Mandatory Versus Discretionary Spending: the Status Quo Effect

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Abstract

Do mandatory spending programs such as Social Security and Medicare improve efficiency? To address this question, We analyze a model with two parties allocating a fixed budget to a public good and private transfers each period over an infinite horizon. We compare two institutions: one in which the public good spending is discretionary and the other in which it is mandatory. We model mandatory spending as an endogenous status quo since it is enacted by law and remains in effect until changed. Mandatory programs always result in higher public good spending. Over-provision of the public good can arise as a transient state when parties are highly polarized, but in steady states, the level of public good spending is either below or equal to the efficient level, and is always closer to the efficient level than when public good spending is discretionary. The party that places a higher value on the public good benefits from mandatory programs; more surprisingly, the party that places a lower value on the public good also benefits from mandatory programs, provided that parties are patient, persistence of power is low, and polarization is low. Under these conditions, mandatory programs ex ante Pareto dominate discretionary programs.

Keywords: budget negotiations, mandatory programs, dynamic bargaining, endogenous status quo, public goods

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1 Introduction

Government budgets are primarily decided through negotiations. Institutions governing budget negotiations play an important role in fiscal policy outcomes. These institutions vary across countries and time, and examining their effects is an important step towards understanding these variations.[1] In this paper, we are interested in the role of a particular institution: mandatory spending programs.

Mandatory spending is expenditure that is governed by formulas or criteria set forth in enacted law, rather than by periodic appropriations. As such, unless explicitly changed, the previous year's spending bill applies to the current year. By contrast, discretionary spending is expenditure that is governed by annual or other periodic appropriations. Examples of mandatory spending in the U.S. include entitlement programs such as Social Security and Medicare, while discretionary spending consists of mostly military spending. As Figure 1 shows, mandatory spending has been growing as a share of GDP in the U.S.. In 2011, mandatory spending was $2 trillion compared to discretionary spending of $1.3 trillion. Because of these trends, mandatory spending has been at the heart of recent budget negotiations and is consistently ranked as a top issue by the public and policymakers.[2]

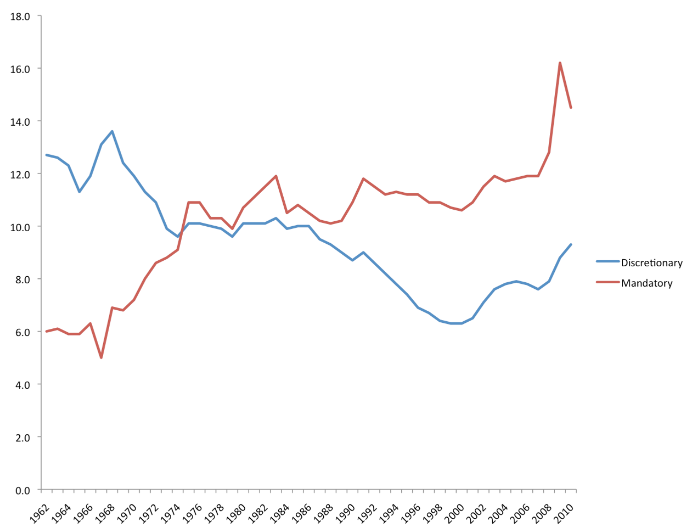


Figure 1: US mandatory versus discretionary spending as % of GDP, 1962-2010

We take a first step towards understanding the effects of mandatory spending programs on budget negotiations and their implications for the efficient provision of public goods.[3] In

[1]See International Budget Practices and Procedures Database of the OECD, which is available at www.oecd.org/gov/budget/database.

[2]See http://www.people-press.org/2012/06/14/debt-and-deficit-a-public-opinion-dilemma/.

[3]The definition of a public good requires that it is non-excludable and non-rivalrous. However, our model only requires that the good be non-excludable, and as such, is also applicable to a common pool resource. Entitlement programs such as Social Security and Medicare are often thought of as a common pool resource.

our model, two parties decide how to allocate an exogenously given budget each period over an infinite horizon. The parties must determine the allocation to spending on a public good and private transfers for each party. Parties potentially differ in the value they attach to the public good and we refer to the degree of such differences as the level of *polarization* between the parties. Each period a party is randomly selected to make a budget proposal. The probability that the last period's proposer is selected to be the proposer in the current period captures the *persistence of political power*. The proposer makes a take-it-or-leave-it budget offer. If the other party accepts the offer, it is implemented; otherwise, the status quo prevails. We compare two institutions that govern the status quo: a political system in which public good spending is discretionary, in which case the status quo public good allocation is normalized to zero each period; and a political system in which public good spending is mandatory, in which case the status quo public good allocation in any period is given by what was implemented in the previous period, and hence is endogenous. Under both institutions, we assume that the status quo allocation to private transfers is zero.

Under discretionary public spending, in the unique Markov perfect equilibrium, the party in power under-provides the public good and extracts the maximum private transfer for itself. Under mandatory public spending, the degree of polarization plays an important role in determining equilibrium allocations. We characterize Markov perfect equilibria first when polarization is low and second when polarization is high.

In the low-polarization case, the levels of public good spending proposed by both parties are either below or equal to the efficient level in both transient and steady states, and are always closer to the efficient level than when public good spending is discretionary. To understand why, note that mandatory programs create a channel to provide insurance against power fluctuations because they raise the bargaining power of the non-proposing party by raising its status quo payoff. When the status quo level of the public good is low, the party that places a higher value on the public good exploits this insurance effect by proposing a level of public good spending higher than what it would propose without mandatory programs. The incumbent receives all the private transfers when the status quo level of public good spending is below the efficient level, but when the status quo is higher than the efficient level, the incumbent proposes to lower the public good spending to the efficient level, and gives some private transfer to the opposition party so as to pass the budget proposal.

In the high-polarization case, the insurance effect from mandatory programs can lead the party placing a high value on the public good to propose a level of public good spending above the efficient level, creating temporary "over-provision." This is only temporary because of power fluctuations – once the party who places a lower value on the public good comes into power, it will lower the level of public good to the efficient level. Indeed, the unique steady state in the high-polarization case involves the efficient level of public good spending and private transfers only to the incumbent party.

As is typical in dynamic games, we cannot appeal to general theorems on uniqueness of

Markov perfect equilibrium under mandatory public spending; however, we show that under some conditions, there are no steady states other than the ones in the equilibria we characterize, allowing us to conduct comparative statics analysis and make welfare comparisons.

One interesting result is that greater power fluctuations (lower persistence of power) lead to greater efficiency with mandatory programs. This is because greater power fluctuations provides stronger insurance incentives leading to a higher level of the public good in the steady state. This is in contrast to Besley and Coate (1998), who show that power fluctuations undermine incentives to invest in the public good and lead to less efficient outcomes.

Perhaps it is not surprising that the party placing a higher value on the public good benefits from the introduction of mandatory programs. But strikingly, we show that the party placing a lower value on the public good also benefits from mandatory programs, provided that the parties are patient, the persistence of power is low, and polarization is low. Intuitively, if the party with a lower value cares sufficiently about future payoffs, expects power to fluctuate frequently, and places a relatively high value on the public good, then the insurance benefit from mandatory programs is high, making the lower-value party better off. Thus, mandatory programs can be Pareto improving, and this may explain why they are successfully enacted in the first place.

Related literature

The distinction between private goods and public goods goes back to at least Adam Smith (1776), who concluded that public goods must be provided by the government since the market fails to do so. By now there exists a vast literature formally studying public goods, starting with the classic work by Wicksell (1896) and Lindahl (1919).

Our paper adds to the literature on public goods provision with political economy frictions as surveyed in Persson and Tabellini (2000). A subset of this literature analyzes public good provision under alternative political institutions. For example, Lizzeri and Persico (2001) investigate the role of alternative electoral systems in the provision of public goods. Our paper focuses on a particular institution, namely mandatory spending programs.

We consider the determination of public good provision in a legislative bargaining framework, similar to Baron (1996), Leblanc, Snyder, Tripathi (2000), Volden and Wiseman (2007), and Battaglini and Coate (2007, 2008). With the exception of Baron (1996), these papers do not consider mandatory programs. Baron (1996) presents a dynamic theory of bargaining over public goods programs in a majority-rule legislature where the status quo in a session is given by the program last enacted. He models the provision of public good as a unidimensional policy choice, and analyzes the equilibrium outcome under mandatory programs only. Our paper contributes to this literature by incorporating both mandatory programs and discretionary programs and exploring their efficiency implications.

Building on the seminal papers of Rubinstein (1982) and Baron and Ferejohn (1989), most papers on political bargaining study environments where the game ends once an agreement is reached. Starting with the works of Epple and Riordan (1987) and Baron (1996), there is now an active literature on bargaining with an endogenous status quo. These include Baron and Heron (2003), Kalandrakis (2004), Bernheim, Rangel and Rayo (2006), Anesi (2010), Diermeier and Fong (2011), Anesi and Seidmann (2012), Bowen and Zahran (2012), Duggan and Kalandrakis (2012), Dziuda and Loeper (2012), Nunnari (2012), and Piguillem and Riboni (2012). Unlike our paper, these papers consider bargaining over either a unidimensional policy or the division of private benefits. Thus, they do not address how mandatory programs affect the provision of public goods, which is the question at the heart of our paper.

Our work is also related to the literature on power fluctuations, which includes Persson and Svensson (1989), Alesina and Tabellini (1990), Besley and Coate (1998), Grossman and Helpman (1998), Hassler, Storesletten and Zilibotti (2007), Klein, Krusell, Ríos-Rull (2008), Azzimonti (2011), and Song, Storesletten and Zilibotti (2012). These papers show that power fluctuations can lead to political failures. By considering equilibria that are non-Markov, Dixit, Grossman and Gül (2000) and Acemoglu, Golosov, Tsyvinski (2010) establish the possibility of political compromise for the purpose of risk sharing under power fluctuations. In our paper, by contrast, even if parties use Markov strategies, they can reach a certain degree of compromise because with mandatory programs, the party in power cannot fully undo the decisions and allocations of the past. Moreover, we discuss political compromise in the context of public good provision, which has efficiency implications beyond risk sharing.

Mandatory programs generate a dynamic link between policy in a given period and political power in future periods. In that sense, our paper is also related to Bai and Lagunoff (2011), who analyze policy endogenous power.

The rest of the paper is organized as follows. In the next section we describe our model. In Section 3 we consider the social planner's problem. In Section 4 we give the definition of Markov perfect equilibrium for our model. We analyze the case of discretionary public spending in Section 5 and the case of mandatory public spending in Section 6. We discuss equilibrium dynamics in Section 7 and efficiency implications of mandatory programs in Section 8. In Section 9, we conclude and discuss some promising extensions.

