consistency coincide.

It is first convenient to study the structure of the “strategic-form information sets” $S_{-i}(h)$ ($h \in H$) in games with observable actions. In such games, non-terminal histories and information sets coincide; therefore, for every $h \in H$, we have $S(h) = \times_{i \in I} S_i(h)$. For each $i \in I$, $h \in H$, and $a_{-i} \in A_{-i}(h)$, let

$$S_{-i}(h, a_{-i}) = \{s_{-i} \in S_{-i}(h) : s_{-i}(h) = a_{-i}\}$$

denote the set of co-players’ strategy profiles consistent with $h$ that select $a_{-i}$ at $h$. For any $h', h'' \in H$, let $\pi(h', h'') \in H$ be the last common predecessor (longest common prefix) of $h'$ and $h''$. For any $h < h'$, let $\alpha(h, h') = (\alpha_j(h, h'))_{j \in I} \in A(h)$ be the unique action profile such that $(h, \alpha(h, h')) \preceq h'$, and let $f(h, h')$ be the immediate follower of $h$ (weakly) preceding $h'$. Note, by definition, $S_{-i}(f(h, h')) = S_{-i}(h, \alpha_{-i}(h, h'))$. The following lemma is crucial to characterize standard belief systems.
Lemma 1. In games with observable actions, if two histories \( h', h'' \in H \) are not ordered by precedence and \( \bar{h} \) is their longest common predecessor, then \( S_{-i}(h') \subseteq S_{-i}(h'') \) implies that 

(a) \( \alpha_{-i}(\bar{h}, h') = \alpha_{-i}(\bar{h}, h'') \) and 
(b) \( S_{-i}(h'') = S_{-i}(f(\bar{h}, h'')) = S_{-i}(f(\bar{h}, h')) \).

Proof: Take any two histories \( h', h'' \in H \) not ordered by precedence and consider their last common predecessor \( \bar{h} = \pi(h', h'') \). We first prove by contraposition that, if \( S_{-i}(h') \subseteq S_{-i}(h'') \), then only player \( i \) can be active at histories \( h \) such that \( \bar{h} \prec h \prec h'' \). Indeed, suppose that some player \( j \neq i \) is active at such \( h \); then, \( j \) has an action \( a^*_j \in A_j(h) \setminus \{\alpha_j(h, h'')\} \), which implies that there exists \( a^*_{-i} \in A_{-i}(h) \) such that \( a^*_{-i} \notin \alpha_{-i}(h, h'') \). Take any \( s^*_{-i} \in S_{-i} \) such that (i) for all \( \bar{h} \prec h' \), \( s_{-i}(\bar{h}) = \alpha_{-i}(\bar{h}, h') \), so that \( s^*_{-i} \in S_{-i}(h') \), and (ii) \( s_{-i}(h) = a^*_{-i} \), so that \( s^*_{-i} \notin S_{-i}(h'') \) (noting that \( h \not\prec h' \), this does not conflict with (i)). With this, there exists \( s^*_{-i} \in S_{-i}(h') \setminus S_{-i}(h'') \), that is, \( S_{-i}(h') \not\subseteq S_{-i}(h'') \).

Now suppose that \( S_{-i}(h') \subseteq S_{-i}(h'') \). Then \( \alpha_{-i}(\bar{h}, h') = \alpha_{-i}(\bar{h}, h'') \), otherwise \( S_{-i}(h') \) and \( S_{-i}(h'') \) would be disjoint. This implies \( S_{-i}(f(\bar{h}, h')) = S_{-i}(f(\bar{h}, h'')) \). Since only player \( i \) can be active at histories \( h \) with \( \bar{h} \prec h \prec h'' \), we have \( S_{-i}(h'') = S_{-i}(f(\bar{h}, h'')) \).

Using this result, we can characterize standard belief systems by means of a property of independence of beliefs from own behavior.

Definition 6. A system of beliefs \( \mu^i \) satisfies own-action independence (OI) if 

\[
\mu^i(\cdot|h') = \mu^i(\cdot|h'') \quad \text{for all} \quad h' = (h, a') \quad \text{and} \quad h'' = (h, a'') \quad \text{with} \quad a'_{-i} = a''_{-i}.
\]

Denote by \( \Delta^H_{\text{FC, OI}}(S_{-i}) \) and \( \Delta^H_{\text{OI}}(S_{-i}) \) the sets of forward consistent and standard belief systems that satisfy OI.

Remark 6. In games with observable actions, all standard belief systems satisfy OI, i.e., \( \Delta^H_{\text{OI}}(S_{-i}) = \Delta^H(S_{-i}) \). Indeed, if \( h' = (h, a') \) and \( h'' = (h, a'') \) are such that \( a'_{-i} = a''_{-i} \), then \( S_{-i}(h') = S_{-i}(h'') \). Then by Remark 3 the chain rule applies to the pair \( (h', h'') \), which implies, for equal conditioning events, equal conditional measures.

Theorem 1. In games with observable actions, a system of beliefs is standard if and only if it is forward consistent and satisfies OI, that is, \( \Delta^H(S_{-i}) = \Delta^H_{\text{FC, OI}}(S_{-i}) \).
**Proof:** The inclusion $\Delta^H(S_{i-1}) = \Delta^{H}_{OI}(S_{i-1}) \subseteq \Delta^{H/OI}(S_{i-1})$ is obvious. We show the other inclusion. Take any $\mu^i \in \Delta^{H/OI}(S_{i-1})$ and two histories $h', h'' \in H$ with $S_{i-1}(h') \subseteq S_{i-1}(h'')$. By Remark 3, it is enough to show that the forward chain rule relates $\mu^i(\cdot|h'')$ to $\mu^i(\cdot|h')$. If $h'' \preceq h'$ the forward chain rule relates $\mu^i(\cdot|h'')$ to $\mu^i(\cdot|h')$. If $h''$ and $h'$ are not related by precedence, then let $h = \pi(h', h'')$, $a' = \alpha(h, h')$ and $a'' = \alpha(h, h'')$. Since $S_{i-1}(h') \subseteq S_{i-1}(h'')$, Lemma 1 implies (a) $a'_{i-1} = a''_{i-1}$ and (b) $S_{i-1}(h') = S_{i-1}(h, a'') = S_{i-1}(h, a')$. Since $(h, a'') \preceq h''$, the forward chain rule relates $\mu^i(\cdot|(h, a''))$ to $\mu^i(\cdot|h'')$. Since $S_{i-1}(h'') = S_{i-1}(h, a'')$, this implies $\mu^i(\cdot|h'') = \mu^i(\cdot|(h, a''))$. Since $a'_{i-1} = a''_{i-1}$ and $\mu^i$ satisfies OI, we have $\mu^i(\cdot|(h, a'')) = \mu^i(\cdot|(h, a'))$. Finally, since $(h, a') \preceq h'$, the forward chain rule relates $\mu^i(\cdot|(h, a'))$ to $\mu^i(\cdot|h')$. Thus, $\mu^i(\cdot|h'') = \mu^i(\cdot|(h, a'')) = \mu^i(\cdot|(h, a'))$ and the forward chain rule indirectly relates $\mu^i(\cdot|h'')$ to $\mu^i(\cdot|h')$. 

This characterization is useful in constructive proofs when working with “standard” CPSs in games with observable actions: it is often easier to show that a system of beliefs satisfies the forward chain rule and OI than showing directly standard consistency.

Note, OI is key to be able to conclude that $\mu^i(\cdot|(h, a')) = \mu^i(\cdot|(h, a''))$ in the proof. Otherwise, if $\mu^i(S_{i-1}(h, a'_{i-1})|h) = 0$, forward consistency does not guarantee that $i$ revises her beliefs at $(h, a')$ and $(h, a'')$ in the same way and hence we would not be guaranteed that the forward chain rule indirectly relate $\mu^i(\cdot|h')$ and $\mu^i(\cdot|h'')$.

On the other hand, a forward consistent belief system $\mu^i$ may violate OI at two histories $h' = (h, a')$, $h'' = (h, a'')$ with $a'_{i-1} = a''_{i-1}$ only if (a) $a'_i \neq a''_i$, which implies that $i$ is active at $h$, and (b) $\mu^i(S_{i-1}(h, a'_{i-1})|h) = 0$, which implies that at least one co-player of $i$ is active at $h$, since otherwise $S_{i-1}(h, a'_{i-1}) = S_{i-1}(h)$ and thus $\mu^i(S_{i-1}(h, a'_{i-1})|h) = \mu^i(S_{i-1}(h)|h) = 1 > 0$. So, forward consistent belief systems might feature own-action dependence only in games where there are simultaneous moves at some stage, which yields the following result:

**Corollary 1.** In games with perfect information, a system of beliefs is standard if and only if it is forward consistent: $\Delta^H(S_{i-1}) = \Delta^{H/OI}(S_{i-1})$.

Since these results are independent of players’ interactive knowledge of the profile of
payoff functions, they extend to games with incomplete information and observable actions.

As for complete consistency, we typically have that it is stronger than forward/standard consistency, even in games with perfect information, as shows the example below.

![Figure 4:](image)

**Example 1.** Consider histories \( h' = (L, d') \) and \( h'' = (R, d'') \) in the perfect-information game structure depicted in Figure 4. For strategies \( s_{-i} = d'.d''.\ell, s'_{-i} = d'.d''.r, \) and \( s^*_{-i} = a'.a''.\ell \) (where \(-i = j\) is Joe), we have \( s_{-i}, s'_{-i} \in S_{-i}(h') \cap S_{-i}(h'') \), \( s^*_{-i} \not\in S_{-i}(h') \), and \( s^*_{-i} \not\in S_{-i}(h'') \). Take a forward consistent (hence, by Corollary 1, *standard*) belief system \( \mu^i \) of Isa such that \( \mu^i(s^*_{-i}|\emptyset) = 1 \). Isa’s beliefs after \( L \) and \( R \) coincide with her initial belief, because Joe is not active at the root. It follows that \( \mu^i(S_{-i}(h')|L) = 0 \) and \( \mu^i(S_{-i}(h'')|R) = 0 \), so that Isa has to revise her beliefs at both \( h' \) and \( h'' \). Since \( S_{-i}(h') \) and \( S_{-i}(h'') \) are different and not nested, we can set \( \mu^i(s_{-i}|h') = 1 \) and \( \mu^i(s'_{-i}|h'') = 1 \). But this violates the odds-ratio property of Remark 4, which would require in this specific case that \( \mu^i(s_{-i}|h') = \mu^i(s_{-i}|h'') \).

Thus, \( \mu^i \) is a standard belief system that is not completely consistent. ▲

5 A characterization of complete consistency

In this section, we show that complete consistency is characterized by a strong coherency condition relating probability ratios across different information sets.
We already noted that complete consistency implies constant odds ratios between pair of opponents’ strategy profiles at all information sets consistent with both strategy profiles. Yet, this condition is not sufficient for complete consistency, as shown by the following example.

Figure 5:

Example 2. Consider the game structure depicted in Figure 5. Note that there is no pair \( (h_i, h'_i) \) of third-stage information sets of Isa such that \( S_j(h_i) \cap S_j(h'_i) \) contains two distinct actions/strategies of Joe. Thus, if belief system \( \mu^i \) is forward consistent it also satisfies the “odds-ratio” property of Remark 4, which applies only if either \( h_i \) or \( h'_i \), is the initial information set of Isa. Suppose that Isa is initially certain of termination, that is, \( \mu^i(t|\emptyset) = 1 \), then forward consistency and the “odds-ratios” property are trivially satisfied by \( \mu^i \). Yet, \( \mu^i \) need not be completely consistent. To see this, let \( xy = \{(x, xy), (y, xy)\} \) denote the third-stage information set of Isa after she chose \( xy \) and Joe chose either \( x \) or \( y \) (both different from \( t \)). Since \( \mu^i(t|\emptyset) = 1 \), forward consistency puts no constraints on beliefs at such information sets. Then we may have \( \mu^i(a|[ab]) > \frac{1}{2} \), \( \mu^i(b|[bc]) > \frac{1}{2} \), and \( \mu^i(c|[ac]) > \frac{1}{2} \), which violates complete consistency. Indeed, any complete CPS \( \tilde{\mu}^i \in \Delta^*(S_j) \) defines \( i \)'s belief conditional on \( \{a, b, c\} \subset S_j \), and hence the chain rule relates beliefs conditional on \( \{a, b\}, \{b, c\} \) and \( \{a, c\} \) to \( \tilde{\mu}^i(\cdot|\{a, b, c\}) \). ▲
A standard odds ratio is computed using the conditional probabilities of two elements $s_{-i}$ and $s'_{-i}$ at the same information set. As such, it does not allow to compare the probabilities of $s_{-i}$ and $s'_{-i}$ at different information sets. This comparison is made possible by the following notion of “generalized odds ratio.”

**Definition 7.** Given a belief system $(\mu^i(\cdot|h_i))_{h_i \in H_i}$ and a pair $s_{-i}, s'_{-i} \in S_{-i}$, a (possibly infinite) **generalized odds ratio** between $s_{-i}$ and $s'_{-i}$ is the product of a finite concatenation of odds ratios, namely:

$$q = \frac{\mu^i(s_{-i}^1|h_1)}{\mu^i(s'_{-i}^1|h_1)} \cdot \frac{\mu^i(s_{-i}^2|h_2)}{\mu^i(s'_{-i}^2|h_2)} \cdot \cdots \cdot \frac{\mu^i(s_{-i}^{n_i-1}|h_{n_i})}{\mu^i(s'_{-i}^{n_i}|h_{n_i})},$$

where $s_{-i}^k, s'_{-i}^k \in S_{-i}(h_k)$ for all $k \in \{1, \ldots, n\}$, with $s_{-i}^0 = s_{-i}, s_{-i}^n = s'_{-i}$, and if one odds ratio is infinite, no odds ratio is zero.

For instance, if player $i$ deems $s_{-i}$ twice more likely than another strategy $s'_{-i}$ at some information set where both $s_{-i}$ and $s'_{-i}$ are still possible, and deems $s'_{-i}$ twice more likely than $s''_{-i}$ at some other information set where $s'_{-i}$ and $s''_{-i}$ are still possible, then we can say that there is a generalized odds ratio of 4 between $s_{-i}$ and $s''_{-i}$ in player $i$’s belief system.

A generalized odds ratio can be zero or infinite (but not indeterminate). A zero generalized odds ratio between $s_{-i}$ and $s'_{-i}$ can be given the following interpretation: If player $i$ was to make an indirect assessment of the relative likelihood of $s_{-i}$ and $s'_{-i}$ by looking at the relative likelihoods between these and other strategies at different information sets, she must conclude that $s_{-i}$ has zero probability whenever $s'_{-i}$ is deemed possible. Whether she actually does so depends on the degree of consistency in her belief system. Complete consistency implies that all generalized odds ratios for any pair of co-players’ strategy profiles are equal.\(^{10}\)

**Theorem 2.** A belief system $\mu^i$ is completely consistent if and only if for every pair $s_{-i}, s'_{-i} \in S_{-i}$, all generalized odds ratios of $\mu^i$ between $s_{-i}$ and $s'_{-i}$ are identical.

\(^{10}\)The (2020) working-paper version of Siniscalchi (2021) provides an alternate characterization of complete consistency closely connected to ours.
Proof. The part showing that constant generalized odds ratios (GORs) given \( \mu^i \) implies complete consistency of \( \mu^i \) is rather technical and is proved by Catonini (2021). We provide a direct proof that complete consistency implies constant GORs. Fix a pair \((s_{-i}, s'_{-i})\) and consider two GORs

\[
q^1 = \frac{\mu^i(s_{-i}|h^1_i)}{\mu^i(s'_{-i}|h^1_i)} \cdot \frac{\mu^i(s^1_{-i}|h^2_i)}{\mu^i(s^1_{-i}|h^2_i)} \cdot \ldots \cdot \frac{\mu^i(s^{n-1}_{-i}|h^n_i)}{\mu^i(s'_{-i}|h^n_i)},
\]

\[
q^2 = \frac{\mu^i(s_{-i}|h^m_i)}{\mu^i(s'_{-i}|h^m_i)} \cdot \frac{\mu^i(s^{m-1}_{-i}|h^{m-1}_i)}{\mu^i(s^{m-1}_{-i}|h^{m-1}_i)} \cdot \ldots \cdot \frac{\mu^i(s^{n+1}_{-i}|h^{n+1}_i)}{\mu^i(s'_{-i}|h^{n+1}_i)},
\]

where we use a labeling convention so that \( m > n \). We want to show that \( q^1 = q^2 \). If they are both zero or both infinite, we are done. Otherwise, suppose without loss of generality that \( q^2 \geq q^1 \), so that \( q^1 \neq \infty \) and \( q^2 > 0 \). Calling \( s^n_{-i} = s'_{-i} \) and \( s^0_{-i} = s_{-i} \), let

\[
q := \frac{q^1}{q^2} = \frac{\mu^i(s^0_{-i}|h^1_i)}{\mu^i(s_{-i}|h^1_i)} \cdot \ldots \cdot \frac{\mu^i(s^{n-1}_{-i}|h^n_i)}{\mu^i(s_{-i}|h^n_i)} \cdot \frac{\mu^i(s^n_{-i}|h^{n+1}_i)}{\mu^i(s'_{-i}|h^{n+1}_i)} \cdot \ldots \cdot \frac{\mu^i(s^{m-1}_{-i}|h^{m+1}_i)}{\mu^i(s^0_{-i}|h^{m+1}_i)}.
\]

Thus, all denominators are positive. Clearly, \( q^1 = q^2 \) if \( q = 1 \). To show \( q = 1 \), we proceed as follows: Let \( \bar{\mu}^i = (\bar{\mu}^i(\cdot|h_i))_{2^{|S_{-i}\{\emptyset}\}} \) be a complete CPS that extends \( \mu^i \). Let

\[
C = \cup_{k=1}^m S_{-i}(h^k_i).
\]

Since \( \bar{\mu}^i(C|C) = 1 \), there is \( k \in \{1, ..., m\} \) such that \( \bar{\mu}^i(S_{-i}(h^k_i)|C) > 0 \). Since \( \bar{\mu}^i(s^k_{-i}|S_{-i}(h^k_i)) = \mu^i(s^k_{-i}|h^k_i) > 0 \), by the chain rule \( \bar{\mu}^i(s^k_{-i}|C) > 0 \). By definition of the odds ratios, \( s^k_{-i} \in S_{-i}(h^{k+1}_i) \) (whenever \( k + 1 \leq m \)), hence \( \bar{\mu}^i(S_{-i}(h^{k+1}_i)|C) > 0 \). Expanding this reasoning we obtain that \( \bar{\mu}^i(S_{-i}(h^p_i)|C) > 0 \) for every \( p = k, ..., m \). Now, note that \( s^m_{-i} = s^0_{-i} \), and that \( s^m_{-i} \in S_{-i}(h^1_i) \). So, \( \bar{\mu}^i(S_{-i}(h^m_i)|C) > 0 \) implies \( \bar{\mu}^i(s^0_{-i}|C) > 0 \) by the chain rule with \( \bar{\mu}^i(\cdot|S_{-i}(h^m_i)) \) and \( \bar{\mu}^i(s^0_{-i}|S_{-i}(h^m_i)) > 0 \). Thus, \( \bar{\mu}^i(S_{-i}(h^1_i)|C) > 0 \), and repeating the above argument we have \( \bar{\mu}^i(S_{-i}(h^p_i)|C) > 0 \) for all \( p = 1, ..., m \). Therefore, by the chain rule

\[
\frac{\mu^i(s^{p-1}_{-i}|h^p_i)}{\mu^i(s^p_{-i}|h^p_i)} = \frac{\bar{\mu}^i(s^{p-1}_{-i}|C)}{\bar{\mu}^i(s^p_{-i}|C)}.
\]
Hence,

\[ q = \frac{\bar{\mu}_i(s_0^i|C)}{\bar{\mu}_i(s_1^i|C)} \cdot \frac{\bar{\mu}_i(s_1^i|C)}{\bar{\mu}_i(s_2^i|C)} \cdot \ldots \cdot \frac{\bar{\mu}_i(s_{m-1}^i|C)}{\bar{\mu}_i(s_m^i|C)} = 1. \]

Thus, complete consistency assumes a very high degree of coherency concerning how players revise their beliefs at different information sets. Yet this coherency need not be viewed as a consequence of conscious calculations of a fully introspective player about his conditional beliefs. A player may still be partially introspective about how she would revise her beliefs at unexpected contingencies, yet complete consistency could be a “wired-in” property of belief formation. Catonini (2021) relates complete consistency to the possibility of being “dutch-booked”, suggesting that individuals violating it may be willing to undertake detrimental courses of actions.

6 Rational planning

In this section, we define standard criteria of optimality for strategies given a system of beliefs and develop a novel interpretation in terms of “partial planning”, which does not require players to be fully introspective about their conditional beliefs, but only requires players to know their current beliefs. We then relate this interpretation to the degree of consistency that belief systems may feature.

Recall that \( H_i(s_i) \) is the collection of information sets that may occur if \( i \) implements strategy \( s_i \). For any \( s_i, h_i \in H_i(s_i) \), and belief system \( \mu^i \), the conditional expected payoff of \( s_i \) given \( h_i \) under \( \mu^i \) is

\[ U_i(s_i, \mu^i(\cdot|h_i)) = \sum_{s'_{-i} \in S_{-i}(h_i)} u_i(\zeta(s_i, s'_{-i}))\mu^i(s'_{-i}|h_i). \]

**Remark 7.** Fix \( s_i, \mu^i, \) and \( h_i \in H_i(s_i) \) arbitrarily. By Remark 1, expected payoff \( U_i(s_i, \mu^i(\cdot|h_i)) \) is independent of how \( s_i \) is specified at information sets that cannot occur
under $s_i$: $U_i(s_i, \mu^i(\cdot|h_i)) = U_i(s'_i, \mu^i(\cdot|h_i))$ for every $s'_i$ that selects the same actions as $s_i$ on the sub-domain $H_i(s_i)$, that is, for every $s'_i \equiv s_i$.

For every $h_i \in H_i$, let $\rho_i[s_i/h_i]$ denote the “$h_i$-replacement” strategy that, at every $h'_i \prec h_i$, chooses the unique action $\alpha_i(h'_i, h_i)$ leading from $h'_i$ toward $h_i$, and coincides with $s_i$ at all other information sets, that is,

$$
\forall h'_i \in H_i, \rho_i[s_i/h_i](h'_i) = \begin{cases} 
\alpha_i(h'_i, h_i) & \text{if } h'_i \prec h_i, \\
\quad s_i(h'_i) & \text{if } h'_i \not\prec h_i.
\end{cases}
$$

By definition, $\rho_i[s_i/h_i] \in S_i(h_i)$ and $U_i(\rho_i[s_i/h_i], \mu^i(\cdot|h_i))$ is well posed for all $s_i \in S_i$ and $h_i \in H_i$. The most demanding notion of optimality of behavior consists in planning and implementing a strategy that maximizes expected payoff starting from every information set.

**Definition 8.** A strategy $\bar{s}_i$ is **sequentially optimal** under belief system $\mu^i$, written $\bar{s}_i \in BR^*_i(\mu^i)$, if

$$
\forall h_i \in H_i, \forall s'_i \in S_i(h_i), U_i(\rho_i[\bar{s}_i/h_i], \mu^i(\cdot|h_i)) \geq U_i(s'_i, \mu^i(\cdot|h_i)).
$$

Note that, if belief system $\mu^i$ is not forward consistent, the set of sequentially optimal strategies under $\mu^i$ may be empty because of conflicts between expected-payoff maximization at different ordered information sets.

**Example 3.** In game $\Gamma'''$ of Figure 3, the chain rule implies that Isa must hold the same belief about Joe at the first and second node, because at both nodes she has no information about Joe. Thus, if she initially believes $\mu^i (\ell|\emptyset) > \frac{1}{2}$ and the chain rule holds, strategy $c'.c''$ is sequentially optimal. If instead $\mu^i (\ell|\emptyset) > \frac{1}{2}$ and $\mu^i (\ell|c') < \frac{1}{3}$, violating the chain rule, then the maximization problem at the root is still solved by $c'.c''$, but the one at history/node $(c')$ is solved by $c'.d''$, and no strategy is sequentially optimal under $\mu^i$. ▲
The aforementioned issue of conflicting conditional preferences is circumvented by an “intra-personal equilibrium” property that can always be satisfied in finite games with perfect recall: one-step optimality. For any strategy \( s_i \in S_i \), information set \( h_i \in H_i \), and action \( a_i \in A_i(h_i) \), let \( \lambda_i [s_i/h_i, a_i] \) denote the “local \((h_i, a_i)\)-replacement” strategy of \( i \) that chooses \( a_i \) at \( h_i \) and coincides with \( \rho_i[s_i/h_i] \) at all other information sets of \( i \), that is,

\[
\forall h_i' \in H_i, \lambda_i [s_i/h_i, a_i] (h_i') = \begin{cases} 
    a_i & \text{if } h_i' = h_i, \\
    \alpha_i(h_i', h_i) & \text{if } h_i' \prec h_i, \\
    s_i(h_i') & \text{if } h_i' \not\succ h_i.
\end{cases}
\]

**Definition 9.** A strategy \( \bar{s}_i \in S_i \) is **one-step optimal** under belief system \( \mu^i \) if

\[
\forall h_i \in H_i, \forall a_i \in A_i(h_i), \ U_i(\rho_i[\bar{s}_i/h_i], \mu^i(\cdot|h_i)) \geq U_i(\lambda_i[\bar{s}_i/h_i, a_i], \mu^i(\cdot|h_i)).
\]

One-step optimality can be interpreted as the result of a *folding-back planning* algorithm: A fully introspective player \( i \) who knows her belief system \( \mu^i \) first considers the information sets \( h_i \) with height 0 in \( H_i \), that is, those where she makes a last move;\(^\text{11}\) for any such information set \( h_i \), she plans to choose an expected-payoff-maximizing action given belief \( \mu^i(\cdot|h_i) \) about the co-players (breaking ties arbitrarily). In step \( \ell > 0 \) of the algorithm, as she considers any information set \( h_i \) with height \( \ell \) in \( H_i \), that is, one that followed by at most \( \ell \) moves of her, she plans to choose an expected-payoff-maximizing action given belief \( \mu^i(\cdot|h_i) \) about the co-players and the prediction that she would behave as already planned in earlier steps of the algorithm at the following information sets of height \( k < \ell \) (again, breaking ties arbitrarily). Note that this algorithm is equivalent to one-step optimality even if the chain rule does not hold.

**Example 4.** Go back to game \( \Gamma''' \). If \( \mu^i(\ell|\emptyset) > \frac{1}{2} \) and \( \mu^i(\ell|c') < \frac{1}{4} \), the only one-step

\(^{11}\)For the sake of this discussion, the distinction between information sets where \( i \) is active or inactive is immaterial.
optimal strategy is $d'.d''$: Isa understands how she would change her beliefs and that she
would choose $d''$ at the second node, if reached; thus, she chooses $d'$ at the root. In other
words, $d''$ is just a conditional prediction of Isa about herself, not something that she initially
deems optimal conditional on $c'$. ▲

Finiteness and perfect recall imply that a one-step optimal (pure) strategy can always be
found by folding back:

**Remark 8.** For every belief system $\mu^i$, at least one strategy is one-step optimal under
$\mu^i$.

When the chain rule holds, conditional preferences are dynamically consistent and—by a relatively standard dynamic programming argument—one can show that a strategy is
sequentially optimal if and only if can be obtained by folding-back planning. This implies that
sequential and one-step optimality are equivalent, a result known as the *one-shot-deviation principle*. Notably, Perea (2002) proves that the one-shot-deviation principle holds also for
the weakest form of chain rule considered here, forward consistency.\(^{12}\)

**Theorem 3.** (Perea 2002) For every forward consistent belief system $\mu^i$, a strategy is
sequentially optimal under $\mu^i$ if and only if it is one-step optimal under $\mu^i$.

Clearly, since $\Delta^H_i(S_{-i}) \subseteq \Delta^{H_{i}}(S_{-i}) \subseteq \Delta^{F_{i}}(S_{-i})$, the one-shot deviation principle also
holds for standard and completely consistent belief systems. Intuitively, the reason why the
equivalence between sequential and one-step optimality holds despite possible violations of
strong versions of the chain rule is the following. Under a forward consistent belief system,
player $i$ may hold conflicting beliefs at information sets $h_i'$ and $h''_i$ only if they follow (a)
different actions of her and (b) unexpected actions of the co-players, such as $h'_i = \{(L, C)\}$
and $h''_i = \{(R, C)\}$ for Isa in game $\Gamma'$, if she initially assigns probability 0 to action $C$ of Joe.
In this case, the decision problems of $i$ at $h'_i$ and $h''_i$ are mutually independent; furthermore,

\(^{12}\) Perea shows the result for two-person games, but the extension to $n$-person games is straightforward. The
proof is available upon request.
from the perspective of earlier information sets (the root for Isa in \( \Gamma' \)), planning at \( h'_i \) and \( h''_i \) does not affect expected payoffs, because \( i \) deems both \( h'_i \) and \( h''_i \) unreachable.

Theorem 3 and Remark 8 imply the following:

**Corollary 2.** For every forward consistent belief system \( \mu^i \), at least one strategy is sequentially optimal under \( \mu^i \), that is, \( BR^*_i(\mu^i) \neq \emptyset \).