2 Model

Consider a stylized economy and political system with two parties labeled H and L. Time is infinite and indexed by $t = 0, 1 \ldots$. Each period the two parties decide how to allocate an exogenously given dollar. The budget consists of an allocation to spending on a public good, g^t, and private transfers for each party, x_H^t and x_L^t. Denote by $b^t = (g^t, x_H^t, x_L^t)$ the budget implemented at time t. Let $B = \{\mathbf{y} \in \mathbb{R}_+^3 : \sum_{i=1}^3 y_i \leq 1\}$. Feasibility requires that $b^t \in B$. The stage utility for party i from the budget b^t is

$$u_i(b^t) = x_i^t + \theta_i \ln(g^t),$$

where θ_i captures the relative value of the public good for party $i \in \{H, L\}$ of public goods.[4,5] We assume $\theta_H \geq \theta_L \geq 0$ and $\theta_H + \theta_L < 1$, which ensures that the efficient level of public good spending is lower than the size of the budget, as we show later.

The parties have a common discount factor δ. Party i seeks to maximize its discounted dynamic payoff from a infinite sequence of budgets, $\sum_{t=0}^{\infty} \delta^t u_i(b^t)$.

Political system

We consider a political system with unanimity rule. Each period a party is randomly selected to make a proposal for the allocation of the dollar. The probability of being proposer is Markovian. Specifically, p is the probability that party i is the proposer in period $t+1$ if it was the proposer in period t, which we interpret as the persistence of political power.

At the beginning of period t, the identity of proposing party is realized. The proposing party makes a proposal for the budget, denoted by z^t. If the responding party agrees to the proposal, it becomes the implemented budget for the period, so $b^t = z^t$; otherwise, $b^t = s^t$, where s^t is the status quo budget.

Let $S \subseteq B$ be the set of feasible status quo budgets, and let $\zeta : B \to S$ be an exogenous function that determines the status quo in period $t+1$ as a function of the budget in period t. So $s^{t+1} = \zeta(b^t)$ for all t. The set S and the function ζ are determined by the rules governing mandatory and discretionary programs. For example, if no mandatory programs are allowed, then $S = \{(0, 0, 0)\}$ and $\zeta(b) = (0, 0, 0)$ for all $b \in B$. That is, in the event that the responding party rejects the proposal, no spending on either public good or private transfers will occur that period. At the other extreme where all spending is in the form of mandatory programs, $S = B$ and the status quo is $\zeta(b) = b$. That is, disagreement on a new budget implies the last period's budget is implemented.

We compare two institutions: one in which all spending is discretionary (that is, $\zeta(b) = (0, 0, 0)$), and the other in which spending on the public good is mandatory, but private transfers are discretionary (that is, $\zeta(b) = (g, 0, 0)$ for any $b = (g, x_H, x_L)$). We find it reasonable to think of the U.S. federal budget as allocating private transfers through discretionary spending and public goods through mandatory programs. This is because private transfers in the form of earmarks designated for particular districts are typically appropriated annually, whereas social programs such as Social Security and Medicare are funded through mandatory programs and provide benefits from which constituents of any particular party cannot be excluded. As mentioned in the introduction, although Social Security and Medicare do not satisfy the "non-rivalrous" criterion, they satisfy the"non-excludable" criterion

[4] Our results would go through if instead we assumed $u_i(b^t) = x_i^t + \theta_i \ln(\alpha_i g^t)$ for some constant $\alpha_i > 0$. We can think of α_i as the fraction of the common pool resource party i extracts in a second stage game after the total allocation to the public good is agreed upon. In that sense, our results apply to settings where g^t is non-excludable but not necessarily non-rivalrous.

[5] We assume log utility for tractability. This functional form is commonly used, for example, in Azzimonti (2011) and in Song et al. (2012). In the numerical analysis we conducted using CRRA utility functions, we obtained qualitatively same results.

and are therefore often thought of as a common pool resource. Our model applies when g is a common pool resource; for expositional convenience, we refer to g as a "public good."

3 Social planner's problem: Pareto efficient allocations

As a benchmark, consider the Pareto efficient allocations. These solve

$$\max_{\{b^t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u_L(b^t)$$
$$\text{s.t.} \sum_{t=0}^{\infty} \delta^t u_H(b^t) \geq \overline{U} \text{ and } b^t \in B \text{ for all } t.$$

We find that any Pareto efficient allocation with $x_L^{t'} > 0$ and $x_H^{t''} > 0$ for some t' and t'' must have $g^t = \theta_H + \theta_L$ for all t.[6] (Note that $g^t = \theta_H + \theta_L$ also uniquely satisfies the Samuelson condition for efficient provision of public goods.) We henceforth refer to $\theta_H + \theta_L$ as the efficient level of the public good.

For contrast, consider party i's ideal allocation in any period, which solves $\max_{b \in B} u_i(b)$. Let us call the level of public good that solves this problem the *dictator* level for party i. Clearly party i would not choose to allocate any spending to party j, hence the dictator level solves $\max_g 1 - g + \theta_i \ln(g)$. This is maximized at $\theta_i < \theta_H + \theta_L$. So party i's ideal level of the public good in any period results in under-provision of the public good. In a political system that is a dictatorship in every period, this is the level of public good allocated.[7]

4 Markov perfect equilibrium

We consider stationary Markov perfect equilibria.[8] A Markov strategy depends only on payoff-relevant events, and a stationary Markov strategy does not depend on calendar time. In our model, the payoff-relevant state in any period is the status quo s. Thus, a (pure) stationary Markov strategy for party i is a pair of functions $\sigma^i = (\pi^i, \alpha^i)$, where $\pi^i : S \to B$ is a proposal strategy of party i and $\alpha^i : S \times B \to \{0, 1\}$ is an acceptance strategy of party i. Party i's proposal strategy $\pi^i = (\gamma^i, \chi_H^i, \chi_L^i)$ associates with each status quo s an amount of

[6] A proof is available in a supplementary Appendix.

[7] If it is the same party who is the dictator in every period, then clearly in every period it chooses $g = \theta_i$; if different parties become the dictator in different periods, then whenever party i is the dictator, it still chooses $g = \theta_i$ in any Markov perfect equilibrium, but it is possible to have $g \neq \theta_i$ in a non-Markovian equilibrium, similar to Dixit et al. (2000) and Acemoglu et al. (2011).

[8] By focusing on stationary Markov perfect equilibria, we rule out punishment strategies that depend on payoff irrelevant past events. This is a commonly used solution concept in political bargaining models. See, for example, Battaglini and Coate (2008), Diermeier and Fong (2011), Dziuda and Loeper (2012). This solution concept seems reasonable in the context of political bargaining where there is turnover within parties since stationary Markov equilibria are simple and do not require coordination.

public good spending, denoted by $\gamma^i(s)$, an amount of private spending for party H, denoted by $\chi_H^i(s)$, and an amount of private spending for party L, denoted by $\chi_L^i(s)$. Party i's acceptance strategy $\alpha^i(s,z)$ takes the value 1 if party i accepts the proposal z offered by party $j \neq i$ when the status quo is s, and 0 otherwise. A stationary Markov equilibrium is a subgame perfect Nash equilibrium in stationary Markov strategies. We henceforth refer to a stationary Markov equilibrium simply as an equilibrium.

To each strategy profile $\sigma = (\sigma_H, \sigma_L)$, and each party i, we can associate two functions $V_i(\cdot;\sigma)$ and $W_i(\cdot;\sigma)$. The value $V_i(s;\sigma)$ represents the dynamic payoff of party i if i is the *proposer* in the current period and the value $W_i(s;\sigma)$ represents the dynamic payoff of party i if i is the *responder* in the current period, when the status quo is s and the strategy profile σ will be played from the current period onwards.

We restrict attention to equilibria in which (i) $\alpha^i(s,z) = 1$ when party i is indifferent between s and z; and (ii) $\alpha^i(s,\pi^j(s)) = 1$ for all $s \in S$, $i, j \in \{H, L\}$ with $j \neq i$. That is, the responder accepts any proposal that it is indifferent between accepting and rejecting, and the equilibrium proposals are always accepted.[9] Given the restriction that equilibrium proposals are always accepted, in these equilibria the implemented budget is the proposed budget.

Call a strategy profile σ and associated payoff quadruple (V_H, W_H, V_L, W_L) a *strategy-payoff pair*. In what follows, we suppress the dependence of the payoff quadruple on σ for notational convenience. Given the restrictions that parties accept when indifferent and equilibrium proposals are always accepted, a strategy-payoff pair is an *equilibrium strategy-payoff pair* if and only if

(E1) Given (V_H, W_H, V_L, W_L), for any proposal $z = (g', x'_H, x'_L) \in B$ and status quo $s = (g, x_H, x_L) \in S$, the acceptance strategy $\alpha^i(s,z) = 1$ if and only if

$$x'_i + \theta_i \ln(g') + \delta[(1-p)V_i(\zeta(z)) + pW_i(\zeta(z))] \geq K_i(s) \tag{1}$$

where $K_i(s) = x_i + \theta_i \ln(g) + \delta[(1-p)V_i(s) + pW_i(s)]$ denotes the dynamic payoff of i from the status quo $s = (g, x_H, x_L)$.

[9] Any equilibrium is payoff equivalent to some equilibrium (possibly itself) that satisfies (i) and (ii). We take two steps to show this: first, any equilibrium is payoff equivalent to some equilibrium that satisfies (i); second, any equilibrium that satisfies (i) is payoff equivalent to some equilibrium that satisfies (i) and (ii).

To prove the first step, consider an equilibrium σ^E that does not satisfy (i). Then there exists a status quo s' and a proposal $z' = (g', x'_H, x'_L)$ such that the responder i is indifferent between s' and z' but $\alpha^i(s', z') = 0$. If z' gives the proposer j a lower payoff than $\pi^j(s')$, then σ^E is payoff equivalent to the equilibrium which is the same as σ^E except that $\alpha^i(s', z') = 1$ because j would not propose z' when the status quo is s'. If z' gives the proposer a strictly higher payoff than $\pi^j(s')$, then there exists a proposal z'' that gives the responder a higher payoff than z' does and gives the proposer a strictly higher payoff than $\pi^j(s')$. That is, z'' is a strictly better proposal than $\pi^j(s')$, contradicting that σ^E is an equilibrium.

To prove the second step, consider an equilibrium σ^E that satisfies (i) but not (ii). Then there exists a status quo s' such that $\alpha^i(s', \pi^j(s')) = 0$, implying that the proposer receives the status quo payoff by proposing $\pi^j(s')$ when the status quo is s'. By condition (i), the status quo is a proposal that is accepted. It follows that σ^E is payoff equivalent to the equilibrium which is the same as σ^E except that $\pi^j(s') = s'$.

(E2) Given (V_H, W_H, V_L, W_L) and α^j, for any status quo $s = (g, x_H, x_L) \in S$, the proposal strategy $\pi^i(s)$ of party $i \neq j$ satisfies:

$$\pi^i(s) \quad \in \quad \underset{z=(g',x'_H,x'_L)\in B}{\arg\max} \; x'_i + \theta_i \ln(g') + \delta[pV_i(\zeta(z)) + (1-p)W_i(\zeta(z))] \tag{2}$$

$$\text{s.t.} \quad x'_j + \theta_j \ln(g') + \delta[(1-p)V_j(\zeta(z)) + pW_j(\zeta(z))] \geq K_j(s). \tag{3}$$

(E3) Given $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$, the payoff quadruple (V_H, W_H, V_L, W_L) satisfies the following functional equations for any $s = (g, x_H, x_L) \in S$, $i, j \in \{H, L\}$ with $j \neq i$:

$$V_i(s) = \chi_i^i(s) + \theta_i \ln(\gamma^i(s)) + \delta[pV_i(\zeta(\pi^i(s))) + (1-p)W_i(\zeta(\pi^i(s)))], \tag{4}$$

$$W_i(s) = \chi_i^j(s) + \theta_i \ln(\gamma^j(s)) + \delta[(1-p)V_i(\zeta(\pi^j(s))) + pW_i(\zeta(\pi^j(s)))]. \tag{5}$$

We establish existence of equilibria by construction. We begin by considering the benchmark model of all discretionary, and then consider the model in which spending on the public good is mandatory and private transfers are discretionary.

5 Discretionary public spending

Suppose all spending is discretionary, implying that the status quo level of public good spending as well as private transfers is zero. That is, $\zeta(b) = (0, 0, 0)$ for any $b \in B$. Because of log utility in the public good, the responder's status payoff $K_i(s)$ is $-\infty$ for any status quo s, and hence the responder's acceptance constraint is not binding. The proposer therefore sets the public good at the dictator level θ_i every period and there is under-provision of the public good. This leads to the first proposition.[10]

Proposition 1. *If all spending is discretionary, then the public good is provided at the dictator level, and there is under-provision of the public good in equilibrium.*

One implication of Proposition 1 is that with only discretionary spending, the equilibrium allocation to the public good follows a Markov process. Specifically, if i is the proposer in the current period, spending on the public good next period is θ_i with probability p (if i is the proposer in the next period), and θ_j with probability $1-p$ (if j is the proposer in the next period). In Section 7, we compare this long-run behavior of spending on the public good under discretionary programs to the long-run behavior under mandatory programs, and assess the efficiency implications in Section 8.

[10]Because of log utility in g, Proposition 1 holds for arbitrary status quo rules for private transfers, as long as public spending is discretionary.

6 Mandatory public spending

In this section, we consider the case in which only the public good spending is mandatory, that is, $\zeta(b) = (g, 0, 0)$ for any $b = (g, x_H, x_L) \in B$. In the rest of this section, to lighten notation, we suppress the dependence of π^i and α^i on the components of the status quo other than g, and write $\pi^i(g)$ and $\alpha^i(g, z)$ instead of $\pi^i(s)$ and $\alpha^i(s, z)$. We also refer to the status quo public good level as the status quo. To obtain some intuition for the equilibrium under mandatory public spending and discretionary private spending, we first analyze a one-period model with an exogenous status quo and then analyze the infinite horizon game.

6.1 A one-period model

Suppose that party i is the proposer and seeks to maximize its utility $u_i(z) = x'_i + \theta_i \ln(g')$ in one period, given an exogenous status quo g and unanimity rule. Party i's one-shot problem that is analogous to (E2) is

$$\begin{aligned} \pi^i(g) \quad &\in \quad \underset{z=(g',x'_H,x'_L)\in B}{\arg\max} \quad x'_i + \theta_i \ln(g') \\ & \qquad\qquad \text{s.t.} \quad x'_j + \theta_j \ln(g') \geq K_j(g), \end{aligned}$$

where $K_j(g) = \theta_j \ln(g)$.

Proposition 2. *In the one-period model with mandatory public spending and discretionary private spending, the proposal strategy for party $i \in \{H, L\}$ is*

$$\gamma^i(g) = \begin{cases} \theta_i & \text{for } g \leq \theta_i, \\ g & \text{for } \theta_i \leq g \leq \theta_H + \theta_L, \\ \theta_H + \theta_L & \text{for } \theta_H + \theta_L \leq g \leq 1, \end{cases}$$

$$\chi^i_j(g) = \begin{cases} 0 & \text{for } g \leq \theta_H + \theta_L, \\ \theta_j[\ln(g) - \ln(\theta_H + \theta_L)] & \text{for } \theta_H + \theta_L \leq g \leq 1, \end{cases}$$

and $\chi^i_i(g) = 1 - \gamma^i(g) - \chi^i_j(g)$.

We relegate the proof of Proposition 2 to the Appendix. Henceforth all omitted proofs are in the Appendix. We illustrate $\gamma^i(g)$ in Figure 2 for the one-period problem.

Notice that when the status quo level of the public good is below proposer i's static ideal θ_i, proposer i has a constant choice of $\gamma^i(g)$ equal to its static ideal. Intuitively, when the status quo is below some threshold, the responder's acceptance constraint does not bind, and hence the proposer is able to set its ideal level of the public good and extract the remainder of the budget as a transfer for itself. When the status quo is above this threshold, the responder's acceptance constraint binds. For some intermediate range of the status quo, it is optimal for the proposer to maintain the level of the public good at the status quo

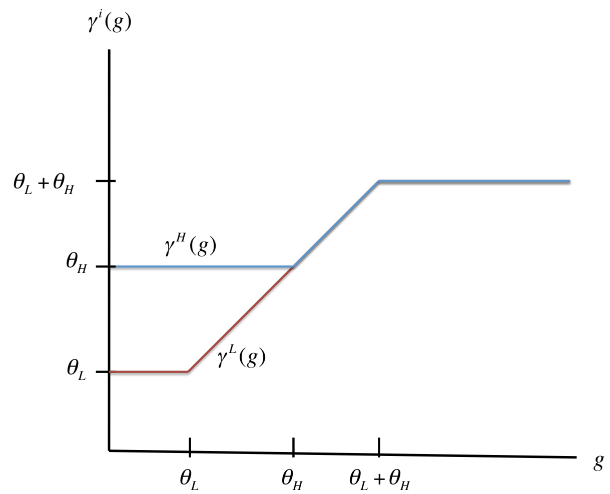


Figure 2: $\gamma^i(g)$ in one-period problem

and extracts the remaining budget as a transfer. For status quos above the efficient level $\theta_H + \theta_L$, since the sum of the marginal benefit of the public good is lower than the sum of the marginal benefit of transfers, the proposer does best by lowering the level of the public good to the efficient level, giving the responder a transfer to make the responder indifferent, and extracting the remainder of the budget for itself. Hence $\gamma^i(g)$ is constant at the efficient level when the status quo is above the efficient level. These strategies give the following payoffs to the proposer i and responder j respectively in the one-period model.

$$V_i(g) = \begin{cases} 1 - \theta_i + \theta_i \ln(\theta_i) & \text{if } g \leq \theta_i, \\ 1 - g + \theta_i \ln(g) & \text{if } \theta_i \leq g \leq \theta_H + \theta_L, \\ 1 - \theta_H - \theta_L - \theta_j \ln(g) + (\theta_H + \theta_L) \ln(\theta_H + \theta_L) & \text{if } \theta_H + \theta_L \leq g, \end{cases}$$

and

$$W_j(g) = \begin{cases} \theta_j \ln(\theta_i) & \text{if } g \leq \theta_i, \\ \theta_j \ln(g) & \text{if } \theta_i \leq g. \end{cases}$$

As shown above, the proposer's equilibrium payoff $V_i(g)$ is constant and maximized when the status quo is below its static ideal. This is where the responder's constraint is not binding and the proposer obtains the highest payoff possible. The proposer's payoff is decreasing for status quos higher than its ideal because now the responder's constraint is binding, and the responder's status quo payoff is increasing in the status quo. Similarly, the responder's payoff $W_j(g)$ is constant for status quos below the proposer's ideal and increasing for status quos above the proposer's ideal, where it is equal to the status quo payoff.

Given the equilibrium payoffs in the one-period problem take different functional forms for different regions, the analysis of the T-period problem, even for $T = 2$, is cumbersome. Partly because of this, we do not analyze a T-period problem. Rather, we analyze the infinite-horizon problem by exploiting the recursive structure.

6.2 The infinite-horizon model

Now consider the infinite-horizon model. From the equilibrium conditions (E2), it must be the case that, for all $i, j \in \{H, L\}, j \neq i$ and any status quo g, the proposal $\pi^i(g)$ is a solution to the following maximization problem,

$$\pi^i(g) \in \underset{z=(g', x'_H, x'_L)\in B}{\arg\max} \; x'_i + \theta_i \ln(g') + \delta[pV_i(g') + (1-p)W_i(g')] \tag{6}$$

$$\text{s.t.} \quad x'_j + \theta_j \ln(g') + \delta[(1-p)V_j(g') + pW_j(g')] \geq K_j(g), \tag{7}$$

where

$$K_j(g) = \theta_j \ln(g) + \delta[(1-p)V_j(g) + pW_j(g)]. \tag{8}$$

From (E3), V_i and W_i satisfy the following functional equations:

$$V_i(g) = \chi_i^i(g) + \theta_i \ln(\gamma^i(g)) + \delta[pV_i(\gamma^i(g)) + (1-p)W_i(\gamma^i(g))], \tag{9}$$

$$W_i(g) = \chi_i^j(g) + \theta_i \ln(\gamma^j(g)) + \delta[(1-p)V_i(\gamma^j(g)) + pW_i(\gamma^j(g))]. \tag{10}$$

We construct equilibria by the "guess and verify" method. The form of the parties' equilibrium strategies and payoffs in the one-period model are a natural starting place to consider the solution to the infinite-horizon model; however, we expect the solution to the infinite-horizon model to take into account continuation strategies and payoffs. We provide here some brief intuition about how this may alter strategies. Consider the choice of the proposer when the responder's constraint is not binding. In the one-period model, the proposer chooses its static ideal. In the infinite-horizon model the proposer takes into account the fact that it may not be the proposer in the next period; hence he may wish to provide *insurance* for itself by setting the value of the public good above its static ideal. We find this insurance effect to be present in the infinite-horizon model.

This insurance effect appears to have the desirable property that it increases the equilibrium level of the public good compared to discretionary spending, but is it possible that it causes parties to increase the level of the public good *above* the efficient level? The answer is yes for some parameter values. In particular, define the level of *polarization* as the ratio $\frac{\theta_H}{\theta_L}$. Below we divide the characterization of the equilibrium of the infinite-horizon model into the *low-polarization* case and the *high-polarization* case. In the case of low polarization we show that the insurance effect leads party H to propose levels of public good spending that are higher than what it proposes when such spending is discretionary, but there is no over-provision of the public good in equilibrium. In the high-polarization case we do observe over-provision of the public good. We make these statements precise in the equilibrium characterization below.

To simplify the characterization, we use the recursive structure of the dynamic payoffs to establish Lemma 1, which shows that when the responder's acceptance constraint (7) binds, the responder's dynamic payoff $W_i(g)$ and its status quo payoff $K_i(g)$ can be expressed entirely in terms of $V_i(g)$, its dynamic payoff if it was the proposer.