We now turn to a seemingly less demanding notion of rationality.

**Definition 10.** A strategy \( \bar{s}_i \) is **weakly sequentially optimal** under belief system \( \mu^i \), written \( \bar{s}_i \in BR_i(\mu^i) \), if

\[
\forall h_i \in H_i(\bar{s}_i), \forall s_i \in S_i(h_i), \quad U_i(\bar{s}_i, \mu^i(\cdot|h_i)) \geq U_i(s_i, \mu^i(\cdot|h_i)).
\]

Like sequential optimality, also weak sequential optimality may be unsatisfiable if \( \mu^i \) is not forward consistent. Focusing on forward consistent belief systems, weak sequential optimality can be given the following “forward-planning” interpretation. Suppose player \( i \) forms her beliefs according to \( \mu^i \in \Delta^H_{\mathcal{F}}(S_{-i}) \) as the play unfolds, although she may not know how she would revise her beliefs upon observing unexpected information, i.e., she may not be fully introspective. At the root, she plans to follow a (possibly partial) strategy that maximizes her expected payoff restricted to the collection of information sets \( H_i(\mu^i|\emptyset) \) she deems possible under \( \mu^i(\cdot|\emptyset) \). Let \( \bar{s}_{i,\emptyset} \) be such a strategy. One way to compute \( \bar{s}_{i,\emptyset} \) is to perform a folding-back planning algorithm on the restricted collection \( H_i(\mu^i|\emptyset) \), that is, starting from information sets that are terminal within \( H_i(\mu^i|\emptyset) \), and then focusing on behavior in the subcollection of information sets in \( H_i(\mu^i|\emptyset) \) that are possible if such partial plan is implemented; indeed, by Remark 7, the specification of \( \bar{s}_{i,\emptyset} \) at information sets in \( H_i(\mu^i|\emptyset) \) that cannot occur under \( \bar{s}_{i,\emptyset} \) is immaterial for the maximization problem. Since conditional beliefs within \( H_i(\mu^i|\emptyset) \) are obtained by updating, player \( i \) would have no incentive to deviate from \( \bar{s}_{i,\emptyset} \) at any \( h_i \in H_i(\mu^i|\emptyset) \).
Example 5. Consider again game $\Gamma''$ and let $\mu^i$ be a forward consistent belief system of Isa such that $\mu^i(\ell|\emptyset) < \frac{1}{2}$, then it is optimal to go down immediately, $\bar{s}_{a,\emptyset}(\emptyset) = d'$, $(c')$ is inconsistent with $\bar{s}_{a,\emptyset}$, and the specification of $\bar{s}_{i,\emptyset}$ at $(c')$ does not matter. ▲

If, implementing $\bar{s}_{i,\emptyset}$, player $i$ unexpectedly obtains information $h_i$, where $h_i$ is a first follower in $H_i \setminus H_i(\mu^i|\emptyset)$ of the information sets in $H_i(\mu^i|\emptyset)$, then $i$ comes up with a new belief $\mu^i(\cdot|h_i) \in \Delta(S_{-i}(h_i))$ and plans to follow a (possibly partial) continuation strategy $\bar{s}_{i,h_i}$ within the collection $H_i(\mu^i|h_i)$ of information sets weakly following $h_i$ that she deems possible under $\mu^i(\cdot|h_i)$. If an hypothetical external observer knew $\mu^i$ (and how $i$ breaks ties), she could determine $\bar{s}_i$ for all information sets consistent with $\bar{s}_i$, that is, those in $H_i(\bar{s}_i)$, and $\bar{s}_i$ would be weakly sequentially optimal under $\mu^i$.13

The following result shows that the two notions of planning – weak sequential optimality and sequential optimality – are behaviorally equivalent.

Lemma 2. A strategy is weakly sequentially optimal under a forward consistent belief system $\mu^i$ if and only if it is behaviorally equivalent to some strategy that is sequentially optimal under $\mu^i$.

Proof: Suppose that $\bar{s}_i$ is weakly sequentially optimal under $\mu^i \in \Delta_{\mathcal{F}}^H_i(S_{-i})$. By Corollary 2, there is some strategy $\hat{s}_i$ that is sequentially optimal under $\mu^i$. Let $\bar{s}_i'$ denote the strategy coincides with $\bar{s}_i$ on $H_i(\bar{s}_i)$ and with $\hat{s}_i$ on $H_i \setminus H_i(\bar{s}_i)$. By construction, $\bar{s}_i'$ is behaviorally equivalent to $\bar{s}_i$. We must show that $\bar{s}_i'$ is sequentially optimal under $\mu^i$.

Since $\bar{s}_i$ is weakly sequentially optimal under $\mu^i$, Remark 7 implies that

$$\forall h_i \in H_i(\bar{s}_i), \forall s_i \in S_i(h_i), \quad U_i(\bar{s}_i', \mu^i(\cdot|h_i)) = U_i(\bar{s}_i, \mu^i(\cdot|h_i)) \geq U_i(s_i, \mu^i(\cdot|h_i)).$$

13 The strategies obtained by “folding-back” on the collections $H_i(\mu^i|h_i)$ of information sets deemed possible by beliefs $\mu^i(\cdot|h_i)$ ($h_i \in H_i$) form a subset of $BR_i(\mu^i)$, those weakly sequentially optimal under $\mu^i$ (and a superset of $BR^*_i(\mu^i)$). But, for each one of them, there is a structurally equivalent strategy in $BR_i(\mu^i)$. We omit the details.
Since \( \hat{s}_i \) is sequentially optimal under \( \mu^i \) and \( \bar{s}'_i \) coincides with \( \hat{s}_i \) on \( H_i \setminus H_i(\bar{s}_i) \),

\[
\forall h_i \in H_i \setminus H_i(\bar{s}_i), \forall s_i \in S_i(h_i), \quad U_i(\rho_i[\bar{s}'_i/h_i], \mu^i(\cdot|h_i)) = U_i(\rho_i[\hat{s}_i/h_i], \mu^i(\cdot|h_i)) \geq U_i(s_i, \mu^i(\cdot|h_i)).
\]

It follows that \( \bar{s}'_i \) is sequentially optimal under \( \mu^i \). ■

Clearly, the equivalence result also holds for standard and completely consistent belief systems.

**Example 6.** Strategy \( R.a'.a'' \) of Isa in game \( \Gamma' \) of Figure 1 is not sequentially optimal under any standard belief system, because there is no common value \( \mu^i(C.\ell|L,C) = \mu^i(C.\ell|R,C) \) that makes \( a' \) a best reply given \( L,C \) (which requires \( \mu^i(C.\ell|L,C) \leq \frac{3}{5} \)), and \( a'' \) a best reply given \( R,C \) (which requires \( \mu^i(C.\ell|R,C) \geq \frac{1}{2} \)). Yet, there is a continuum of standard belief systems that make the behaviorally equivalent strategy \( R.d'.a'' \) sequentially optimal, including all the \( \mu^i \in \Delta^{H_i}(S_{-i}) \) with \( \mu^i(\{Q.\ell,Q.r\}) = 1 \) and \( \mu^i(C.\ell|L,C) = \mu^i(C.\ell|R,C) \geq \frac{1}{2} \) (there are more if \( \varepsilon > 0 \)). ▲

To summarize, the analysis and discussion of this section clarifies two points:

1. In order to rationally decide what to do at an information set \( h_i \), player \( i \) only has to know her current belief at \( h_i \) and what she should believe at followers she deems possible given \( h_i \), that is, information sets \( h'_i \in H_i(\mu^i|h_i) \). The resulting plan is dynamically consistent (optimal at each \( h'_i \in H_i(\mu^i|h_i) \), or at least the elements of \( H_i(\mu^i|h_i) \) possible under the plan itself) as long as the forward chain rule holds. Therefore, partial introspection and the forward chain rule embedded in forward consistent belief systems are sufficient for rational planning.

2. **Forward planning** (iterated whenever a player is surprised, as explained above) determines reduced rather than full strategies, as anticipated in Section 2.
7 Belief change and strategic reasoning: equivalence results

In this section, we argue that the behavioral implications of rationality and different versions of “common belief in rationality” do not depend on coherency restrictions of conditional beliefs across information sets beyond those implied by forward consistency. The literature on the epistemic foundations of solution concepts shows that such behavioral implications are characterized by extensions of the rationalizability idea from simultaneous-move to sequential games.

7.1 Strong rationalizability

The idea of common belief in rationality has been extended from simultaneous-move games to sequential games in different ways, depending on how players are assumed to revise their beliefs after unexpected moves. In this section, we analyze Strong Rationalizability,\(^\text{14}\) which is based on the idea that players “strongly believe” in the rationality and strategic sophistication of co-players. A player strongly believes an event if she would be always certain of that event unless observing evidence in direct contradiction with it. Strong Rationalizability captures the implications of rationality and common strong belief in rationality, a formalization of the “best rationalization principle”: players always ascribe to their co-players the highest degree of strategic sophistication consistent with their past behavior, even when surprised by their behavior. This form of reasoning is also often referred to as “forward-induction.” See Battigalli (1996) and Battigalli & Siniscalchi (2002).

A belief system \(\mu^i\) strongly believes event \(E_{-i} \subseteq S_{-i}\) if \(\mu^i(E_{-i}|h_i) = 1\) for every \(h_i \in H_i\) such that \(E_{-i} \cap S_{-i}(h_i) \neq \emptyset\). Similarly, a complete CPS \(\mu^i \in \Delta^* (S_{-i})\) strongly believes \(E_{-i}\) if \(\mu^i(E_{-i}|C_{-i}) = 1\) for every conditioning event \(C_{-i} \subseteq S_{-i}\) such that \(C_{-i} \cap E_{-i} \neq \emptyset\). We let

\(^{14}\)Strong Rationalizability used to be called “extensive-form rationalizability” (Pearce 1984, Battigalli 1997). We avoid this terminology because there are different ways to define “rationalizability” for the extensive-form analysis of sequential games; see, e.g., Initial Rationalizability, which captures rationality and common initial belief in rationality (Ben-Porath 1997, Battigalli & Siniscalchi 1999) and Backwards Rationalizability, which we analyze in the next section.
$SB_i(E_{-i})$ denote the set of either belief systems or complete CPSs (which one will be clear from the context) that strongly believe $E_{-i}$.

To define Strong Rationalizability, it is convenient to use weak sequential optimality as our notion of best reply.\footnote{Battigalli & De Vito (2021) define strong rationalizability and provide epistemic justifications using the notion of sequential best reply to a (standard) belief system. They discuss the equivalence between the two approaches.} When we consider a complete CPS $\bar{\mu}^i \in \Delta^* (S_{-i})$, with a slight abuse of notation, we let $BR_i(\bar{\mu}^i)$ denote the set of strategies of $i$ that are weakly sequentially optimal under the (completely consistent) belief system $\mu^i$ derived from $\bar{\mu}^i$ (that is, with $\mu^i(\cdot|H_i) = \bar{\mu}^i(\cdot|S_{-i}(H_i))$ for all $H_i \in H_i$). Depending on which kind of beliefs we consider for the definition, we let $\Delta_i$ denote either forward consistent, or standard, or completely consistent belief systems, or complete CPSs, which yields four distinct definitions of strong rationalizability.\footnote{One may wonder why we consider separate definitions for completely consistent belief systems and complete CPSs. Indeed, (i) completely consistent belief systems are the “projection” of complete CPSs, and (ii) for any complete CPS $\bar{\mu}^i \in \Delta^* (S_{-i})$, the set of (weakly) sequentially optimal strategies under $\bar{\mu}^i$ depends only on the conditional beliefs at information sets $(\mu^i(\cdot|S_{-i}(H_i)))_{H_i \in H_i}$. Yet, the strong-belief requirement is more demanding for complete CPSs than for belief systems, which might, in principle, cause a difference between the two solution concepts.}

**Definition 11.** Fix $\Delta = (\Delta_i)_{i \in I} \in \left\{ (\Delta^H_i(S_{-i}))_{i \in I}, (\Delta^H_i(S_{-i}))_{i \in I}, (\Delta^H_i(S_{-i}))_{i \in I}, (\Delta^*(S_{-i}))_{i \in I} \right\}$.

Let $(S_i^{\Delta,0})_{i \in I} = (S_i)_{i \in I}$ and define inductively for each $m \in \mathbb{N}_0$ and $i$:

$$S_i^{\Delta,m+1} = \{ s_i \in S_i : \exists \mu^i \in \Delta_i \cap (\bigcap_{k=0}^m SB_i(S_{-i}^{\Delta,k})), s_i \in BR_i(\mu^i) \}.$$ 

The set of **strongly rationalizable** strategies for $i$ (given $\Delta$) is $S_i^{\Delta,\infty} = \bigcap_{k=0}^\infty S_i^{\Delta,k}$.

The sequence $(S_i^{\Delta,m})_{m=0}^\infty$ is by definition a weakly decreasing sequence of events. Notice that at step $m+1$, we require strong belief in each $S_{-i}^{\Delta,k}$ with $k \leq m$, not just $S_{-i}^{\Delta,m}$. The reason for this demanding requirement is that the strong belief operator is not monotone: suppose that $E_{-i} \subset F_{-i}$; although $\mu^i(E_{-i}|h_i) = 1$ implies $\mu^i(F_{-i}|h_i) = 1$ for any given $h_i$, strong belief in $E_{-i}$ does not imply strong belief in $F_{-i}$, because $E_{-i}$ might be consistent with a strictly smaller collection of information sets than $F_{-i}$; in this case, strong belief in $F_{-i}$ implies
restrictions on conditional beliefs not implied by strong belief in $E_{-i}$. The best rationalization principle is captured by strong belief in the whole sequence $(S_{-i}^{\Delta,m})_{m=0}^{\infty}$: once a degree $m$ of strategic rationality is contradicted at information set $h_i$ (i.e., $S_{-i}(h_i) \cap S_{-i}^{\Delta,m} = \emptyset$), player $i$ interprets her information on co-players’ past behavior by assuming they are implementing plans of the highest degree of strategic rationality compatible with such information, i.e., she infers that $s_{-i} \in S_{-i}^{\Delta,k^*}$ for the highest $k^*$ such that $S_{-i}(h_i) \cap S_{-i}^{\Delta,k^*} \neq \emptyset$. This form of strategic sophistication shapes how players revise their beliefs upon reaching unexpected histories.

The main result of this section is that these definitions obtained with different notions of belief change are all equivalent.

**Theorem 4.** The strong rationalizability procedures of Definition 11 coincide.

Before turning to the proof, we introduce some preliminary results.