Lemma 1. *If* $W_i(g) = K_i(g)$, *then*

$$W_i(g) = K_i(g) = \frac{1}{1-\delta p}[\theta_i \ln(g) + \delta(1-p)V_i(g)]. \tag{11}$$

Proof: Suppose $W_i(g) = K_i(g)$. Then $W_i(g) = \theta_i \ln(g) + \delta[(1-p)V_i(g) + pW_i(g)]$. Rearranging gives (11). ■

Lemma 1 conveniently transforms the dynamic payoff for party i into one with a single value function $V_i(g)$, rather than two - $V_i(g)$ and $W_i(g)$ - when party i's constraint is binding.

For the upcoming analysis, it is useful to define f_i as party i's dynamic payoff when the public spending in the current period is g and party i receives the remaining surplus:

$$f_i(g) = 1 - g + \theta_i \ln(g) + \delta[pV_i(g) + (1-p)W_i(g)]. \tag{12}$$

Low-polarization case

We look for an equilibrium strategy-payoff pair $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$ and (V_H, W_H, V_L, W_L) with the following properties that bear some resemblance to the one-period solution:

(G1) There exist g_L^* and g_H^* with $g_L^* < g_H^* < \theta_H + \theta_L$ such that $g_i^* \in \arg\max f_i(g)$ for $i \in \{H, L\}$ and if $g \leq g_i^*$, then $\pi^i(g) = \pi^i(g_i^*)$, and specifically $\gamma^i(g) = g_i^*$.

(G2) If $g \in [g_i^*, \theta_H + \theta_L]$, then $\gamma^i(g) = g$ and $W_j(g) = K_j(g)$ for $i, j \in \{H, L\}$ with $i \neq j$.

(G3) For any $i, j \in \{H, L\}$ with $j \neq i$, if $g \geq \theta_H + \theta_L$, then $\gamma^i(g) = \theta_H + \theta_L$, $W_j(g) = K_j(g)$, and the proposer's equilibrium payoff $V_i(g)$ takes the form $V_i(g) = C_i \ln(g) + D_i$.

Guess (G1) says that when the status quo is sufficiently low, each proposer proposes a constant level of public good spending that maximizes its dynamic payoff, with the public good spending proposed by L being lower than that proposed by H. This is reasonable since when the status quo is sufficiently low, the responder's acceptance constraint should be slack at the proposer's dynamic ideal level of public good spending. Furthermore, since the static ideal public good level for H is higher than that for L, one would expect that the dynamic ideal for party H is higher than that for party L.

Guess (G2) says that when the status quo is higher than the cutoff specified in (G1), but lower than the efficient level $\theta_H + \theta_L$, then the proposer maintains the status quo public goods spending, and the responder's acceptance constraint binds.

Guess (G3) says that when the status quo is higher than the efficient level, then the proposer proposes public good spending that is equal to the efficient level and makes transfers to the responder so that the responder is just willing to accept. The functional form guess of V_i is motivated by the fact that per-period utility functions are linear in $\ln(g)$.

Suppose $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$ and (V_H, W_H, V_L, W_L) is an equilibrium strategy-payoff pair that satisfies (G1)-(G3). In the next few lemmas we establish some properties of σ and (V_H, W_H, V_L, W_L), and in Proposition 3 we use these to characterize the equilibrium.

Consider first the proposer's problem (6) without imposing the responder's acceptance constraint (7). Since g enters the problem only through the constraint (7), the proposer's value function is independent of g, and we denote proposer i's highest payoff without the constraint (7) by $V_i^* = \max_g f_i(g)$. Clearly, if z is a solution to proposer i's problem without the acceptance constraint, then $z = (g', x_H, x_L)$ where $x_i = 1 - g'$ for some $g' \in \arg\max f_i(g)$.

Since V_i^* is proposer i's highest payoff without the constraint (7), it follows that $V_i^* \geq V_i(g)$ for any g. Denote $W_L(g_H^*)$ by W_L^* and denote $W_H(g_L^*)$ by W_H^*.

Lemma 2. *Under (G1), for all $i, j \in \{H, L\}$ with $j \neq i$, (i) if $g \leq g_i^*$, then $V_i(g) = V_i^*$, $\chi_i^i(g) = 1 - g_i^*$, $\chi_j^i(g) = 0$, and (ii) if $g \leq g_j^*$, then $W_i(g) = W_i^*$.*

Proof: Part (i): By (G1), $g_i^* \in \arg\max f_i(g)$. Since responder j accepts the proposal $(g_i^*, 1 - g_i^*, 0)$ when the status quo is $g = g_i^*$, it follows that $V_i(g_i^*) \geq V_i^*$. Since $V_i^* \geq V_i(g)$ for any g , it follows that $V_i(g_i^*) = V_i^*$, $\chi_i^i(g_i^*) = 1 - g_i^*$, and $\chi_j^i(g_i^*) = 0$. The rest of (i) follows immediately from (G1).

Part (ii) follows from (10). ∎

Lemma 2 says that party i's dynamic payoff as the proposer is constant and maximized for $g \leq g_i^*$ when the responder's constraint is not binding. Next consider when the responder's acceptance constraint is binding. To begin, we characterize these dynamic payoffs over the range $g \in [g_i^*, \theta_H + \theta_L]$.

Lemma 3. *Under (G1) and (G2), if $g \in [g_L^*, g_H^*]$, then*

$$V_L(g) = \frac{1}{1 - \delta p}[1 - g + \theta_L \ln(g) + \delta(1 - p)W_L^*], \tag{13}$$

and if $g \in [g_H^*, \theta_H + \theta_L]$, then

$$V_i(g) = \frac{(1 - \delta p)(1 - g)}{(1 - \delta)(1 + \delta - 2\delta p)} + \frac{\theta_i}{1 - \delta} \ln(g) \tag{14}$$

for all $i \in \{H, L\}$.

Proof: Under (G2), if $g \in [g_i^*, \theta_H + \theta_L]$, then $\gamma^i(g) = g$. Since the responder accepts the proposal $(g, 1 - g, 0)$ if the status quo is g, this implies that $\chi_j^i(g) = 0$ for $g \in [g_i^*, \theta_H + \theta_L]$ and therefore

$$V_i(g) = 1 - g + \theta_i \ln(g) + \delta[pV_i(g) + (1 - p)W_i(g)]. \tag{15}$$

By Lemma 2, if $g \in [g_L^*, g_H^*]$, then $W_L(g) = W_L^*$. Substituting in (15) and rearranging terms, we get (13). Under (G2), if $g \in [g_H^*, \theta_H + \theta_L]$, then $W_i(g) = K_i(g)$ and by Lemma 1, equation (11) holds. Substituting (11) in (15) and rearranging terms, we get (14). ∎

Lemma 3 gives the functional form for proposer i in a range that includes its dynamic ideal level of the public good g_i^*. We are now in a position to fully characterize g_i^*.

Lemma 4. *Under (G1) and (G2), $g_L^* = \theta_L$ and $g_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$.*

Lemma 4 formalizes the intuition given at the beginning of this subsection. It says that party L's dynamic ideal g_L^* is equal to its static ideal θ_L, while party H's dynamic ideal g_H^* is strictly higher than its static ideal θ_H. To understand this result, note that the proposer's choice of the public good level has a static effect on the current-period payoff and a dynamic effect on the continuation payoff because it determines next period's status quo. Furthermore, the dynamic effect creates two competing incentives for the incumbent: the incentive to raise the public good level for fear that the opposition party comes into power next period, and the incentive to lower the public good level to lower the bargaining power of the opposition party if the incumbent stays in power next period. In the low-polarization case, the dynamic effect of party L's proposal is zero because even if party H becomes the proposer next period, it would choose its dynamic ideal, which is sufficiently high. On the other hand, party H is indeed concerned that party L would set the level of public good too low should party L come into power, and the insurance incentive arising from this dynamic concern leads party H to propose g_H^* strictly higher than its static ideal θ_H. Clearly, a necessary condition for an equilibrium to exist that satisfies (G1)-(G3) is that $g_H^* < \theta_H + \theta_L$. By Lemma 4, this is satisfied if $\frac{\theta_H}{\theta_L} < \frac{1-\delta p}{\delta(1-p)}$. Since this condition implies that the parties' preferences regarding the value of public good are sufficiently similar, we call this the "low-polarization" case.

We now characterize the proposer's dynamic payoff over the remainder of the range of g. By (G3), the dynamic payoffs are given by $V_i(g) = C_i \ln(g) + D_i$ for $g \geq \theta_H + \theta_L$. Lemma 5 characterizes the values of C_i and D_i.

Lemma 5. *Under (G3),*

$$C_i = \frac{-(1-\delta p)\theta_j + \delta(1-p)\theta_i}{(1-\delta)(1+\delta-2\delta p)}, \tag{16}$$

$$D_i = \frac{(1-\delta p)(1-\theta_L-\theta_H+(\theta_H+\theta_L)\ln(\theta_H+\theta_L))}{(1-\delta)(1+\delta-2\delta p)}, \tag{17}$$

for $i, j \in \{H, L\}$ with $j \neq i$.

Recall that we guess in (G3) that $\gamma^i(g) = \theta_H + \theta_L$ for all $g \geq \theta_H + \theta_L$. To ensure that this holds in equilibrium – in particular, the responder accepts the proposal – we need $\alpha^j(g, (\theta_H + \theta_L, x_H, x_L)) = 1$ with $x_j = 1 - \theta_L - \theta_H$, $x_i = 0$ for all $g \geq \theta_H + \theta_L$, that is, the responder would agree to bring the public spending to the efficient level of $(\theta_H + \theta_L)$ after receiving the rest of the surplus as private transfers. In what follows, we derive a condition under which this holds in equilibrium, and we discuss what happens if the condition is violated at the end of this subsection.

Note that $\alpha^j(g, (\theta_H + \theta_L, x_H, x_L)) = 1$ with $x_j = 1 - \theta_L - \theta_H$, $x_i = 0$ is satisfied if

$$1 - (\theta_H + \theta_L) + \theta_j \ln(\theta_H + \theta_L) + \delta[(1-p)V_j(\theta_H + \theta_L) + pW_j(\theta_H + \theta_L)] \geq K_j(g).$$

Substituting for $K_j(g)$ and $W_j(g)$ using Lemma 1 and substituting for $V_j(g) = C_j \ln(g) + D_j$ for $g \geq \theta_H + \theta_L$, the inequality simplifies to

$$1 - (\theta_H + \theta_L) \geq \frac{\theta_j(1-\delta p) - \theta_i \delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}[\ln(g) - \ln(\theta_H + \theta_L)]. \tag{18}$$

Since the right-hand side of inequality (18) is higher when $j = H$ than when $j = L$, it follows that if the inequality holds for $j = H$, then it holds for $j = L$ as well. Moreover, the right-hand side of (18) is increasing in g, implying that if the inequality holds for $g = 1$, then it holds for all $g \geq \theta_H + \theta_L$. Call the following inequality condition $(*)$.

$$1 - (\theta_H + \theta_L) \geq \frac{\theta_H(1-\delta p) - \theta_L \delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}(-\ln(\theta_H + \theta_L)). \tag{$*$}$$

We are now ready to establish the equilibrium characterization result in the low-polarization case. For brevity, we use θ_i^* to denote $\frac{1+\delta-2\delta p}{1-\delta p}\theta_i$ for $i \in \{H, L\}$ for the rest of the paper.