**Lemma 3.** Fix a strategy $s_i$, a subset $E_{-i} \subseteq S_{-i}$, and a forward consistent belief system $\mu^i$ that strongly believes $E_{-i}$. Then there exists a complete CPS $\bar{\mu}^i$ such that (i) $\mu^i(\cdot|h_i) = \bar{\mu}^i(\cdot|S_{-i}(h_i))$ for all $h_i \in H_i(s_i)$ and (ii) $\bar{\mu}^i$ strongly believes $E_{-i}$.

**Proof:** See the Appendix.

This first lemma establishes that we can map a forward consistent belief system that strongly believes in an event into a complete CPS that (i) coincides with the original belief system at all information sets consistent with a strategy $s_i$ and (ii) strongly believes in the event. Property (i) implies that weak sequential optimality of $s_i$ under $\mu^i$ is preserved by the transformation.

Lemma 3, however, can only be used for strong belief in one event, while the strong rationalizability procedures require strong belief in a sequence of events. Yet, the strong rationalizability procedures with complete CPSs and FBSs can be characterized by procedures that require strong belief only in one event at each step:\textsuperscript{17}

\textsuperscript{17}As a corollary of Theorem 4, we show in the Appendix that all our procedures are characterized by the one-memory procedure.
Definition 12. Given $\Delta = (\Delta_i)_{i \in I}$ as in Definition 11, let $(\bar{S}_i^{\Delta,0})_{i \in I} = (S_i)_{i \in I}$ and define inductively for each $m \in \mathbb{N}_0$ and $i$:

$$\bar{S}_i^{\Delta,m+1} = \{ s_i \in \bar{S}_i^{\Delta,m} : \exists \mu^i \in \Delta_i \cap SB_i(\bar{S}_{-i}^{\Delta,m}), s_i \in BR_i(\mu^i) \}.$$ 

Let $\bar{S}_i^{\Delta,\infty} = \cap_{k=0}^{\infty} \bar{S}_i^{\Delta,k}$.

Lemma 4. Let $\Delta = (\Delta^H_i (S_{-i}))_{i \in I}$ or $\Delta = (\Delta^* (S_{-i}))_{i \in I}$. Then $\bar{S}_i^{\Delta,m} = \bar{S}_i^{\Delta,m}$ for every $i$ and $m \in \mathbb{N}$.

Proof: See the Appendix.

This result allows us to exploit Lemma 3 for the proof of Theorem 4. To see why Lemma 4 is true, note that a key difference between the definitions of $\bar{S}_i^{\Delta,m+1}$ and $\bar{S}_i^{\Delta,m+1}$ is that at step $m + 1$, the latter requires $s_i \in \bar{S}_i^{\Delta,m}$. But then, recursively, $s_i \in \bar{S}_i^{\Delta,k}$ for all $k \leq m$, and thus for each $k \leq m$ we can find a belief system $\mu^{i,k}$ that strongly believes $\bar{S}_{-i}^{\Delta,k}$ and justifies $s_i$ as a weak sequential best reply. Then, we show that we can compose these belief systems to construct one that strongly believes the whole sequence $(\bar{S}_{-i}^{\Delta,k})_{k=1}^{m}$ and justifies $s_i$ as a weak sequential best reply. We are now in a position to prove the theorem.

Proof of Theorem 4: We begin to show that the procedures obtained with complete CPSs and forward consistent belief systems (FBSs), denoted respectively $(S^{C,k})_{k=0}^{\infty}$ and $(S^{F,k})_{k=0}^{\infty}$, coincide at each step. Then we conclude by showing that for any notion of belief system in between forward consistency and complete consistency, the induced procedure is equal to the one of complete CPSs and FBSs.

Claim 1: $S^{C,m} = S^{F,m}$ for all $m \in \mathbb{N}_0$.

Proof: The basis step for $m = 0$ holds by definition: $S^{C,0} = S_i = S^{F,0}$ for every $i$. Suppose by way of induction that for some $m \geq 0$, $S^{C,m} = S^{F,m}$ for every $i$. We show that $S^{C,m+1} = S^{F,m+1}$, or equivalently, by Lemma 4, that $\bar{S}_i^{C,m+1} = \bar{S}_i^{F,m+1}$. Note that by the
inductive hypothesis and Lemma 4, \( \tilde{S}_{i}^{C,m} = \tilde{S}_{i}^{F,m} \) for every \( i \). Fix \( i \in I \). We show both inclusions.

Take any \( s_i \in \tilde{S}_{i}^{C,m+1} \). Then \( s_i \in \tilde{S}_{i}^{C,m} \), and thus \( s_i \in \tilde{S}_{i}^{F,m} \) since \( \tilde{S}_{i}^{F,m} = \tilde{S}_{i}^{C,m} \). Moreover, there exists a complete CPS \( \mu^i \) that strongly believes \( \tilde{S}_{i}^{C,m} \), hence \( \tilde{S}_{i}^{F,m} \), such that \( s_i \in BR_i(\mu^i) \). The belief system \( \bar{\mu}^i \) induced by \( \mu^i \) strongly believes \( \tilde{S}_{i}^{F,m} \), is forward consistent, and we have \( s_i \in BR_i(\bar{\mu}^i) \). This shows that \( s_i \in \tilde{S}_{i}^{F,m+1} \).

Conversely, take any \( s_i \in \tilde{S}_{i}^{F,m+1} \). Then \( s_i \in \tilde{S}_{i}^{F,m} \), hence \( s_i \in \tilde{S}_{i}^{C,m} \) since \( \tilde{S}_{i}^{F,m} = \tilde{S}_{i}^{C,m} \). Moreover, there exists an FBS \( \mu^i \) that strongly believes \( \tilde{S}_{i}^{F,m} \), hence strongly believes \( \tilde{S}_{i}^{C,m} \), such that \( s_i \in BR_i(\mu^i) \). It follows by Lemma 3 that there exists a complete CPS \( \bar{\mu}^i \) that strongly believes \( \tilde{S}_{i}^{C,m} \) such that \( \bar{\mu}^i(\cdot|_{\tilde{S}_{i}^{F,m}}) = \mu^i(\cdot|_{\tilde{S}_{i}^{F,m}}) \) for every \( h_i \in H_i(s_i) \), which implies, since \( s_i \in BR_i(\mu^i) \), that \( s_i \in BR_i(\bar{\mu}^i) \). Therefore \( s_i \in \tilde{S}_{i}^{C,m+1} \). □

**Claim 2:** Let \( \Delta = (\Delta_i)_{i \in I} \) be such that for every \( i \in I \), \( \Delta_i^{H_i}(S_{-i}) \subseteq \Delta_i \subseteq \Delta_i^{H_i}(S_{-i}) \). Then \( S_{i}^{\Delta,m} = S_{i}^{C,m} \) for every \( m \).

**Proof:** The proof is by induction on \( m \in \mathbb{N}_0 \). The basis step is immediate: \( S_{i}^{C,0} = S_{i}^{\Delta,0} = S_{i}^{F,0} \) for every \( i \). Suppose by way of induction that for some \( m \in \mathbb{N}_0 \), we have for every \( k \leq m \) and \( i \in I \) that \( S_{i}^{C,k} = S_{i}^{\Delta,k} = S_{i}^{F,k} \). Fix \( i \in I \). By inspection of the definitions and the inductive hypothesis, because \( \Delta_i \subseteq \Delta_i^{H_i}(S_{-i}) \) it is immediate that \( S_{i}^{\Delta,m+1} \subseteq S_{i}^{F,m+1} \). Moreover, it is easy to see that \( S_{i}^{C,m+1} \subseteq S_{i}^{\Delta,m+1} \). Suppose \( s_i \in S_{i}^{C,m+1} \). Then there is a complete CPS \( \mu^i \) that strongly believes \( S_{-i}^{C,k} \) and \( S_{-i}^{\Delta,k} \) are such that \( s_i \in BR_i(\mu^i) \). But then the belief system \( \bar{\mu}^i \) induced by \( \mu^i \) is completely consistent (hence \( \mu^i \in \Delta_i \) since \( \Delta_i^{H_i}(S_{-i}) \subseteq \Delta_i \)), strongly believes \( S_{-i}^{\Delta,m} \) and \( s_i \in BR_i(\bar{\mu}^i) \). Hence \( s_i \in S_{i}^{\Delta,m+1} \). To conclude, we have that \( S_{i}^{C,m+1} \subseteq S_{i}^{\Delta,m+1} \subseteq S_{i}^{F,m+1} \). Since by Claim 1 we have \( S_{i}^{F,m+1} = S_{i}^{C,m+1} \), it follows that \( S_{i}^{C,m+1} = S_{i}^{\Delta,m+1} = S_{i}^{F,m+1} \). □

Since \( \Delta \) defined as standard belief systems or completely consistent belief systems satisfy the assumption of Claim 2, the result follows. ■

Theorem 4 implies, in particular, that the set of strategies that can be justified as weak
sequential best replies to at least one belief system is invariant to consistency restrictions beyond the forward chain rule.

7.2 Backwards rationalizability

Backwards Rationalizability characterizes the behavioral implications of rationality and common belief in continuation rationality starting from every information set (cf. Perea 2014, Battigalli & De Vito 2021). In generic games with perfect information, backwards rationalizability coincides with subgame perfect equilibrium and can therefore be computed by backward induction. Catonini & Penta (2021) show that in more general games backwards rationalizability can be computed by means of a convenient Backwards Procedure.\footnote{In a related vein, Perea (2014) introduces the Backward Dominance procedure, a generalization of the backward induction algorithm which is more permissive than Backwards Rationalizability, also in terms of induced paths.} A result by Chen & Micali (2013) on the order-independence of iterated dominance procedures in sequential games implies that the set of strongly rationalizable paths is (weakly) contained in the set of backwards rationalizable path. Therefore, backwards rationalizability (possibly computed with the backward procedure) can help find the strongly rationalizable paths.

In this section, we focus on games with observable actions\footnote{Catonini & Penta (2021) define Backwards Rationalizability for games with incomplete information. Their definition coincides with the definition of Battigalli & De Vito (2021) in games with complete information.} and consider the definition of backwards rationalizability given by Battigalli & De Vito (2021),\footnote{Perea (2014) only considers reduced strategies. Battigalli & De Vito (2021) consider full strategies.} which is behaviorally equivalent to the one of Perea (2014) for this class of games, as it yields the same reduced strategies.\footnote{} To justify this solution concept, Battigalli & De Vito (2021) start from the following epistemic assumptions: Each player is fully introspective about her belief system and plans rationally by folding back, thus obtaining a strategy that is sequentially optimal under her belief system. The player is rational if her behavior complies with her strategy. Players initially believe that co-players are rational. The key assumption is that when a player observes unexpected behavior she keeps believing that the co-players planned rationally and that, even if in the past they deviated from their plans, they will nonetheless comply with their
plans in the subgame (according to strong rationalizability, instead, surprised players change their beliefs about the plans that their supposedly rational co-players are implementing). The assumptions concerning higher levels of mutual belief in rationality have a similar flavor.

With this, we want to express the idea that a player in a subgame believes that her co-players are going behave according to plans having some properties, even if such properties were violated in the past. For example, a nonterminal history \( h \in H \) may be unreachable if the co-players’ are rational, and yet player \( i \) may believe that in the subgame with root \( h \) the co-players will implement continuation strategies consistent with their rationality in the subgame. With this in mind, given a subset of \( j \)’s strategies \( E_j \subseteq S_j \) and a history \( h \in H \), let

\[
\chi^h_j(E_j) = \{ s_j \in S_j(h) : \exists s'_j \in E_j, \forall h' \in H, h \preceq h' \Rightarrow s'_j(h') = s_j(h) \}
\]

denote the set of strategies \( s_j \) consistent with \( h \) that behave like some strategy in \( E_j \) in the subgame with root \( h \). As we consider all the co-players, for each profile of strategy subsets \( (E_j)_{j \neq i} \), we obtain \( \chi^h_{-i}(E_{-i}) = \times_{j \neq i} \chi^h_j(E_j) \), where \( E_{-i} = \times_{j \neq i} E_j \). With this, we define the set of belief systems \( \mu^i \) that “persistently believe” in the projection of \( E_{-i} = \times_{j \neq i} E_j \) in each subgame:

\[
PB_i(E_{-i}) = \{ \mu^i \in \times_{h \in H} \Delta(S_{-i}(h)) : \forall h \in H, \mu^i(\chi^h_{-i}(E_{-i})|h) = 1 \}.
\]

Note that, unlike the strong belief operator \( SB_i(\cdot) \), operator \( PB_i(\cdot) \) is monotone. Depending on which kind of belief systems we use for the definition of backwards rationalizability, we let \( \Delta_i \) denote the set of either forward consistent, or standard, or completely consistent belief systems. The main result of this section is that the notions of backwards rationalizability obtained by considering these different kinds of belief system are behaviorally equivalent.

Recall that \( BR^*_i(\mu^i) \) is the set of strategies of \( i \) that are sequentially optimal under belief system \( \mu^i \); given the observability of actions, it is the set of strategies whose continuations maximize \( i \)'s expected payoff in each subgame with root \( h \in H \) given belief \( \mu^i(\cdot|h) \).
Definition 13. Let $\Delta_i = \Delta^{H_i}_{S_{-i}}$, or $\Delta_i = \Delta^{H_i}_{S_{-i}}$, or $\Delta_i = \Delta^{C_i}_{S_{-i}}$ for each $i$, and let $\Delta = (\Delta_i)_{i \in I}$. With this, let $(\hat{S}_i^\Delta,0)_{i \in I}$; suppose $(\hat{S}_i^\Delta,m)_{i \in I}$ has been defined; then, for each $i$, let

$$\hat{S}_i^\Delta,m+1 = \{ s_i \in S_i : \exists \mu^i \in \Delta_i \cap PB_i(\hat{S}_i^\Delta,m), s_i \in BR_i^*\}.$$ 

Strategies in $\hat{S}_i^\Delta,\infty = \cap_{k=0}^\infty \hat{S}_i^\Delta,k$ are **backwards rationalizable** for player $i$.

Theorem 5. The backwards rationalizability procedures of Definition 13 coincide.

Proof: See the Appendix. ■

7.3 Beliefs systems defined on active information sets

In the foundational works on the theory of games of von Neumann & Morgenstern (1944) and Kuhn (1953), and in most of the following game theoretic work, the information of players is specified only at information sets where they are active. Indeed, it was either argued or taken as self-evident that the information of inactive players is irrelevant for the strategic analysis of games. The results of this section confirm to some extent this intuition.

Consider systems of beliefs where players form their beliefs only at the information sets where they are active, i.e., with domain $\hat{H}_i = \{ h_i \in H_i : |A_i(h_i)| \geq 2 \} \cup \{ \emptyset \}$. (We assume $\emptyset \in \hat{H}_i$ even if $i$ is not active at the initial history.)

A unifying definition of weak sequential optimality for both $\hat{H}_i$-based and $H_i$-based belief systems consists in requiring maximization of conditional expected utility only at active information sets. Next we show that strong rationalizability coincides with both models of beliefs defined only on active information sets and those defined on all information sets.

We show that we can always extend a forward consistent belief system on active information sets to a forward consistent belief system on all information sets whose restriction to active information sets is equal to the original belief system. Then it is immediate that the strong rationalizability procedures with both kind of belief systems are equal.
Lemma 5. Fix a set $E_{-i}$ and an $\hat{H}_i$-based FBS $\mu^i$ that strongly believes $E_{-i}$. There exists an $H_i$-based FBS $\tilde{\mu}^i$ that strongly believes $E_{-i}$ such that $\mu^i(\cdot|h_i) = \tilde{\mu}^i(\cdot|h_i)$ for all $h_i \in \hat{H}_i$.

Proof: Fix an $\hat{H}_i$-based FBS $\tilde{\mu}$ that strongly believes $E_{-i}$ and an $H_i$-based FBS $\mu$ that strongly believes $E_{-i}$.

For each $h_i \in \hat{H}_i$, denote by $\mathcal{H}^i_0(h_i)$ the set of all information sets $h'_i$ that weakly follow $h_i$ such that $\mu(S_{-i}(h'_i)|h_i) > 0$ and by $\mathcal{H}^i_0(h_i)$ the set of all $h'_i$ that weakly follow $h_i$ such that $\tilde{\mu}(S_{-i}(h'_i)|h_i) = 0$. Furthermore, let $\mathcal{H}^i_0(h_i)$ be the collection of earliest $h_i^*$ in $\mathcal{H}^i_0(h_i)$. For every $h_i \in H_i$, let $H_i(h_i)$ be the information sets that weakly follow $h_i$, and $\hat{H}_i^*(h_i)$ the earliest active information sets that follow $h_i$, and by $[h_i, \hat{H}_i^*(h_i))$ the information sets that weakly follow $h_i$ and precede some element of $\hat{H}_i^*(h_i)$. Also, let $H_i^-(h_i, E_{-i})$ be the information sets in $\hat{H}_i^*(h_i)$ consistent with $E_{-i}$. Denote by $H_i^-(h_i, E_{-i})$ the earliest information sets in $[h_i, \hat{H}_i^*(h_i))$ that are not in $[h_i, \hat{H}_i^*(h_i, E_{-i})$.

Now we construct a partition $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\} \cup \{\overline{\mathcal{H}}\}$ of $H_i$ as follows: $\mathcal{H}_1^* = \{\emptyset\}$ and $\mathcal{H}_1 = \mathcal{H}_+(\emptyset)$, and for $m \in \mathbb{N}$, let

$$\mathcal{H}^*_m = \bigcup_{h_i \in \mathcal{H}_m} \mathcal{H}^*_0(h_i)$$

and from this, let $\mathcal{H}^*_m = \bigcup_{h_i \in \mathcal{H}^*_m} [h_i, \hat{H}_i^*(h_i))$ and

$$\mathcal{H}^+_m = \bigcup_{h_i \in \mathcal{H}^*_m} (\mathcal{H}^*_m, \mathcal{H}_+(h_i^*))$$

and we set $\mathcal{H}^*_m = \mathcal{H}^*_m \cup \mathcal{H}^+_m$.

Let $\mathcal{H}^* = \bigcup_{m \in \mathbb{N}} \mathcal{H}^*_m$ and $\mathcal{H} = \bigcup_{m \in \mathbb{N}} \mathcal{H}_m$. Finally, let $\overline{\mathcal{H}} = H_i \setminus \mathcal{H}$. Naturally $H_i = H_i \cup \overline{\mathcal{H}}$ and the sets in $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\} \cup \overline{\mathcal{H}}$ are pairwise disjoint, so it is indeed a partition of $H_i$. Note that $\overline{\mathcal{H}} \cap \hat{H}_i = \emptyset$.

We use the arrays of measures $\tilde{\mu}$ and $\tilde{\mu}$ to construct an $H_i$-based FBS $\mu$ that strongly believes $E_{-i}$ such that $\mu(\cdot|h_i) = \tilde{\mu}(\cdot|h_i)$ for all $h_i \in \hat{H}_i$.

Define $\mu$ as follows: For each $m$ and $h_i^* \in \mathcal{H}^*_m$, let $\mu(h_i^*)$ be the equiprobable convex
combination of the measures \((\mu(\cdot|h_i))_{h_i \in \hat{H}_i(h_i^*,S_{-i})}\) (if any), and for all \(h_i \in H_i(h_i^*)\) such that 
\(\mu(S_{-i}(h_i)|h_i^*) > 0\), derive \(\mu(\cdot|h_i)\) by conditioning from \(\mu(\cdot|h_i^*)\). This derivation preserves 
strong belief in \(E_{-i}\) and ensures that \(\mu(\cdot|h_i) = \bar{\mu}(\cdot|h_i)\) for every \(h_i \in \hat{H}_i^*(h_i^*,E_{-i})\). Derive all 
measures at the remaining information sets of \(\mathcal{H}_m\) with the same operation as we did for \(h_i^*\) 
starting from each element of \(\bar{h}_i \in H_i^-(h_i^*,E_{-i})\). We covered all \(\mathcal{H}_m\) in a way that satisfies 
the forward chain rule, and we have by construction \(\mu(\cdot|h_i) = \bar{\mu}(\cdot|h_i)\) for every \(h_i \in \mathcal{H}_m \cap \hat{H}_i\). 
Finally, set \(\mu(\cdot|h_i) = \bar{\mu}(\cdot|h_i)\) for every \(h_i \in \bar{H}\), which preserves strong belief of \(\mu\) in \(E_{-i}\). It 
is easy to see that our \(\mu\) satisfies all the desired properties.

Note, \(\hat{H}_i\)-based standard belief belief systems, whose set is denoted by \(\Delta^\hat{H}_i(S_{-i})\), is obtained 
from CPSs on \((S_{-i}, \hat{H}_i)\), where \(\hat{H}_i = \{S_{-i}(h_i) \subseteq S_{-i} : h_i \in \hat{H}_i\}\). Extendibility for such belief 
systems is not always possible because when \(\hat{H}_i \subset \mathcal{H}_i\), these belief systems may embody less 
coherency restrictions across information sets not ordered by precedence than the standard 
one. Yet, our results show that this is irrelevant for the predictions. Indeed, notice that:

**Lemma 6.** Fix an \(\hat{H}_i\)-based FBS \(\mu^i\) that strongly believes \(E_{-i}\) and a strategy \(s_i \in BR_i(\mu^i)\). 
Then there is a complete CPS \(\bar{\mu}^i\) such that \(\bar{\mu}^i\) strongly believes \(E_{-i}\) and \(s_i \in BR_i(\bar{\mu}^i)\).

**Proof:** Fix a \(\hat{H}_i\)-based FBS \(\mu\) that strongly believes \(E_{-i}\) and \(s_i \in BR_i(\mu)\). By Lemma 5 
there is an \(H_i\)-based FBS \(\bar{\mu}\) that strongly believes \(E_{-i}\) with \(\bar{\mu}(\cdot|h_i) = \mu(\cdot|h_i)\) for all \(h_i \in \hat{H}_i\), 
so \(s_i \in BR_i(\bar{\mu})\). Lemma 3 in turn implies that there is a complete CPS \(\bar{\mu}\) that strongly 
believes \(E_{-i}\) with \(\bar{\mu}(\cdot|S_{-i}(h_i)) = \mu(\cdot|h_i)\) for all \(h_i \in H_i(s_i)\), so that \(s_i \in BR_i(\mu)\). 

Since \(\Delta^\hat{H}_i(S_{-i}) \subseteq \Delta^\bar{H}_i(S_{-i}) \subseteq \Delta^\bar{H}_i(S_{-i})\), the equivalence between strong rationalizability 
with \(\hat{H}_i\)-based belief systems and with complete CPSs is shown with the same sandwitch 
argument used in the proof of Theorem 4, which by Theorem 4 further implies equivalence 
with the other procedures.

**Theorem 6.** The strong rationalizability procedures with our notions of belief systems 
defined on all or only active information sets coincide.
8 Relation to previous work

Most of the literature on rationalizability in sequential games defined the algorithms using forward consistent belief systems, such as Battigalli & Siniscalchi (2003) on strong $\Delta$-rationalizability, or Perea (2014) on backwards rationalizability, among others. A large body of the literature on strong rationalizability, such as Pearce (1984), Battigalli (1997), Shimoji & Watson (1998), and Shimoji (2004) use a notion of “consistent updating systems” that is equivalent to forward consistent belief systems in terms of sequential optimality. Complete CPSs were used by Battigalli (1996) to define strong rationalizability.

On the other hand, most works on the epistemic foundations of those same solution concepts use “standard” CPSs, Battigalli & Siniscalchi (1999, 2002), Battigalli & De Vito (2021). From a technical point of view, our results establish the bridge between the predictions of solution concepts that were defined with forward consistent belief systems, and the epistemic foundations that have been established for procedures defined with “standard” CPSs.

Finally, our results also connect to known characterizations of rationalizability procedures in sequential games. Shimoji & Watson’s (1998) characterization of strong rationalizability in terms of iterated conditional dominance was proved with forward consistent systems of beliefs. Theorem 4 implies that the characterization holds for all our three kinds of belief systems, and similarly for Shimoji’s (2004) result of generic equivalence between strong rationalizability and iterated admissibility.

9 Conclusion

In dynamic games, players must update or completely change their beliefs about the behavior of co-players upon the arrival of new information regarding their past moves. Various restrictions on how players revise their beliefs have been used in the literature, but the very meaning of these requirements and their implications for optimal planning and strategic reasoning remain unclear. We put forward and analyze three ordered notions of coherency
among beliefs at different information sets, and we offer interpretations in terms of players’ level of introspection.

The weakest notion, forward consistency, only requires belief updating by conditioning, as long as possible, along every path of play. Forward consistency takes the viewpoint of a player who, at each information set, only asks herself how the co-players will behave in the contingencies that she currently deems possible. When surprised by moves that she previously deemed impossible, such a player might form different beliefs depending on her own past moves. Our intermediate notion of belief system, standard belief system, rules this out by requiring that the beliefs at different information sets satisfy the chain rule of probability; hence, same information (about others) implies same belief. Standard belief systems are isomorphic to conditional probability systems a la Renyi (1955), defined over the collection of conditional events that correspond to some information set. Our strongest notion of coherency, complete consistency, requires instead that the beliefs at the information sets are consistent with a conditional probability system over the power set of the space of uncertainty, which also contains conditional events that cannot be observed at any point in the game. Such virtual events can be interpreted as “mental exercises” of a fully introspective player, who asks herself “what would I believe if I were to observe this event”, and derives her beliefs at the actual information sets accordingly. Complete consistency implies that the relative probabilities of pairs of co-players’ strategy profiles are mutually consistent across all information sets where such profiles are possible, a property that standard belief systems may violate, as we show by example.

Although complete consistency is much more restrictive than forward consistency for players’ belief systems, the main result of the paper shows that weak sequential optimality and strong rationalizability are invariant to the choice of any of the three notions of belief system. Perhaps even more surprisingly, also backwards rationalizability, an algorithm that makes use of (non-weak) sequential rationality, is invariant to the chosen notion of belief system. The key observation for these results is that forward consistency is sufficient to identify a player’s courses of action that are optimal also under a completely consistent belief
system, along every path.

Appendix A: Proofs of Section 7.1

In this section, we provide the omitted proofs of the section on strong rationalizability. In order to show that strong rationalizability with FBSs and complete CPSs is characterized by the one-memory procedure, we proceed with a preliminary lemma.

Lemma 7. Fix a chain of events $E^n_{−i} \subseteq \ldots \subseteq E^1_{−i} \subseteq E^0_{−i}$. Take any strategy $s_i \in S_i$ and two complete CPSs (or FBSs) $\mu$ and $\mu^0$ such that $\mu$ strongly believes $E^n_{−i}$, $\ldots$, $E^1_{−i}$, $\mu^0$ strongly believes $E^0_{−i}$, $s_i \in BR_i(\mu)$ and $s_i \in BR_i(\mu^0)$. Then the array $\mu^*$ defined for all $C \in \mathcal{S}_i$ as $\mu^*(\cdot|C) = \mu(\cdot|C)$ if $E^1_{−i} \cap C \neq \emptyset$ and $\mu^*(\cdot|C) = \mu^0(\cdot|C)$ if $E^1_{−i} \cap C = \emptyset$ is a complete CPS that strongly believes $E^n_{−i}$, $\ldots$, $E^0_{−i}$ such that $s_i \in BR_i(\mu^*)$. The same is true for FBSs for the array $\mu^*$ defined for all $h_i \in H_i$ as $\mu^*(\cdot|h_i) = \mu(\cdot|h_i)$ if $E^1_{−i} \cap S_{−i}(h_i) \neq \emptyset$ and $\mu^*(\cdot|h_i) = \mu^0(\cdot|h_i)$ if $E^1_{−i} \cap S_{−i}(h_i) = \emptyset$.

Proof: We show the result for complete CPSs, the proof for FBSs is analogous. Take $\mu^*$ as described in the statement of the lemma.

Claim 1: $\mu^*$ is a complete CPS. Take any $C_{−i}, D_{−i} \in \mathcal{S}_{−i}$ such that $D_{−i} \subseteq C_{−i}$. If $D_{−i} \cap E^1_{−i} \neq \emptyset$, then also $C_{−i} \cap E^1_{−i} \neq \emptyset$ and so we have $\mu^*(\cdot|C_{−i}) = \mu(\cdot|C_{−i})$ and $\mu^*(\cdot|D_{−i}) = \mu(\cdot|D_{−i})$. Hence the chain rule relates $\mu^*(\cdot|C_{−i})$ to $\mu^*(\cdot|D_{−i})$ since $\mu$ is a complete CPS. If $D_{−i} \cap E^1_{−i} = \emptyset$ and $C_{−i} \cap E^1_{−i} \neq \emptyset$, then $\mu^*(\cdot|C_{−i}) = \mu(\cdot|C_{−i})$ and strong belief of $\mu$ in $E^1_{−i}$ implies $\mu(E^1_{−i}|C_{−i}) = 1$. Then, because $D_{−i} \cap E^1_{−i} = \emptyset$ we have $\mu(D_{−i}|C_{−i}) = 0$. So, the chain rule relates $\mu^*(\cdot|C_{−i})$ to $\mu^*(\cdot|D_{−i})$. If $C_{−i} \cap E^1_{−i} = \emptyset$, then $D_{−i} \cap E^1_{−i} = \emptyset$, so that $\mu^*(\cdot|C_{−i}) = \mu^0(\cdot|C_{−i})$ and $\mu^*(\cdot|D_{−i}) = \mu^0(\cdot|D_{−i})$ and the chain rule relates $\mu^*(\cdot|C_{−i})$ to $\mu^*(\cdot|D_{−i})$ since $\mu^0$ is a complete CPS. So, $\mu^*$ is a complete CPS.