Proposition 3. *Suppose* $\frac{\theta_H}{\theta_L} < \frac{1-\delta p}{\delta(1-p)}$ *and condition* $(*)$ *holds. Then, there exists an equilibrium strategy-payoff pair* $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$ *and* (V_H, W_H, V_L, W_L) *that satisfies (G1)-(G3). Specifically, for* $i, j \in \{H, L\}$, $j \neq i$,

$$\gamma^i(g) = \begin{cases} g_i^* & \text{for } g \leq g_i^*, \\ g & \text{for } g_i^* \leq g \leq \theta_H + \theta_L, \\ \theta_H + \theta_L & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

$$\chi_j^i(g) = \begin{cases} 0 & \text{for } g \leq \theta_H + \theta_L, \\ \frac{\theta_j(1-\delta p) - \theta_i\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} \ln(\frac{g}{\theta_H + \theta_L}) & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

and $\chi_i^i(g) = 1 - \gamma^i(g) - \chi_j^i(g)$, *where* $g_L^* = \theta_L$ *and* $g_H^* = \theta_H^*$.

Figure 3 is the numerical output from value function iterations. It illustrates the parties' proposal strategies for the public good in an equilibrium that satisfies (G1)-(G3). We include the illustration of parties' proposal strategies for transfers in the Appendix.[11]

Equilibrium when condition $(*)$ **fails:** Denote by z_j^e the proposal $(\theta_H + \theta_L, x_H, x_L)$ where $x_i = 0$ and $x_j = 1 - \theta_H - \theta_L$. Recall that in Proposition 3, we assume that condition $(*)$ holds, which ensures that the responder j accepts the proposal z_j^e even when the status quo is high. What happens if condition $(*)$ fails, that is, if $\alpha^j(g, z_j^e) = 0$ for g sufficiently high? In that case, instead of proposing $g' = \theta_H + \theta_L$, party i proposes $g' > \theta_H + \theta_L$, $x_i' = 0$, and $x_j' = 1 - g'$ such that party j is just willing to accept. Figure 4 illustrates the parties'

[11] In the low-polarization case when parameters satisfy condition $(*)$, all numerical output we have obtained satisfy (G1)-(G3).

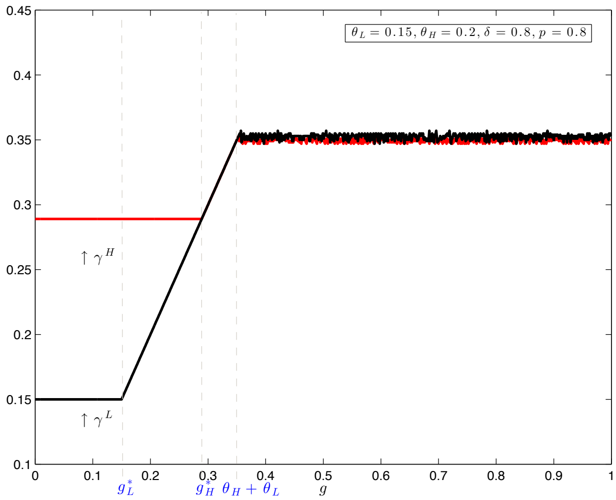


Figure 3: $\gamma^i(g)$ in low-polarization case when $(*)$ holds

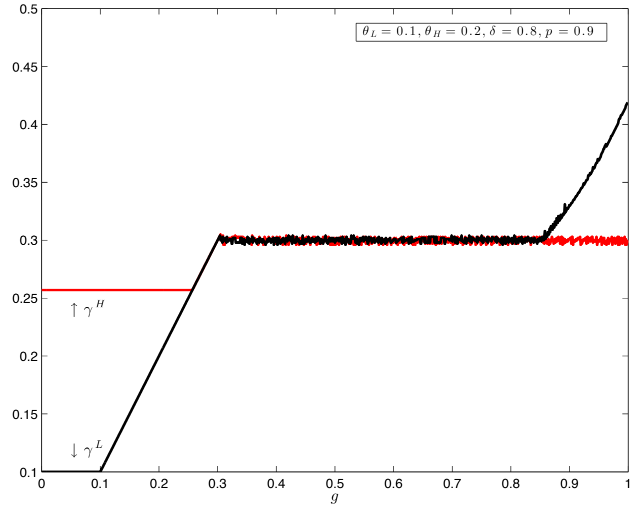


Figure 4: $\gamma^i(g)$ in low-polarization case when $(*)$ fails

proposal strategies when condition $(*)$ fails. In the figure (G1)-(G2) are still satisfied, but for very high status quos, (G3) is violated. As we show in Section 7, the failure of condition $(*)$ does not affect the set of steady states.

High-polarization case

Now suppose $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$, so polarization is high. Figure 5 below illustrates a numerical output from value function iteration when this condition holds.

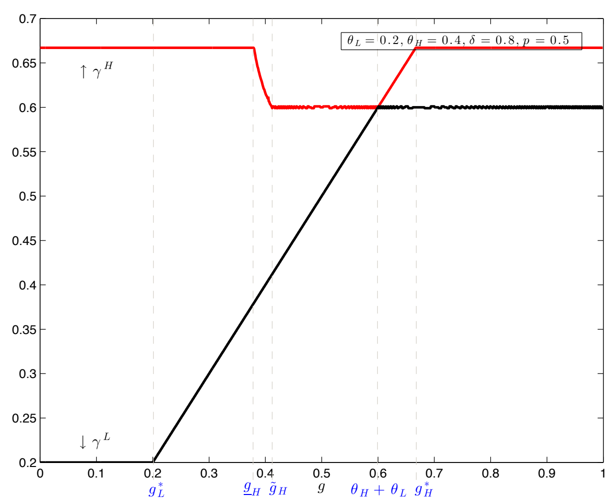


Figure 5: $\gamma^i(g)$ in high-polarization case

Figure 5 suggests equilibrium strategies that at first glance look very different from the low-polarization case; however, upon further examination, we find parallels. First consider the strategy illustrated for party L. This strategy is in fact similar to party L's strategy in the low-polarization case: a constant value g_L^* (in this case 0.2) is chosen at low levels of the status quo, for intermediate values of the status quo, the public good is chosen to be equal to the status quo, and for status quos above the efficient level ($\theta_H + \theta_L = 0.6$), the efficient level of the public good is chosen.

For party H, the condition for high-polarization, $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$, necessitates that g_H^* characterized in the low-polarization case (which is 0.67 for these parameter values) is now strictly above the efficient level, 0.6. It is not surprising that at low values of the status quo, below the point $\underline{g}_H$ in Figure 5, party H still chooses the public good spending to be equal to its dynamic ideal g_H^*. Interestingly, Figure 5 shows that g_H^* is also chosen at very high levels of the status quo, which suggests that party L's acceptance constraint is slack when the status quo is very high. The intuition for setting the level of the public good above the static ideal is the same as before: party H's insurance motive dominates, but under high polarization, what is dynamically optimal for party H is higher than the efficient level.

Between $\underline{g}_H$ and a higher threshold $\tilde{g}_H$, the level of public good proposed by party H is between its dynamic ideal g_H^* and the efficient level $\theta_H + \theta_L$. This is because the acceptance constraint for party L binds and party H cannot propose its dynamic ideal, but party L's status quo payoff is low enough that party H does not have to propose the efficient level. As the status quo increases, party L's status quo payoff also increases, and party H has to propose a level of the public good closer to the efficient level.

Between $\tilde{g}_H$ and $\theta_H + \theta_L$, the efficient level is proposed by party H. In this range, party L's status quo payoff is high enough that party H finds it optimal to propose the efficient level of the public good and give party L some transfer so that it consents to raising the level of the public good. Finally, between $\theta_H + \theta_L$ and g_H^*, it is optimal for party H to maintain the status quo since it is closer to party H's dynamic ideal, and it satisfies party L's constraint.

It remains to formally characterize an equilibrium with these properties. Motivated by Figure 5, we make the following guesses about an equilibrium strategy-payoff pair. Recall that $f_i(g)$, defined in (12), is party i's dynamic payoff when the public spending in the current period is g and party i receives the remaining surplus.

(G1′) There exist g_L^* and g_H^* with $g_L^* < \theta_H + \theta_L < g_H^*$ such that $g_i^* \in \arg\max f_i(g)$ for $i \in \{H, L\}$.

(G2′) If $g \leq g_L^*$, then $\pi^L(g) = \pi^L(g_L^*)$ and specifically $\gamma^L(g) = g_L^*$; if $g \in [g_L^*, \theta_H + \theta_L]$, then $\gamma^L(g) = g$; if $g \geq \theta_H + \theta_L$, then $\gamma^L(g) = \theta_H + \theta_L$. If $g \geq g_L^*$, then $W_H(g) = K_H(g)$.

(G3′) There exist $\underline{g}_H$ and $\tilde{g}_H$ that satisfy $g_L^* \leq \underline{g}_H < \tilde{g}_H < \theta_H + \theta_L$ such that (i) $\pi^H(g) = \pi^H(g_H^*)$ for $g \leq \underline{g}_H$ and $g \geq g_H^*$; (ii) if $g \in [\underline{g}_H, g_H^*]$ then $W_L(g) = K_L(g)$; (iii) if $g \leq \tilde{g}_H$

or if $g \geq \theta_H + \theta_L$, then $\chi_L^H(g) = 0$; and (iv)

$$\gamma^H(g) = \begin{cases} g_H^* & \text{for } g \leq \underline{g}_H, \\ g' \in [\theta_H + \theta_L, g_H^*] & \text{for } \underline{g}_H \leq g \leq \tilde{g}_H, \\ \theta_H + \theta_L & \text{for } \tilde{g}_H \leq g \leq \theta_H + \theta_L, \\ g & \text{for } \theta_H + \theta_L \leq g \leq g_H^*, \\ g_H^* & \text{for } g_H^* \leq g. \end{cases}$$

where g' is a function of g satisfying $\theta_L \ln(g') + \delta[(1-p)V_L(g') + pW_L(g')] = K_L(g)$.

(G4′) If $\gamma^i(g) = \theta_H + \theta_L$, then $V_i(g)$ is piecewise linear in g and $\ln(g)$.

In (G2′), we guess that $\gamma^L(g) = \theta_H + \theta_L$ for all $g \geq \theta_H + \theta_L$. This is analogous to the low-polarization case and we need a condition similar to $(*)$ to guarantee that it holds in equilibrium. This condition, which we call $(**)$, is given below. Recall that $\theta_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$.

$$1 - (\theta_H + \theta_L) + \frac{\theta_H}{1-\delta}\ln(\theta_H + \theta_L) \geq \frac{\delta(1-p)(\theta_H+\theta_L-\theta_H^*)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\delta(1-p)\theta_H}{(1-\delta p)(1-\delta)}\ln(\theta_H^*). \tag{$**$}$$

The derivation of condition $(**)$ is similar to that of condition $(*)$ and can be found in Section 10.5.2 in the Appendix.

In (G3′), we guess that $g_L^* \leq \underline{g}_H$. In Lemma 10 in the Appendix, we find the value of $\underline{g}_H$ under (G1′)-(G4′) and we denote this value by ψ. We also show in the Appendix that $\psi \geq \theta_L^*$ guarantees that $g_L^* \leq \underline{g}_H$ in equilibrium.

Proposition 4. *If $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$, $\psi \geq \theta_L^*$ and condition $(**)$ holds, then there exists an equilibrium strategy-payoff pair that satisfies (G1′)-(G4′).*

Equilibrium when condition $(**)$ fails: Figure 6 illustrates the parties' proposal strategies when condition $(**)$ fails. The figure suggests that (G1′)-(G4′) are still satisfied in equilibrium, except that for very high status quos, $\gamma^L(g) > \theta_H + \theta_L$, similar to the low-polarization case. As we show in Section 7, the failure of condition $(**)$ does not affect the set of steady states.

Equilibrium when condition $\psi < \theta_L^*$: Figure 7 illustrates the parties' proposal strategies when $\psi < \theta_L^*$. The figure suggests that two kinds of equilibria arise. In panel (a), the equilibrium strategies still satisfy (G1′)-(G4′) with the exception that $g_L^* > \underline{g}_H$. In this case, party L's dynamic ideal is $g_L^* = \theta_L^* > \theta_L$, an analog to party H's dynamic ideal $g_H^* = \theta_H^*$. Intuitively, when $g_L^* > \underline{g}_H$, party L's choice of public good has a non-zero dynamic effect because if party H comes into power in the next period, its proposal will depend on the status quo if the status quo is above the threshold $\underline{g}_H$. This dynamic effect results in party L's dynamic ideal g_L^* being higher than its static ideal θ_L. In panel (b), party H's strategy again satisfies the guesses, but party L's strategy violates (G2′). In particular, instead of proposing $g' = g$ when $g \in [g_L^*, \theta_H + \theta_L]$, now party L proposes a constant level $g' = \theta_L^*$

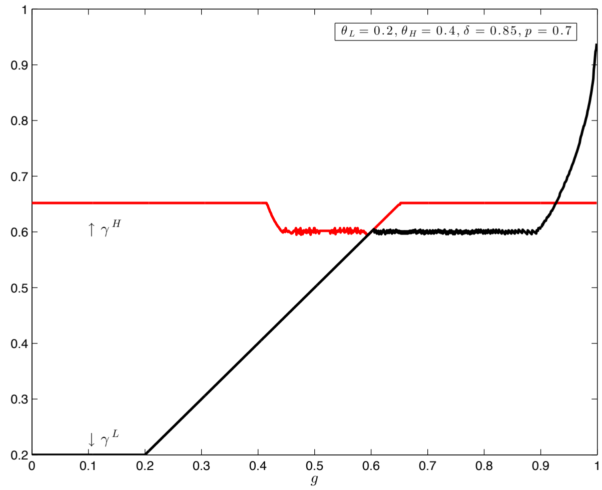
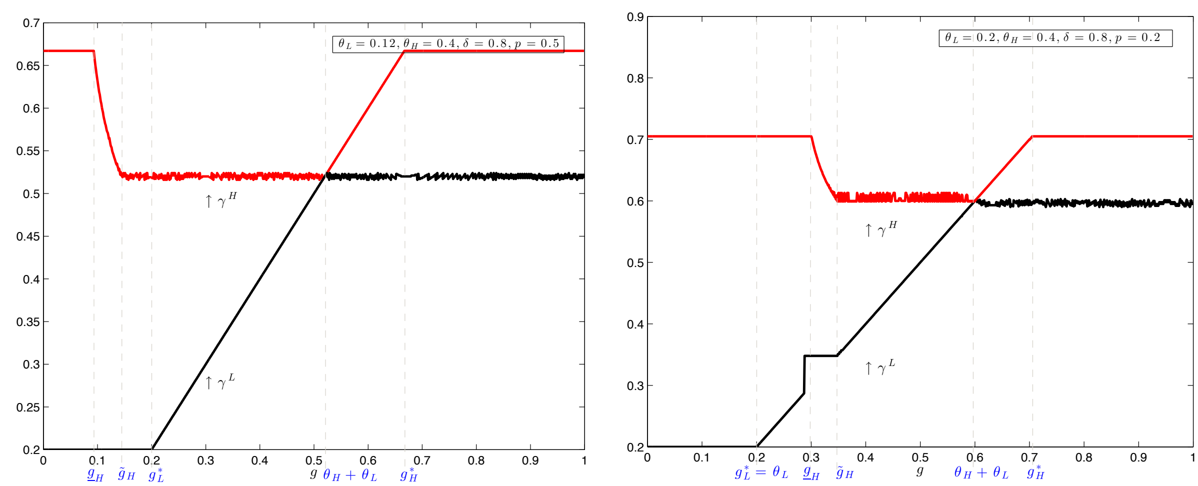


Figure 6: $\gamma^i(g)$ in high-polarization case when condition $(**)$ does not hold

when the status quo is in a subinterval of $[g_L^*, \theta_H + \theta_L]$ (the "kink"). This is because in this subinterval proposing $g' = \theta_L^*$ is a local maximizer for party L's optimization problem without the acceptance constraint and it is optimal to make this proposal (leaving party H's acceptance constraint slack).

Although the details of party L's strategy violate certain aspects of (G2$'$) when $\psi < \theta_L^*$, the efficiency implications and the set of steady states are still the same, as illustrated in Figure 7 and will be formalized in Section 7. For this reason, we omit a full equilibrium characterization when $\psi < \theta_L^*$.



(a) $g_L^* = \theta_L^* > \theta_L$ (b) $g_L^* = \theta_L$

Figure 7: $\gamma^i(g)$ in high-polarization case when $\psi < \theta_L^*$

7 Equilibrium dynamics

We next discuss equilibrium dynamics. Let g^0 denote the initial level of public good spending. As we show in Proposition 5 below, there is a unique steady state, denoted by g^s, corresponding to each g^0. Recall that for an equilibrium satisfying (G1)-(G3) in the low-polarization case, $g_H^* = \theta_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$.

Proposition 5. *In an equilibrium that satisfies (G1)-(G3) in the low-polarization case, if $g^0 \leq \theta_H^*$, then $g^s = \theta_H^*$; if $g^0 \in [\theta_H^*, \theta_H + \theta_L]$, then $g^s = g^0$; if $g^0 \geq \theta_H + \theta_L$, then $g^s = \theta_H + \theta_L$. In an equilibrium that satisfies (G1′)-(G4′) in the high-polarization case, $g^s = \theta_H + \theta_L$ for any g^0.*

The proposition says that in the low-polarization case, starting from a level of the public good below the efficient level, the steady state is still below the efficient level, but above what would be implemented with only discretionary programs. Starting from a level of the public good above the efficient level, the steady state is at the efficient level. This is because when the status quo is above the efficient level, parties find it optimal to reduce spending on the public good to the efficient level. But once public good spending is at the efficient level, any allocation that exhausts the budget is on the Pareto frontier; hence any proposal that improves the payoff of the proposer must reduce the payoff of the responder. Because public good spending is mandatory, the responder's bargaining power prevents the proposer from reducing its payoff, and hence this is a steady state.

Proposition 5 says that in the high-polarization case, the only steady state involves public good spending equal to the efficient level $\theta_H + \theta_L$. The dynamics leading to this unique steady state may be non-monotone. Specifically, if the initial status quo is below $\tilde{g}_H$ and L is the initial proposer, L chooses $\gamma^L(g) \in [\theta_L, \tilde{g}_H]$ and this level persists until party H next comes to power. When party H is next in power, party H sets a higher level of the pubic good $\gamma^H(g) \in [\theta_H + \theta_L, g_H^*]$, and the public good spending remains at this level until party L next comes to power. When party L returns to power, he finds it optimal to reduce the level of the public good to the efficient level, which is then sustained. Hence in the high-polarization case, the level of the public good can potentially overshoot the steady state level even if the initial state is low.

Proposition 5 says that in the equilibrium we constructed, the set of steady states is $[\theta_H^*, \theta_H + \theta_L]$ in the low-polarization case, and it is the singleton $\{\theta_H + \theta_L\}$ in the high-polarization case. In the next proposition, we show that there are no other steady states in any other equilibrium under certain conditions.

Suppose σ and (V_H, W_H, V_L, W_L) is an equilibrium strategy-payoff pair. Let G^s denote the set of steady states, that is, for any $g \in G^s$, $\gamma^i(g) = g$ for $i \in \{H, L\}$. Let G denote the set of public good spending levels g such that when the status quo is g the acceptance constraint binds regardless of the responder.

Proposition 6. *Let $g \in G^s$, and suppose that (i) V_H and V_L are differentiable on an open set C such that $g \in C \subseteq G$, and (ii) the responders' acceptance constraints satisfy Kuhn-Tucker Constraint Qualification. Then $g \in [\theta_H^*, \theta_H + \theta_L]$ in the low-polarization case, and $g = \theta_H + \theta_L$ in the high-polarization case.*

We next discuss comparative statics on the set of steady states in the low-polarization case. Since the highest steady state is constant at the efficient level, comparative statics on the set of steady states is driven by comparative statics on the lowest steady state, which is given by party H's dynamic ideal level of the public good $g_H^* = \theta_H^*$.

Proposition 7. *In the low-polarization case, the lowest steady state θ_H^* is decreasing in the persistence of power p and is increasing in the discount factor δ.*

The intuition for this result is simple. The static ideal level of public good spending for party H is equal to θ_H, but dynamic considerations create incentives for party H to set a level of the public good above its static ideal to increase its status quo payoff in the event that it loses (proposing) power. As party H becomes more confident that it will still be in power in the next period, its incentive to insure itself decreases, and hence it sets a level of the public good closer to its static ideal, knowing that it will likely be able to set the same level in the next period and receive transfers. Similarly, as party H's discount factor increases, it puts more weight on future payoffs, and hence is more sensitive to being out of power in the future. To insure itself against being out of power in the next period, it increases the level of the public good in the current period. This means that more patience or less persistence in political power results in steady states closer to the efficient level.

8 Efficiency implications of mandatory programs

One objective of this paper is to examine the efficiency implications of mandatory programs. In this section we explore this. First recall that if the public good spending is discretionary, then in any Markov perfect equilibrium, the level of public spending is equal to θ_i if party i is the proposer in that period. By Proposition 3, the equilibrium level of public good spending proposed by party i is in $[g_i^*, \theta_H + \theta_L]$ under mandatory programs in the low-polarization case. Since $g_i^* \geq \theta_i$ for all $i \in \{H, L\}$, the level of public good spending is higher when it is mandatory than when it is discretionary, independent of the status quo. Since over-provision of public good does not happen in equilibrium in the low-polarization case, this means that the equilibrium level of public good spending is closer to the efficient level when it is mandatory than when it is discretionary. In the high-polarization case, however, the level of public good spending proposed by party H can be as high as g_H^*, which is now higher than $\theta_H + \theta_L$. Hence *over-provision* of the public good is possible, but as shown in Proposition 5, it is only a transient state.