Claim 2: $\mu^*$ strongly believes $E^n_{−i}$, $\ldots$, $E^0_{−i}$. Fix $C \in \mathcal{S}_{−i}$ such that $C \cap E^k_{−i} \neq \emptyset$ for some $k > 0$. Then $C \cap E^1_{−i} \neq \emptyset$ since $E^k_{−i} \subseteq E^1_{−i}$, and so $\mu^*(\cdot|C) = \mu(\cdot|C)$. Since $\mu$ strongly believes $E^k_{−i}$, we have $\mu^*(E^k_{−i}|C) = \mu(E^k_{−i}|C) = 1$. Now fix $C$ such that $C \cap E^0_{−i} \neq \emptyset$. If $C \cap E^1_{−i} \neq \emptyset$,
then \( \mu^*(|C) = \mu(|C) \), and since \( \mu \) strongly believes \( E^1_{-i} \), we have \( \mu^*(E^0_{-i} | C) \geq \mu^*(E^1_{-i} | C) = \mu(E^1_{-i} | C) = 1 \). If \( C \cap E^1_{-i} = \emptyset \), then \( \mu^*(|C) = \mu^0(|C) \), and since \( \mu^0 \) strongly believes \( E^0_{-i} \), we have \( \mu^*(E^0_{-i} | C) = \mu^0(E^0_{-i} | C) = 1 \).

**Claim 3:** \( s_i \) is a weak sequential best reply to \( \mu^* \). Take any \( h \in H_i(s_i) \). We have either \( \mu^*(|S_{-i}(h)) = \mu(|S_{-i}(h)) \) or \( \mu^*(|S_{-i}(h)) = \mu^0(|S_{-i}(h)) \). In both cases \( s_i^\mu \) maximizes player \( i \)'s expected payoff at \( h \) by weak sequential optimality of \( s_i \) under both \( \mu \) and \( \mu^0 \). □

**Proof of Lemma 4:** Let \( \Delta \) denote FBSs or complete CPSs. We prove that \( S^\Delta_{i,m} = \tilde{S}^\Delta_{i,m} \) for every \( i \) and \( m \in \mathbb{N} \). The basis step is immediate, as \( S^\Delta_{i,1} = \tilde{S}^\Delta_{i,1} \) for every \( i \) by inspection of the definitions. Suppose by way of induction that \( S^\Delta_{i,k} = \tilde{S}^\Delta_{i,k} \) for every \( i \) and \( k \leq m \).

The inductive hypothesis implies, by inspection of the definitions, that \( S^\Delta_{i,m+1} \subseteq \tilde{S}^\Delta_{i,m+1} \) for every \( i \). We show the other inclusion. Take any \( s_i \in \tilde{S}^\Delta_{i,m+1} \). Then because \( (\tilde{S}^\Delta_{i,k})_{k \in \mathbb{N}} \) is a weakly decreasing sequence of sets, we have \( s_i \in \tilde{S}^\Delta_{i,k} \) for every \( k \leq m+1 \). This implies that for each \( k \leq m \), there is \( \mu^k \in \Delta_i \) such that \( \mu^k \) strongly believes \( \tilde{S}^k_{i} \) and \( s_i \in BR_{i}(\mu^k) \).

We construct \( \mu^* \) that strongly believes the chain \( (\tilde{S}_{-i}^m, \ldots, \tilde{S}_{-i}^1) \) such that \( s_i \in BR_{i}(\mu^* \star) \) recursively applying Lemma 7. Take first \( \mu = \mu^m \) and \( \mu^0 = \mu^{m-1} \) to obtain \( \mu^*, \mu^{m-1} \) that strongly believes \( (\tilde{S}_{-i}^m, \tilde{S}_{-i}^{m-1}) \) such that \( s_i \in BR_{i}(\mu^*, \mu^{m-1}) \); for all \( k = m, \ldots, 1 \), recursively define \( \mu^{*,k-1} \) with the same operation with \( \mu = \mu^{*,k} \) and \( \mu^0 = \mu^{k-1} \). Then letting \( \mu^* := \mu^*, \mu^0 \) we have that \( \mu^* \in \Delta_i \) strongly believes \( (\tilde{S}_{-i}^m, \ldots, \tilde{S}_{-i}^{1}) = (S_{-i}^m, \ldots, S_{-i}^{1}) \), where the equality holds by the inductive hypothesis, and \( s_i \in BR_{i}(\mu^*) \). Thus, \( s_i \in S^\Delta_{i,m+1} \). □

**Proof of Lemma 3:** Fix a strategy \( s_i \), a forward consistent belief system \( \mu = (\mu(|h_i))_{h_i \in H_i} \) that strongly believes \( E_{-i} \), and some probability measure \( \hat{\nu} \in \Delta(S_{-i}) \) with \( \text{Supp} \hat{\nu} = E_{-i} \).

In this proof, we consider a set of \( \#H_{i}(s_i) + 1 \) indexed measures obtained from \( \hat{\nu} \) and \( \mu \) restricted to \( H_i(s_i) \). Specifically, each measure \( \mu(|h_i) \) with \( h_i \in H_i(s_i) \) is indexed by the conditioning information set \( h_i \), and \( \hat{\nu} \) is indexed by an arbitrary symbol \( \hat{h}_i \) (where \( \hat{h}_i \notin H_i \)). Formally, we consider the set of \( \#H_{i}(s_i) + 1 \) distinct pairs

\[
\{(\mu(|h_i), h_i)\}_{h_i \in H_i(s_i)} \cup \{((\hat{\nu}, \hat{h}_i)\},
\]

44
so that, by definition, \( h'_i \neq h''_i \) implies \( (\mu(\cdot|h'_i), h'_i) \neq (\mu(\cdot|h''_i), h''_i) \) even if \( \mu(\cdot|h'_i) = \mu(\cdot|h''_i) \), and \( (\mu(\cdot|h_i), h_i) \neq (\hat{\nu}, \hat{h}_i) \) even if \( \mu(\cdot|h_i) = \hat{\nu} \). Yet, to ease notation, we just write \( \mu(\cdot|h_i) \) instead of pair \((\mu(\cdot|h_i), h_i)\), and \( \hat{\nu} \) instead of pair \((\hat{\nu}, \hat{h}_i)\), since our measures are already labeled differently. We put a linear order \(<\) on this indexed set in such a way that for all \( h_i, h'_i \in H_i(s_i) \), \( (C1)\) if \( h_i \prec h'_i \), then \( \mu(\cdot|h_i) < \mu(\cdot|h'_i) \); \( (C2)\) if \( S_{-i}(h_i) \cap E_{-i} \neq \emptyset \), then \( \mu(\cdot|h_i) < \hat{\nu} \); \( (C3)\) if \( S_{-i}(h_i) \cap E_{-i} = \emptyset \), then \( \hat{\nu} < \mu(\cdot|h_i) \).

**Claim A:** There is an order \(<\) on \( \{ (\mu(\cdot|h_i), h_i) \}_{h_i \in H_i(s_i)} \cup \{ (\hat{\nu}, \hat{h}_i) \} \) that satisfies \((C1)-(C3)\).

**Proof:** Construct an order \(<\) as follows: Let \( Z(E_{-i}) = \{ z \in Z : \exists s_{-i} \in E_{-i}, \exists s_i \in S_i, \zeta(s_i, s_{-i}) = z \} \) denote the set of terminal histories that can realize if the co-players implement strategy profiles in \( E_{-i} \). Order \( Z \) as \( z_1 < \ldots < z_{|Z|} \) such that each \( z_k \in Z(E_{-i}) \) precedes all \( z \notin Z(E_{-i}) \). Begin to order all \( \mu(\cdot|h_i) \) with \( h_i \) such that \( S_{-i}(z_1) \subseteq S_{-i}(h_i) \) according to information-set precedence, that is, if \( h_i \prec h'_i \) then \( \mu(\cdot|h_i) < \mu(\cdot|h'_i) \). Notice that this covers all cases since for every different \( h_i, h'_i \) with \( S_{-i}(z_1) \subseteq S_{-i}(h_i), S_{-i}(h'_i) \) either \( h_i \prec h'_i \) or \( h'_i \prec h_i \).

For each \( k = 2, \ldots, |Z| \), do the same with \( z_k \) for the indexed measures not already ordered in the previous steps \( j < k \). Indexed measures ordered at previous steps come before those ordered at subsequent steps. Place \( \hat{\nu} \) directly after the last \( \mu(\cdot|h_i) \) with \( h_i \) consistent with \( E_{-i} \), that is, with \( S_{-i}(h_i) \cap E_{-i} \neq \emptyset \).

Fix \( h_i, h'_i \in H_i(s_i) \). Suppose \( \mu(\cdot|h_i) \) is ordered at step \( k \) and \( \mu(\cdot|h'_i) \) at step \( \ell \). The fact that the resulting order satisfies \((C1)-(C3)\) is implied by the following two claims.

**Claim A.1:** If \( h_i \prec h'_i \), then \( \mu(\cdot|h_i) < \mu(\cdot|h'_i) \). If \( k = \ell \) or \( k < \ell \), the claim follows by construction of the order. Since \( \mu(\cdot|h'_i) \) is ordered at step \( \ell \), we have \( S_{-i}(z_\ell) \subseteq S_{-i}(h'_i) \), and since \( h_i \prec h'_i \), by perfect recall \( S_{-i}(h'_i) \subseteq S_{-i}(h_i) \). Thus, \( S_{-i}(z_\ell) \subseteq S_{-i}(h_i) \), and so \( \mu(\cdot|h_i) \) is ordered at most at step \( \ell \), that is to say \( k \leq \ell \).

**Claim A.2:** If \( S_{-i}(h_i) \cap E_{-i} \neq \emptyset \) and \( S_{-i}(h'_i) \cap E_{-i} = \emptyset \), then \( \mu(\cdot|h_i) < \mu(\cdot|h'_i) \). If \( k < \ell \) the result follows. First, note that \( z_\ell \notin Z(E_{-i}) \).

---

\(^{21}\)If both \( h'_i \) and \( h''_i \) precede \( z \), then there is \( x' \in h'_i \) such that \( x' \prec z \), and \( x'' \in h''_i \) such that \( x'' \prec z \). But then either \( x' \preceq x'' \) or \( x'' \preceq x' \), which by perfect recall implies either \( h'_i \preceq h''_i \) or \( h''_i \preceq h'_i \).

\(^{22}\)To see this, suppose \( z_\ell \in Z(E_{-i}) \). Then there exists \( s_{-i} \in E_{-i} \) such that \( z_\ell = \zeta(s_i, s_{-i}) \) for some
to show that $z_k \in Z(E_{-i})$. Since $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$, for some $s_i$ and $s^*_i \in S_{-i}(h_i) \cap E_{-i}$ we have $z^* = \zeta(s_i, s^*_i) \in Z(E_{-i})$. Since $z^*$ is reachable from $h_i$, there exists $x^* \in h_i$ such that $x^* \prec z^*$, so that $S_{-i}(z^*) \subseteq S_{-i}(x^*) \subseteq S_{-i}(h_i)$. Thus, $z_k \leq z^*$ and so $z_k \in Z(E_{-i})$. \[ \square \]

Now, fix a complete CPS $\tilde{\mu}$ and construct an array $(\bar{\mu}(\cdot|C_{-i}))_{C_{-i} \in \mathcal{S}_{-i}} \in \times_{C_{-i} \in \mathcal{S}_{-i}} \Delta(C_{-i})$ as follows: for each $C_{-i} \in \mathcal{S}_{-i}$, derive $\bar{\mu}(\cdot|C_{-i})$ by conditioning the first indexed measure in the ordering, if any, that gives positive probability to $C_{-i}$; otherwise, set $\bar{\mu}(\cdot|C_{-i}) = \tilde{\mu}(\cdot|C_{-i})$. We show that this array is a complete CPS satisfying the required properties:

**Claim 1:** $\bar{\mu}$ is a complete CPS. Fix $C_{-i}, D_{-i} \in \mathcal{S}_{-i}$ such that $D_{-i} \subseteq C_{-i}$ and $\bar{\mu}(D_{-i}|C_{-i}) > 0$. If no measure in the ordering gives positive probability to $C_{-i}$, the same is true by monotonicity for $D_{-i} \subseteq C_{-i}$. So, we have $\bar{\mu}(\cdot|C_{-i}) = \bar{\mu}(\cdot|C_{-i})$ and $\bar{\mu}(\cdot|D_{-i}) = \bar{\mu}(\cdot|D_{-i})$. Since $\bar{\mu}$ is a complete CPS and $D_{-i} \subseteq C_{-i}$, the chain rule relates $\bar{\mu}(\cdot|C_{-i}) = \bar{\mu}(\cdot|C_{-i})$ to $\bar{\mu}(\cdot|D_{-i}) = \bar{\mu}(\cdot|D_{-i})$. If $\bar{\mu}(\cdot|C_{-i})$ is derived from some $\nu$ in the ordered set, every measure $\bar{\nu}$ that precedes $\nu$ must assign probability 0 to $C_{-i}$, hence also to $D_{-i} \subseteq C_{-i}$. Moreover, since $\bar{\mu}(\cdot|C_{-i})$ is derived from $\nu$ by conditioning on $C_{-i}$, we have $\nu(D_{-i}) = \bar{\mu}(D_{-i}|C_{-i}) \nu(C_{-i}) > 0$. Hence, $\bar{\mu}(\cdot|D_{-i})$ is derived from $\nu$ by conditioning on $D_{-i}$. Since $\bar{\mu}(\cdot|C_{-i})$ and $\bar{\mu}(\cdot|D_{-i})$ are derived from the same measure and $D_{-i} \subseteq C_{-i}$, the chain rule relates $\bar{\mu}(\cdot|C_{-i})$ to $\bar{\mu}(\cdot|D_{-i})$.

**Claim 2:** $\bar{\mu}(\cdot|S_{-i}(h_i)) = \mu(\cdot|h_i)$ for all $h_i \in H_i(s_i)$. Fix any $h_i \in H_i(s_i)$. Note that $\bar{\mu}(\cdot|S_{-i}(h_i))$ is derived from some indexed measure $\nu$ in the ordered set that weakly precedes $\mu(\cdot|h_i)$ since $\mu(S_{-i}(h_i)|h_i) = 1 > 0$. We show that $\nu = \mu(\cdot|h_i)$ for some $\tilde{h}_i \in H_i(s_i)$ such that $\tilde{h}_i \preceq h_i$. If $S_{-i}(h_i) \cap E_{-i} \neq \emptyset$, then by (C2) $\mu(\cdot|h_i) < \bar{\nu}$, and since $\nu \leq \mu(\cdot|h_i)$ we have $\nu \neq \bar{\nu}$. If $S_{-i}(h_i) \cap E_{-i} = \emptyset$, we have $\bar{\nu}(S_{-i}(h_i)) = 0$, and since $\nu(S_{-i}(h_i)) > 0$, it must be $\nu \neq \bar{\nu}$. Thus, we have $\nu = \mu(\cdot|h_i)$ for some $\tilde{h}_i \in H_i(s_i)$. Now we show that $\tilde{h}_i \preceq h_i$. Since $\mu(\cdot|h_i)$ is derived from $\mu(\cdot|\tilde{h}_i)$, there must exist $s^*_i \in S_{-i}(h_i) \cap S_{-i}(\tilde{h}_i)$. Since $h_i, \tilde{h}_i \in H_i(s_i)$, we have $s_i \in S(h_i)$ and $s_i \in S(\tilde{h}_i)$. Then, by perfect recall $(s_i, s^*_i) \in S(h_i)$ and $(s_i, s^*_i) \in S(\tilde{h}_i)$, and $S(h_i) \cap S(\tilde{h}_i) \neq \emptyset$ implies that either $\tilde{h}_i \preceq h_i$ or $h_i \preceq \tilde{h}_i$. Since $\mu(\cdot|h_i) \leq \mu(\cdot|\tilde{h}_i)$, by (C1) it must be $\tilde{h}_i \preceq h_i$. Since $\tilde{h}_i \preceq h_i$ and $\mu(S_{-i}(h_i)|h_i) > 0$, by the
forward chain rule of $\mu$ we have that $\mu(\cdot|h_i)$ is derived from $\mu(\cdot|\bar{h}_i)$ by conditioning, which implies $\mu(\cdot|h_i) = \bar{\mu}(\cdot|S_{-i}(h_i))$.

**Claim 3:** $\bar{\mu}$ strongly believes $E_{-i}$. Take any $C_{-i} \in \mathcal{F}_{-i}$ such that $C_{-i} \cap E_{-i} \neq \emptyset$. Then $\bar{\mu}(\cdot|C_{-i})$ is derived from some $\nu \leq \hat{\nu}$, as $\hat{\nu}(C_{-i}) = \hat{\nu}(C_{-i} \cap E_{-i}) > 0$ since $\text{Supp} \hat{\nu} = E_{-i}$. If $\nu = \hat{\nu}$, then $\bar{\mu}(E_{-i}|C_{-i}) = 1$, because $\hat{\nu}(E_{-i}) = 1$. If $\nu$ strictly precedes $\hat{\nu}$, then it must be that $\nu = \mu(\cdot|\bar{h}_i)$ for some $\bar{h}_i \in H_i(s_i)$ such that $S_{-i}(\bar{h}_i) \cap E_{-i} \neq \emptyset$, otherwise $\nu$ would follow $\hat{\nu}$ by (C3). Since $\mu$ strongly believes $E_{-i}$, $\mu(E_{-i}|\bar{h}_i) = 1$, hence, $\bar{\mu}(E_{-i}|C_{-i}) = 1$, as $\bar{\mu}(\cdot|C_{-i})$ is derived from $\mu(\cdot|\bar{h}_i)$. ■

As an almost immediate corollary of Theorem 4 we have that all our procedures are characterized by the one-step memory procedure.

**Lemma 8.** Let $\Delta = (\Delta_i)_{i \in I}$ be such that for every $i \in I$, $\Delta_i^{H_i}(S_{-i}) \subseteq \Delta_i \subseteq \Delta_i^{H_i}(S_{-i})$. Then $S^{\Delta,m} = \bar{S}^{\Delta,m}$ for every $m$.

**Proof:** First, we show by induction that for every $m$, $\bar{S}^{C,m} = \bar{S}^{\Delta,m} = \bar{S}^{F,m}$. The basis for $m = 0$ is immediate; using the inductive hypothesis we have by inspection of the definitions that $\bar{S}^{C,m+1} \subseteq \bar{S}^{\Delta,m+1} \subseteq \bar{S}^{F,m+1}$. Since by Theorem 4 and Lemma 4 we have $\bar{S}^{F,m+1} = \bar{S}^{C,m+1}$, it follows that $\bar{S}^{C,m+1} = \bar{S}^{\Delta,m+1} = \bar{S}^{F,m+1}$.

Finally, by Claim 2 of the proof of Theorem 4 we also have $S^{C,m} = S^{\Delta,m} = S^{F,m}$ for every $m$. Since $S^{F,m} = S^{F,m} = S^{\Delta,m}$, it follows that $S^{\Delta,m} = S^{\Delta,m}$ for every $m$. ■

**Appendix B: Proofs of Section 7.2**

In this section, we show the behavioral equivalence result for backwards rationalizability. We proceed analogously to what we did with strong rationalizability, by showing two preliminary lemmas.

It is first convenient to introduce a stronger notion of “counterfactually persistent belief” in $E_{-i}$, that is, the belief at each $h$ that the co-players will follow, or would have followed some strategy profile in $E_{-i}$ starting from every $h' \in H$ that does not precede $h$, including
the histories \( h' \) that are counterfactual given \( h \). Let

\[
\hat{\chi}^h_j(E_j) = \{ s_j \in S_j(h) : \exists \bar{s}_j \in E_j, \forall h' \in H, h' \not\prec h \Rightarrow s_j(h') = \bar{s}_j(h') \}
\]

and

\[
\hat{\mu}^i(E_{-i}) = \{ \mu^i \in \times_{h \in H} \Delta(S_{-i}(h)) : \forall h \in H, \mu^i(\hat{\chi}^h_{-i}(E_{-i})|h) = 1 \},
\]

where \( E_{-i} = \times_{j \neq i} E_j \) and \( \hat{\chi}^h_{-i}(E_{-i}) = \times_{j \neq i} \hat{\chi}^h_j(E_j) \).

We proceed by establishing some preliminary lemmas.

**Lemma 9.** Fix a strategy \( \pi_i \), a subset \( E_{-i} \subseteq S_{-i} \), and a forward consistent belief system \( \bar{\mu}^i \) in \( \hat{\mu}^i(E_{-i}) \). Then there exists a fully consistent \( \mu^i \) such that (i) \( \mu^i(S_{-i}(h)) = \bar{\mu}^i(S_{-i}(h)) \) for all \( h \in H_i(\pi_i) \) and (ii) \( \mu^i \in PB_i(E_{-i}) \).

**Proof:** To economize on constructions, we take the same Myerson CPS \( \mu^i \) as the one of Lemma 3, but assume that \( \bar{\mu}^i \) is in \( \hat{\mu}^i(E_{-i}) \) instead of \( SB_i(E_{-i}) \).

**Claim:** \( \mu^i \in PB_i(E_{-i}) \). Fix any \( h \in H \). If \( \mu^i(S_{-i}(h)) = \bar{\mu}^i(S_{-i}(h)) \) or \( \mu^i(S_{-i}(h)) \) was derived from \( \nu \), we have \( \mu^i(S_{-i}(h)) = 1 \). Suppose now that \( \mu^i(S_{-i}(h)) \) is derived from some \( \bar{\mu}^i(S_{-i}(h)) \) in the ordering; by (C1), it cannot be that \( h \prec \bar{h} \), so we have \( h \not\prec \bar{h} \). Now, note that, since \( \mu^i(S_{-i}(h)) \) is derived from \( \bar{\mu}^i(S_{-i}(h)) \) and \( \bar{\mu}^i \in PB_i(E_{-i}) \), we have \( \text{Supp} \mu^i(S_{-i}(h)) \subseteq \hat{\chi}^h_{-i}(E_{-i}) \). Moreover, we claim that \( h \not\prec \bar{h} \) implies \( \hat{\chi}^h_{-i}(E_{-i}) \subseteq \hat{\chi}^h_{-i}(E_{-i}) \).

Suppose \( h \not\prec \bar{h} \) and fix \( s^*_{-i} \in \hat{\chi}^h_{-i}(E_{-i}) \). Then there exists \( s_{-i} \in E_{-i} \) such that \( s_{-i}(h') = s^*_{-i}(h') \) for all \( h' \not\prec \bar{h} \). Since \( h \not\prec \bar{h} \), every \( h' \succeq h \) satisfies \( h' \not\prec \bar{h} \), and so, \( s_{-i}(h') = s^*_{-i}(h') \) for all \( h' \succeq h \). Hence we have \( s^*_{-i} = s_{-i} \) for some \( s_{-i} \in E_{-i} \), that is to say, \( s^*_{-i} \in \chi^h_{-i}(E_{-i}) \). So, \( \text{Supp} \mu^i(S_{-i}(h)) \subseteq \chi^h_{-i}(E_{-i}) \) for all \( h \in H \), i.e., \( \mu^i \in PB_i(E_{-i}) \). \( \blacksquare \)

Given two belief systems \( \mu^i, \bar{\mu}^i \) and a nonterminal history \( h \in H \), we write \( \mu^i(h) \simeq \bar{\mu}^i(h) \) if \( \mu^i(S_{-i}(h)) = \bar{\mu}^i(S_{-i}(h)) \) for every \( z \succeq h \). We write \( \mu^i \simeq \bar{\mu}^i \) if \( \mu^i(h) \simeq \bar{\mu}^i(h) \) for all \( h \in H \).

**Lemma 10.** Fix a subset \( E_{-i} \subseteq S_{-i} \), a forward consistent belief system \( \bar{\mu}^i \in PB_i(E_{-i}) \),
and a strategy \( \hat{s}_i \). There exists a completely consistent \( \mu^i \in \text{PB}_i(E_{-i}) \) such that \( \mu^i(\cdot|h) \simeq \overline{\mu}^i(\cdot|h) \) for every \( h \in H_i(E_i) \).

**Proof:** Fix a forward consistent belief system \( \overline{\mu}^i \in \text{PB}_i(E_{-i}) \). Note that, for each \( h \), \( \hat{\chi}^h_{-i}(\overline{S}_{-i}) \subseteq \chi^h_{-i}(E_{-i}) \). With this, for each \( h \in H_i \), define a map \( \varsigma^h_{-i} : \chi^h_{-i}(E_{-i}) \to \hat{\chi}^h_{-i}(E_{-i}) \), which is the identity on \( \hat{\chi}^h_{-i}(E_{-i}) \) and is such that for each \( s_{-i} \in \chi^h_{-i}(E_{-i}) \), \( \varsigma^h_{-i}(s_{-i}) = s_{-i} \). Note that for all \( h' \succ h \) and \( s_{-i} \in \chi^h_{-i}(E_{-i}) \), \( s_{-i} \in S_{-i}(h') \) if and only if \( \varsigma^h_{-i}(s_{-i}) \in S_{-i}(h') \). It is convenient to impose the following consistency property: for all \( h' \succ h \) and \( s_{-i} \in S_{-i}(h') \subseteq S_{-i}(h) \), set \( \varsigma^h_{-i}(s_{-i}) = \varsigma^h_{-i}(s_{-i}) \).

Take the belief system \( (\mu^i(\cdot|h))_{h \in H} \) defined as follows: For every \( h \in H \) and \( \hat{s}_{-i} \in \hat{\chi}^h_{-i}(\overline{S}_{-i}) \), let

\[
\mu^i(\hat{s}_{-i}|h) = \overline{\mu}^i(\{s_{-i} : \varsigma^h_{-i}(s_{-i}) = \hat{s}_{-i}\}|h) = \overline{\mu}^i(\varsigma^{h,-1}_{-i}(\hat{s}_{-i})|h).
\]

Notice that for every \( h \in H \), \( \mu^i(S_{-i}(h')|h) = \overline{\mu}^i(S_{-i}(h')|h) \) for all \( h' \succ h \) (\( h' \in \overline{H} \)), which implies \( \overline{\mu}^i \simeq \mu^i \). Also, since \( \overline{\mu}^i \in \text{PB}_i(\overline{S}_{-i}) \), we have

\[
\overline{\mu}^i(\hat{\chi}^h_{-i}(E_{-i})|h) = \overline{\mu}^i(\varsigma^{h,-1}_{-i}(\hat{\chi}^h_{-i}(E_{-i}))|h) = \overline{\mu}^i(\chi^h_{-i}(E_{-i})|h) = 1
\]

for all \( h \in H \). Thus, \( \overline{\mu}^i \in \text{PB}_i(E_{-i}) \). We show that \( \overline{\mu}^i \) is a forward consistent belief system. Fix \( h \prec h' \) and \( E_{-i} \subseteq S_{-i}(h') \) such that \( \overline{\mu}^i(E_{-i}|h) > 0 \). Then, note first that \( \varsigma^{h,-1}_{-i}(E_{-i}) \subseteq S_{-i}(h') \) and that \( \overline{\mu}^i(\varsigma^{h,-1}_{-i}(E_{-i})|h) = \overline{\mu}^i(E_{-i}|h) > 0 \). Thus, the forward chain rule of \( \overline{\mu}^i \) applies and implies:

\[
\overline{\mu}^i(E_{-i}|h') = \overline{\mu}^i(\varsigma^{h',-1}_{-i}(E_{-i})|h') = \overline{\mu}^i(\varsigma^{h,-1}_{-i}(E_{-i})|h') = \frac{\overline{\mu}^i(\varsigma^{h,-1}_{-i}(E_{-i})|h)}{\overline{\mu}^i(S_{-i}(h')|h)} = \frac{\overline{\mu}^i(E_{-i}|h)}{\overline{\mu}^i(S_{-i}(h')|h)}
\]

where the second equality holds by the consistency condition of the maps \( (\varsigma^h_{-i})_{h \in H} \); the third equality by the forward chain rule of \( \overline{\mu}^i \); and the fourth equality by definition of \( \overline{\mu}^i \) and \( \overline{\mu}^i(S_{-i}(h')|h) = \overline{\mu}^i(S_{-i}(h')|h) \).
To summarize, we have a forward consistent belief system $\tilde{\mu}^i \in \text{PB}_i(E_{-i})$ such that $\tilde{\mu}^i(\cdot|h) \simeq \mu^i(\cdot|h)$ for every $h \in H$. Then, by Lemma 9, there exists a completely consistent $\mu^i$ in $\text{PB}_i(E_{-i})$ such that $\mu^i(\cdot|h) = \tilde{\mu}^i(\cdot|h)$ for all $h \in H_i(\bar{s}_i)$. Hence, $\mu^i(\cdot|h) \simeq \tilde{\mu}^i(\cdot|h)$ for all $h \in H_i(\bar{s}_i)$. ■

**Proof of Theorem 5:** Denote by $(\bar{S}^n)_{n \in \mathbb{N}_0}$ and $(\bar{S}^n)_{n \in \mathbb{N}_0}$ the backwards rationalizability algorithms with, respectively, forward consistent and completely consistent belief systems.