How do mandatory programs affect the parties' welfare? The next proposition shows that mandatory programs improve the ex ante welfare of party H. More surprisingly, under some parametric conditions – in particular, when the parties are sufficiently patient and the persistence of power is sufficiently low – they also improve the ex ante welfare of party L. For notational convenience, let

$$w(\delta, p) = \ln\left(\frac{(1+\delta-2\delta p)^2}{\delta(1-p)(1-\delta p)}\right) - \frac{1-\delta p}{\delta(1-p)}.$$

Proposition 8. *Suppose it is equally likely ex ante for party H and party L to become the proposer. Then party H's steady state payoff is higher when public good spending is mandatory than when it is discretionary. Moreover, in the low-polarization case, party L's steady state payoff is higher when public good spending is mandatory than when it is discretionary if $w(\delta, p) > 0$.*

Notice if $\delta = 1$, then $w(\delta, p) = \ln(4) - 1 > 0$. Hence, if the parties are sufficiently patient, then even the party who places a lower weight on public good is better off ex ante if the spending on public good is mandatory.

It is straightforward to verify that $w(\delta, p)$ is decreasing in p and increasing in δ. When $p = 0$, $w(\delta, p) = \ln((1+\delta)^2/\delta) - 1/\delta$, and $w(\delta, 0) = 0$ when $\delta \approx 0.706$. It follows that if $\delta > 0.706$, then there exists $\underline{p} > 0$ such that for all $p < \underline{p}$, $w(\delta, p) > 0$, and even the low party benefits ex ante from mandatory public good spending. Intuitively, when the persistence of power is low, the insurance benefit from mandatory programs is high, making the parties better off.

9 Concluding remarks

In this paper we analyze a model of dynamic bargaining between two political parties over the allocation of a public good and private transfers to understand the efficiency implications of mandatory programs. We find that allocation of the public good through a mandatory program mitigates the problem of under-provision of the public good compared to discretionary programs because it provides a channel for parties to insure themselves against power fluctuations. As a result, mandatory programs provide payoff smoothing for the parties, that is, the difference between each party's payoff when in power and when out of power is smaller under mandatory programs. This leads to higher ex ante dynamic payoffs for both parties, even the one that places a low value on the public good, when the parties are sufficiently patient, not too polarized, and persistence of power is sufficiently low.

Several extensions seem promising for future research. First, in this paper, we focus on a particular status quo rule: spending on the public good is mandatory and private transfers are discretionary. We find this to be a good approximation of the rules governing the U.S. federal budget negotiations, but since there are potentially different rules governing how the status quo evolves, an interesting question is what would be the optimal status quo rule. Separately, if the choice of mandatory versus discretionary programs is endogenous, what would be the outcome?

The persistence of power is parameterized by p, the probability that the proposer last period continues to be the proposer this period, and for simplicity, we assume it to be exogenous in our model. Since success in bringing home "pork" typically results in more favorable electoral outcomes, a second interesting extension is to consider how the efficiency implications of mandatory programs change if power persistence is endogenously determined by the policy choice as in Azzimonti (2011) and Bai and Lagunoff (2011).

In our model, the size of the budget to be allocated in each period is fixed. Another extension would be to investigate the effect of mandatory programs if the size of the cake to be shared among the legislators is endogenous and determined by policy choice, as in a neoclassical growth model à la Battaglini and Coate (2008).

Finally, although parties place different values on the public good, each party's value stays constant over time in our model. If the values of the public good fluctuate over time stochastically, then we expect mandatory programs to have other interesting effects absent in the model with deterministic values. For example, a high level of public good spending that is efficient in times when the the public good is especially valuable becomes inefficient when the value of the public good decreases, and the inertia created by the mandatory program may lead to over-provision of the public good. In some preliminary analysis of a model in which the public good has the same value to both parties but fluctuates stochastically over time, we find that over-provision of the public good can happen when the value of the public good is low but the status quo is high. We plan to pursue this extension and others mentioned above in future work.

10 Appendix

10.1 Proof of Proposition 2

Party i's Lagrangian for this problem is

$$L_i = x_i' + \theta_i \ln(g') + \lambda_1[1 - g' - x_i' - x_j'] + \lambda_2[x_j' + \theta_j \ln(g') - K_j(g)],$$

where $K_j(g) = \theta_j \ln(g)$. The first order conditions are $g', x_i', x_j', \lambda_1, \lambda_2 \geq 0$ and

$$\frac{\theta_i}{g'} - \lambda_1 + \lambda_2 \frac{\theta_j}{g'} \leq 0, \qquad \left[\frac{\theta_i}{g'} - \lambda_1 + \lambda_2 \frac{\theta_j}{g'}\right] g' = 0, \tag{19}$$

$$1 - \lambda_1 \leq 0, \qquad [1 - \lambda_1] x_i' = 0, \tag{20}$$

$$-\lambda_1 + \lambda_2 \leq 0, \qquad [-\lambda_1 + \lambda_2] x_j' = 0, \tag{21}$$

$$1 - g' - x_i' - x_j' \geq 0, \qquad [1 - g' - x_i' - x_j'] \lambda_1 = 0, \tag{22}$$

$$x_j' + \theta_j \ln(g') - K_j(g) \geq 0, \qquad [x_j' + \theta_j \ln(g') - K_j(g)] \lambda_2 = 0. \tag{23}$$

First note that $\lambda_1 \geq 1$ by (20). Hence, for (22) to hold, we must have $1 - g' - x_i' - x_j' = 0$. Next note that $g' > 0$ because otherwise (19) is violated.

There are now four cases to consider.

- $\lambda_2 = 0$: Since $\lambda_1 > 0$, (21) implies that $x_j' = 0$. So we have $x_i' + g' = 1$. Suppose $g' = 1$. Then, since λ_2=0, by (19), $\lambda_1 = \theta_i < 1$, which contradicts (20). Hence, $g' < 1$ and $x_i' = 1 - g' > 0$. By (20), $x_i' > 0$ implies that $\lambda_1 = 1$. Combined with (19), this implies that $g' = \theta_i$, $x_i' = 1 - \theta_i$, and $x_j' = 0$. For the inequality in (23) to hold, we need $g \leq \theta_i$.

- $\lambda_2 > 0$, $x_i' > 0$ and $x_j' > 0$: Then $\lambda_1 = \lambda_2 = 1$. Together with (19), (22) and (23), this implies that

$$\begin{aligned} g' &= \theta_H + \theta_L, \\ x_i' &= 1 - \theta_L - \theta_H - K_j(g) + \theta_j \ln(\theta_H + \theta_L), \\ x_j' &= K_j(g) - \theta_j \ln(\theta_H + \theta_L). \end{aligned}$$

  Since $0 \leq x_i' \leq 1$ and $0 \leq x_j' \leq 1$, for this to be a valid solution we need $0 \leq K_j(g) - \theta_j \ln(\theta_H + \theta_L) \leq 1 - \theta_H - \theta_L$, which holds if $g \geq \theta_H + \theta_L$.

- $\lambda_2 > 0$, $x_i' > 0$ and $x_j' = 0$: Then (23) implies that $g' = g$. Since $x_i' > 0$, $\lambda_1 = 1$, and (19) gives $g' = \theta_i + \lambda_2 \theta_j$. Since $0 < \lambda_2 \leq \lambda_1 = 1$, it follows that this is a valid solution only when $\theta_i < g \leq \theta_H + \theta_L$.

- $\lambda_2 > 0$, $x_i' = 0$ and $x_j' > 0$: Then (21) gives $\lambda_1 = \lambda_2$, and (19) gives $g' = \frac{\theta_i}{\lambda_1} + \theta_j > \theta_j$. Since $\lambda_2 > 0$, (23) implies that $1 - g' + \theta_j \ln(g') = \theta_j \ln(g)$, which is impossible since $g' > \theta_j$.

To summarize, we have the solution given in Proposition 2. ∎

10.2 Proof of Lemma 4

We first show that $g_L^* = \theta_L$. Since $V_L(g)$ and $W_L(g)$ are constant for $g \le g_L^*$ by Lemma 2, it follows that for $g < g_L^*$, $\frac{\partial f_L(g)}{\partial g} = -1 + \frac{\theta_L}{g}$. If $g_L^* > \theta_L$, then $f_L(\theta_L) > f_L(g_L^*)$, contradicting that $g_L^* \in \arg\max f_L(g)$. Hence $g_L^* \le \theta_L$.

By Lemma 3, $V_L'(g) = -\frac{1}{1-\delta p} + \frac{\theta_L}{(1-\delta p)g}$ and by Lemma 2, $W_L'(g) = 0$ for $g \in [g_L^*, g_H^*]$. Substituting these in $f_L'(g)$, we get

$$f_L'(g) = -1 + \frac{\theta_L}{g} + \delta p V_L'(g) = \frac{1}{1-\delta p}(-1 + \frac{\theta_L}{g}),$$

If $g_L^* < \theta_L$, then $f_L(g^*) < f_L(g)$ for any $g \in (g_L^*, \min\{\theta_L, g_H^*\})$, contradicting that $g_L^* \in \arg\max f_L(g)$. Hence, $g_L^* \ge \theta_L$.

Since $g_L^* \le \theta_L$ and $g_L^* \ge \theta_L$, it follows that $g_L^* = \theta_L$.

We next show that $g_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$. If $g \in (g_L^*, g_H^*)$, then $V_H'(g) = 0$ by Lemma 2 and $W_H'(g) = \frac{\theta_H}{(1-\delta p)g}$ by Lemma 1, and therefore

$$f_H'(g) = -1 + \frac{\theta_H}{g} + \delta(1-p)W_H'(g) = -1 + \frac{(1+\delta-2\delta p)\theta_H}{(1-\delta p)g}. \tag{24}$$

If $g_H^* > \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, then (24) implies that $f_H'(g) < 0$ for $g \in (\frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}, g_H^*)$, contradicting that $g_H^* \in \arg\max f_H(g)$. Hence $g_H^* \le \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$.

If $g \in (g_H^*, \theta_H + \theta_L)$, then as shown in (15), $f_H(g) = V_H(g)$, and by (14)

$$f_H'(g) = -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g}. \tag{25}$$

If $g_H^* < \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, then (25) implies that $f_H'(g) > 0$ for $g \in (g_H^*, \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p})$, contradicting that $g_H^* \in \arg\max f_H(g)$. Hence $g_H^* \ge \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$.

Since $g_H^* \le \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$ and $g_H^* \ge \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, it follows that $g_H^* = \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$. ∎

10.3 Proof of Lemma 5

Under (G3), for $i \in \{H, L\}$, $W_i(g) = K_i(g)$ for $g \ge \theta_H + \theta_L$. By Lemma 1, $W_i(g) = K_i(g) = \frac{\theta_j}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_i(g)$.