We show the following inductive hypothesis: for each $i$, (i) $\bar{S}^n_i \subseteq \bar{S}_i^n$, and (ii) for all $h \in H$ there is a map $\zeta_i^{n,h} : \chi_i^h(\bar{S}_i^n) \rightarrow \chi_i^h(\bar{S}_i^n)$ such that, for each $s_i \in \chi_i^h(\bar{S}_i^n)$, $\zeta_i^{n,h}(s_i)(h') = s_i(h')$ for all $h' \in H_i(s_i)$. The basis step is trivial, we show the statement holds at step $n + 1$.

Fix any $i \in I$. First, note that for any consistent CPS $\mu^i$ in $\text{PB}_i(\bar{S}^n_{-i})$, by (i) of the inductive hypothesis and monotonicity of the $\text{PB}_i(\cdot)$ operator, we have $\mu^i \in \text{PB}_i(\bar{S}_i^n_{-i})$, and since a consistent CPS is a forward consistent belief system, we have (i) $\bar{S}^{n+1}_i \subseteq \bar{S}_{i}^{n+1}$.

Now, take any $\bar{h} \in H$ and $s_i \in \chi_i(\bar{S}_{i})^{n+1}$. Then by definition there is a forward consistent belief system $\mu^i \in \text{PB}_i(\bar{S}_i^{n}_{-i})$ such that $s_i[\bar{h}] = s_i[\bar{h}]$ for some $s_i' \in \text{BR}_i^*(\mu^i)$.

Define a belief system $(\hat{\mu}^i(\cdot|h))_{h \in H} \in \chi_{h \in H} \Delta (S_{-i}(h))$ as follows: For $h = \emptyset$ and all $h \neq \emptyset$ such that $\mu^i(S_{-i}(h)[\bar{h}]) = 0$ for all $\bar{h} < h$, set:

$$\forall \bar{s}_{-i} \in \chi_{-i}^h(\bar{S}^n_{-i}), \quad \hat{\mu}^i(\bar{s}_{-i}|h) = \mu^i(\zeta_{n,h}^i(\bar{s}_{-i})|h).$$

We have the following properties: since for every $h' > h$ and $s_{-i} \in \chi_{-i}^h(\bar{S}^n_{-i})$, $s_{-i} \in S_{-i}(h')$ if and only if $\zeta_{n,h}^i(s_{-i}) \in S_{-i}(h')$, we have (a) $\hat{\mu}^i(S_{-i}(h')|h) = 0$ if and only if $\mu^i(S_{-i}(h')|h) = 0$ and (b) $\hat{\mu}^i \simeq \hat{\mu}^i$. Moreover, since $\mu^i \in \text{PB}_i(\bar{S}_i^n)$,

$$\hat{\mu}^i(\chi_{-i}^h(\bar{S}_i^n)|h) = \mu^i(\zeta_{n,h}^i(\chi_{-i}^h(\bar{S}_i^n))|h) = \mu^i(\chi_{-i}^h(\bar{S}_i^n)|h) = 1.$$ 

Thus, deriving all other probability measures by forward conditioning, we indeed obtain a forward consistent belief system $\hat{\mu}^i$. It can be checked using a forward conditioning argument that $\hat{\mu}^i \simeq \mu^i$, so that $\text{BR}_i^*(\mu^i) = \text{BR}_i^*(\hat{\mu}^i)$. Moreover, $\hat{\mu}^i$ belongs to $\text{PB}_i(\bar{S}_i^n)$. To see this,
take any $h' \succ h$. This implies $\chi_{-i}(\hat{S}^n_{-i}) \cap S_{-i}(h') \subseteq \chi_{-i}(\hat{S}^n_{-i})$, and so, if $\hat{\mu}^i(\cdot|h')$ is derived by conditioning from $\hat{\mu}^i(\cdot|h)$, because $\text{Supp} \hat{\mu}^i(\cdot|h) \subseteq \chi_{-i}(\hat{S}^n_{-i})$, we have:

$$\text{Supp} \hat{\mu}^i(\cdot|h') \subseteq \chi_{-i}(\hat{S}^n_{-i}) \cap S_{-i}(h') \subseteq \chi_{-i}(\hat{S}^n_{-i}).$$

To summarize, we have a forward consistent belief system $\hat{\mu}^i$ such that $\hat{\mu}^i \in \mathcal{P}B_i(\hat{S}^n_{-i})$ and $\hat{\mu}^i \simeq \mu^i$. Then by Lemma 10 there exists a consistent CPS $\tilde{\mu}^i \in \mathcal{P}B_i(\hat{S}^n_{-i})$ with $\tilde{\mu}^i(\cdot|\bar{h}) \simeq \mu^i(\cdot|\bar{h})$ for all $h \in H_i(s'_i)$. This in turn implies that there is a strategy $\tilde{s}_i \in B^*_i(\tilde{\mu}^i)$ such that, for each $h \in H_i(s'_i)$ with $h \succeq \bar{h}$, $\tilde{s}_i(h) = s'_i(h) = s_i(h)$. Thus, $\tilde{s}_i \in \hat{S}^{n+1}_i$ and any $\bar{s}_i \in S_i(\bar{h})$ with $\bar{s}_i|\bar{h} = \tilde{s}_i|\bar{h}$ belongs to $\chi^h_i(\hat{S}^{n+1}_i)$, so that we can let $\varsigma^{n+1,h}(s_i) = \bar{s}_i$. ■

**Appendix C: Comparison with Kreps & Wilson**

### 9.1 Complete consistency and sequences

We let $\Delta^*(\Omega) \subseteq \Delta(\Omega)$ denote the set of strictly positive probability measures on $\Omega$.

For any complete CPS $\mu \in \Delta^*(\Omega)$, say that $x$ is $\mu$-infinitely more likely than $y$, written $x \gg^\mu y$, if $\mu(\{x, y\}) = 1$ (i.e., $\mu(\{y\}) = 0$). The chain rule implies that the (typically incomplete) binary relation $\gg^\mu$ is transitive. Therefore, $\mu$ yields an ordered partition \( \left( \Omega^1_0, ..., \Omega^\mu_{L(\mu)} \right) \), where—for $0 \leq k < \ell \leq L(\mu)—each element of $\Omega^\mu_k$ is $\mu$-infinitely more likely of each element of $\Omega^\mu_\ell$. With this, $\mu(\cdot|\Omega^\mu_0) = \mu(\cdot|\Omega)$ is the unconditional belief. More generally, let $k(\mu, C)$ denote the lowest index in \{0, ..., $L(\mu)$\} such that $C \cap \Omega_k \neq \emptyset$; then $\mu(C \cap \Omega_{k(\mu, C)}) = 1$ and

$$\mu(\cdot|C \cap \Omega_{k(\mu, C)}) = \mu(\cdot|C),$$

which implies the following:

**Remark 9.** A complete CPS $\mu \in \Delta^*(\Omega)$ is determined by the binary conditional proba-

\[23\text{See Lemma 4.(i) in Appendix A of Battigalli & De Vito (2021).}\]
bility values \((\mu (\cdot | \{x,y\}))_{x,y \in \Omega}\). Indeed, if two complete CPSs \(\mu\) and \(\nu\) are such that

\[
(\mu (\cdot | \{x,y\}))_{x,y \in \Omega} = (\bar{\mu} (\cdot | \{x,y\}))_{x,y \in \Omega}
\]

then \((\Omega^\mu_0, \ldots, \Omega^\mu_{L(\mu)}) = (\Omega^\nu_0, \ldots, \Omega^\nu_{L(\nu)})\), which implies \(\mu = \nu\).

Ordered partitions can also be used to relate complete CPSs to converging sequences of strictly positive probability measures. First note that if a sequence \((\nu_n)_{n \in \mathbb{N}} \in \Delta^\circ (\Omega)^N\) converges and yields an array \((\mu(\cdot | C))_{\emptyset \neq C \subseteq \Omega} \in \times_{\emptyset \neq C \subseteq \Omega} \Delta (C)\) in the limit, that is,

\[
\forall C, \forall \omega \in C, \mu (\omega | C) = \lim_{n \to \infty} \frac{\nu_n(\omega)}{\nu_n(C)}, \quad (1)
\]

then \((\mu(\cdot | C))_{\emptyset \neq C \subseteq \Omega}\) must be a complete CPS. Next note that, for every complete CPS \(\mu\), the sequence

\[
(\nu_n)_{n \in \mathbb{N}} = \left( K_n \sum_{k=0}^{L(\mu)} n^{-k} \mu (\cdot | \Omega^\mu_k) \right)_{n \in \mathbb{N}} \in \Delta^\circ (\Omega)^N
\]

(where \(K_n = \left( \sum_{k=0}^{L(\mu)} n^{-k} \right)^{-1}\) is a normalizing constant) converges and yields \(\mu\) in the limit. Therefore:

**Remark 10.** An array of probability measures \(\mu = (\mu(\cdot | C))_{\emptyset \neq C \subseteq \Omega} \in \times_{\emptyset \neq C \subseteq \Omega} \Delta (C)\) is a complete CPS if and only if there is a convergent sequence of strictly positive probability measures \((\nu_n)_{n \in \mathbb{N}}\) that yields \(\mu\) in the limit as in eq. (1).

### 9.2 Kreps & Wilson systems of beliefs

An *assessment à la* Kreps & Wilson (1982) is a pair \((\xi, \beta)\), where \(\xi \in \times_{i \in I} (\times_{h_i \in H_i} \Delta (h_i))\) is a system of beliefs over non-terminal nodes conditional the information sets containing them, and \(\beta\) is a profile of behavior (i.e., locally randomized) strategies. Kreps & Wilson’s notion of consistency (used to define sequential equilibrium, a refinement of the Nash equilibrium concept) requires that the beliefs of different players “agree” and that they satisfy what
amounts to a strong across-players independence property. Such restrictions are not germane to our (correlated) rationalizability analysis. Thus, we make them mute by considering personal assessments in two-player games.

A personal assessment of player \( i \) in a two-player game is a pair

\[
(\xi^i, \beta_{-i}) \in (\times_{h_i \in H_i} \Delta(h_i)) \times (\times_{h_{-i} \in H_{-i}} \Delta(\mathcal{A}_{-i}(h_{-i}))),
\]

where \( \xi^i \) is a system of beliefs over nodes/histories in the information sets of \( i \) and \( \beta_{-i} \) is a behavior strategy of the co-player \(-i\), which we may interpret as a probabilistic conjecture of \( i \) about \(-i\). A personal assessment \((\xi^i, \beta_{-i})\) is consistent in the sense of Kreps & Wilson if there exists a converging sequence of strictly positive behavior strategies \((\beta_{-i,n})_{n \in \mathbb{N}} \in (\times_{h_{-i} \in H_{-i}} \Delta^\circ(\mathcal{A}_{-i}(h_{-i})))^\mathbb{N}\) such that \( \beta_{-i} = \lim_{n \to \infty} \beta_{-i,n} \) and

\[
\xi^i(x|h_i) = \lim_{n \to \infty} \frac{\mathbb{P}_{\beta_{-i,n}}(x)}{\mathbb{P}_{\beta_{-i,n}}(h_i)},
\]

for all \( h_i \in H_i \) and \( x \in h_i \), where \( \mathbb{P}_\beta \) denotes the probability of histories induced by randomized strategy profile (pair) \( \beta \), and \( \beta^\circ_i \) is an arbitrary strictly positive randomized strategy of \( i \).\(^{24}\) By Kuhn Theorem (Kuhn 1953, Theorem 4), the latter condition is equivalent to the existence of a converging sequence of strictly positive mixed strategies \((\nu_{-i,n})_{n \in \mathbb{N}} \in \Delta^\circ(S_{-i})^\mathbb{N}\) such that \( \nu_{-i} = \lim_{n \to \infty} \nu_{-i,n} \) and

\[
\xi^i(x|h_i) = \lim_{n \to \infty} \frac{\mathbb{P}_{\nu_{-i,n}}(x)}{\mathbb{P}_{\nu_{-i,n}}(h_i)},
\]

where \( \nu_{-i} \) is the mixed strategy induced by \( \beta_{-i} \) according to Kuhn’s transformation and \( \nu^\circ_i \) is an arbitrary strictly positive randomized strategy.\(^{25}\)

A personal assessment \((\xi^i, \beta_{-i})\) yields a conditional probability measure \( \mathbb{P}_{\xi^i,\beta_{-i},\beta_i}(\cdot|h_i) \)

\(^{24}\)By perfect recall, it does not matter which one, because the factors involving \( \beta^\circ_i \) in the numerator and denominator cancel out.

\(^{25}\)As above, by perfect recall, it does not matter which one, because the factors involving \( \nu^\circ_i \) in the numerator and denominator cancel out.
over paths for every behavior strategy $\beta_i$ of $i$ conditional on every information set $h_i$ of $i$: if $z = (x, a^{\ell(x)+1}, \ldots, a^{\ell(z)})$ with $x \in h_i$ of length $\ell(x)$,

$$
\mathbb{P}_{\xi^i, \beta_{-i}, \beta_i} (z|h_i) = \left( \prod_{k=1}^{\ell(z)-\ell(x)} \prod_{j \in \iota(x)} \beta_j \left( a_j^{\ell(x)+k} | h_j^{\ell(x)+k} \right) \right)^{\ell_i(x)} \xi^i (x|h_i).
$$

Similarly, a system of beliefs $\mu^i$ about the co-player’s strategies yields a conditional probability measure $\mathbb{P}_{\mu^i, s_i} (\cdot|h_i)$ over paths for every strategy $s_i$ of $i$ conditional on every information set $h_i$ of $i$: let $Z(h_i)$ denote the set of terminal nodes/paths following some node in $h_i$, then

$$
\mathbb{P}_{\mu^i, s_i} (z|h_i) = \begin{cases} 
0 & \text{if } s_i \notin S_i(z) \text{ or } z \notin Z(h_i), \\
\mu^i (\Sigma_{-i}(z)) & \text{if } s_i \in S_i(z) \text{ and } z \in Z(h_i).
\end{cases}
$$

Note that pure strategies are special cases of degenerate behavior strategies. With a slight abuse of notation we use the same symbol $s_i$ for a pure strategy of $i$ and the corresponding degenerate behavior strategy of $i$. With this, by Remark 5 we obtain:

**Remark 11.** A belief system $\mu^i$ about the strategies of $-i$ is completely consistent as per Definition 5 if and only if there exists a consistent personal assessment $(\xi^i, \beta_{-i})$ such that

$$
\mathbb{P}_{\mu^i, s_i} (z|h_i) = \mathbb{P}_{\xi^i, \beta_{-i}, s_i} (z|h_i)
$$

for all $h_i \in H_i$, $z \in Z(h_i)$, and $s_i \in S_{-i}(h_i)$.

**References**