Consider any $g \ge \theta_H + \theta_L$ such that $\alpha^j(g, (\theta_H + \theta_L, x_H, x_L)) = 1$ with $x_i = 1 - \theta_L - \theta_H$, $x_j = 0$. Under (G3), $\gamma^i(g) = \theta_H + \theta_L$ and therefore

$$V_i(g) = \chi_i^i(g) + \theta_i \ln(\theta_H + \theta_L) + \delta[pV_i(\theta_H + \theta_L) + (1-p)W_i(\theta_H + \theta_L)]. \tag{26}$$

After substituting for $W_i(\theta_H + \theta_L)$, we have

$$V_i(g) = \chi_i^i(g) + \frac{1+\delta-2\delta p}{1-\delta p}\theta_i \ln(\theta_H + \theta_L) + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}V_i(\theta_H + \theta_L). \tag{27}$$

Since the responder's acceptance constraint is binding at g, we get

$$\chi_j^i(g) = K_j(g) - \frac{\theta_j}{1-\delta p}\ln(\theta_H + \theta_L) - \frac{\delta(1-p)}{1-\delta p}V_j(\theta_H + \theta_L) \tag{28}$$

where

$$K_j(g) = \tfrac{\theta_j}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}V_j(g). \tag{29}$$

Substituting $\chi_i^i(g) = 1 - \chi_j^i(g) - \theta_L - \theta_H$, $V_i(g) = C_i\ln(g) + D_i$, $V_j(g) = C_j\ln(g) + D_j$ and matching the coefficients, we get (16) and (17). ∎

10.4 Proof of Proposition 3

We proceed by first conjecturing an equilibrium strategy-payoff pair and then verifying that it satisfies guesses (G1)-(G3), equilibrium conditions (E1)-(E3), and our assumption on α^i that all proposals made on the equilibrium path are satisfied.

We conjecture an equilibrium strategy-payoff pair such that for any $i, j \in \{H, L\}$ with $j \neq i$, the acceptance strategy $\alpha^i(g, z)$ satisfies (E1), the proposal strategies are

$$\gamma^i(g) = \begin{cases} g_i^* & \text{for } g \leq g_i^*, \\ g & \text{for } g_i^* \leq g \leq \theta_H + \theta_L, \\ \theta_H + \theta_L & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

$$\chi_j^i(g) = \begin{cases} 0 & \text{for } g \leq \theta_H + \theta_L, \\ \frac{\theta_j(1-\delta p) - \theta_i\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}\ln(\frac{g}{\theta_H+\theta_L}) & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

and $\chi_i^i(g) = 1 - \gamma^i(g) - \chi_j^i(g)$, where $g_L^* = \theta_L$ and $g_H^* = \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, and the associated payoff functions are

$$V_L(g) = \begin{cases} V_L^* & \text{for } g < g_L^*, \\ \frac{1}{1-\delta p}[1 - g + \theta_L\ln(g) + \delta(1-p)W_L^*] & \text{for } g_L^* \leq g \leq g_H^*, \\ \frac{(1-\delta p)(1-g)}{(1+\delta-2\delta p)(1-\delta)} + \frac{\theta_L}{1-\delta}\ln(g) & \text{for } g_H^* \leq g \leq \theta_H + \theta_L, \\ C_L\ln(g) + D_L & \text{for } \theta_H + \theta_L < g, \end{cases}$$

$$W_L(g) = \begin{cases} W_L^* & \text{for } g \leq g_H^*, \\ \frac{1}{1-\delta p}[\theta_L\ln(g) + \delta(1-p)V_L(g)] & \text{for } g_H^* \leq g, \end{cases}$$

$$V_H(g) = \begin{cases} V_H^* & \text{for } g < g_H^*, \\ \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(g), & \text{for } g_H^* \leq g \leq \theta_H + \theta_L, \\ C_H\ln(g) + D_H & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

$$W_H(g) = \begin{cases} W_H^* & \text{for } g \leq g_L^*, \\ \frac{1}{1-\delta p}[\theta_H\ln(g) + \delta(1-p)V_H(g)] & \text{for } g_L^* \leq g, \end{cases}$$

where

$$C_i = \frac{-(1-\delta p)\theta_j + \delta(1-p)\theta_i}{(1-\delta)(1+\delta-2\delta p)} \tag{30}$$

$$D_i = \frac{(1-\delta p)(1-\theta_L-\theta_H+(\theta_H+\theta_L)\ln(\theta_H+\theta_L))}{(1-\delta)(1+\delta-2\delta p)}, \tag{31}$$

and

$$W_L^* = \frac{\delta(1-p)}{(1+\delta-2\delta p)(1-\delta)}(1-g_H^*) + \frac{\theta_L}{1-\delta}\ln(g_H^*), \tag{32}$$

$$V_L^* = \frac{1}{1-\delta p}[1-\theta_L+\theta_L\ln(\theta_L)+\delta(1-p)W_L^*], \tag{33}$$

$$V_H^* = \frac{(1-\delta p)(1-g_H^*)}{(1+\delta-2\delta p)(1-\delta)} + \frac{\theta_H}{1-\delta}\ln(g_H^*), \tag{34}$$

$$W_H^* = \frac{1}{1-\delta p}[\theta_H\ln(g_L^*)+\delta(1-p)V_H^*]. \tag{35}$$

This conjecture clearly satisfies (G2) and (G3). (Note that by substituting W_j in (8), we can verify that $W_j(g) = K_j(g)$ for $g \geq g_i^*$.) So we only need to verify that (G1) is satisfied; in particular, that $g_i^* \in \arg\max f_i(g)$ where $f_i(g) = 1-g+\theta_i\ln(g)+\delta[pV_i(g)+(1-p)W_i(g)]$.

Since V_i and W_i are continuous under our conjecture of the equilibrium strategy-payoff pair, f_i is continuous. It is also piecewise differentiable. Specifically,

$$f_L'(g) = \begin{cases} -1+\frac{\theta_L}{g} & \text{for } g < g_H^*, \\ -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} & \text{for } g \in (g_H^*, \theta_H+\theta_L), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L}{g} & \text{for } g > \theta_H+\theta_L, \end{cases}$$

$$f_H'(g) = \begin{cases} -1+\frac{\theta_H}{g} & \text{for } g < g_L^*, \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} & \text{for } g \in (g_L^*, g_H^*), \\ -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g} & \text{for } g \in (g_H^*, \theta_H+\theta_L), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_H}{g} & \text{for } g > \theta_H+\theta_L. \end{cases}$$

Claim 1. *Under our conjecture of the equilibrium strategy-payoff pair, $g_i^* \in \arg\max f_i(g)$ for all $i \in \{H, L\}$.*

Proof: Consider $i = L$ first. Given f_L described above, $f_L'(g) > 0$ if $g < g_L^*$, $f_L'(g) = 0$ if $g = g_L^*$, and $f_L'(g) < 0$ if $g \in (g_L^*, g_H^*)$.

Since $f_L'(g)$ is decreasing for $g \in (g_H^*, \theta_H+\theta_L)$, and at $g = g_H^*$, $-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} = -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)\theta_H} < 0$, it follows that $f_L'(g) < 0$ for $g \in (g_H^*, \theta_H+\theta_L)$.

If $\frac{(1+\delta-2\delta p)\theta_L+\delta(p+\delta-2\delta p)C_L}{1-\delta p} \leq 0$, then $f_L'(g) < 0$ for $g > \theta_H+\theta_L$. If $\frac{(1+\delta-2\delta p)\theta_L+\delta(p+\delta-2\delta p)C_L}{1-\delta p} > 0$, then $f_L'(g)$ is decreasing in g for $\theta_H+\theta_L \leq g$. Since at $g = \theta_H+\theta_L$, $f_L'(g) = \frac{-(1-\delta p)\theta_H+(\delta-\delta p)\theta_L}{(\theta_H+\theta_L)(1+\delta-2\delta p)(1-\delta)} < 0$, it follows that $f_L'(g) < 0$ for $g > \theta_H+\theta_L$.

To summarize, $f_L'(g) > 0$ for $g < g_L^*$, $f_L'(g) = 0$ if $g = g_L^*$, $f_L'(g) > 0$ for $g > g_L^*$, and therefore $g_L^* \in \arg\max f_L(g)$.

Now consider $i = H$. Given f_H described above, $f_H'(g) > 0$ for $g < g_H^*$, $f_H'(g) = 0$ for $g = g_H^*$. By a similar argument as for party L, $f_H(g)$ is decreasing for $g > g_H^*$. Therefore

$g_H^* \in \arg\max f_H(g)$. ■

Claim 1 shows that (G1) is satisfied. We next verify that conditions (E1)-(E3) in the definition of equilibrium strategy-payoff pair are satisfied. Condition (E1) is satisfied by construction.

The values V_L^*, W_L^*, V_H^* and W_H^* satisfy

$$\begin{aligned}
V_L^* &= 1 - g_L^* + \theta_L \ln(g_L^*) + \delta[pV_L^* + (1-p)W_L^*], \\
W_L^* &= \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*], \\
V_H^* &= 1 - g_H^* + \theta_H \ln(g_H^*) + \delta[pV_H^* + (1-p)W_H(g_H^*)], \\
W_H^* &= \theta_H \ln(g_L^*) + \delta[(1-p)V_H^* + pW_H^*].
\end{aligned}$$

These together with Lemmas 2, 3 and 5 show that (E3) is satisfied, i.e., these payoff functions are consistent with the strategy profile.

The remainder of this section shows that (E2) is satisfied. The next claim establishes that $K_i(g)$ is increasing in g, which is useful later in the proof.

Claim 2. *Under our conjecture of the equilibrium strategy-payoff pair, $K_i(g)$ is strictly increasing in g for all $i \in \{H, L\}$.*

Proof: Suppose $g \leq g_L^*$. Then $K_i(g) = \theta_i \ln(g) + \delta[(1-p)V_i^* + pW_i^*]$ and therefore $K_i(g)$ is increasing in g.

Suppose $g \in [g_L^*, g_H^*]$. Then $K_L(g) = \theta_L \ln(g) + \delta[(1-p)V_L(g) + pW_L^*]$ where $V_L(g) = \frac{1}{1-\delta p}[1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*]$. Hence,

$$K_L'(g) = \tfrac{1+\delta-2\delta p}{1-\delta p}\tfrac{\theta_L}{g} - \tfrac{\delta(1-p)}{1-\delta p}. \tag{36}$$

Since $\frac{\theta_L}{\theta_H} > \frac{\delta(1-p)}{1-\delta p}$, we have $K_L'(g) > 0$.

Also, since $K_H(g) = \theta_H \ln(g) + \delta(1-p)V_H^* + \delta p[\frac{\theta_H}{1-\delta p}\ln(g) + \delta(1-p)V_H^*]$, it follows that $K_H(g)$ is increasing in g.

Suppose $g \in [g_H^*, \theta_H + \theta_L]$. Then $K_i(g) = \frac{\theta_i}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_i(g)$. Substituting for $V_i(g)$ and taking the derivative, we get

$$K_i'(g) = \tfrac{1}{1-\delta}\left[\tfrac{-\delta(1-p)}{1+\delta-2\delta p} + \tfrac{\theta_i}{g}\right].$$

Since $\frac{\theta_L}{\theta_H} > \frac{\delta(1-p)}{1-\delta p}$, it follows that $K_i'(g) > 0$ for $g \in [g_H^*, \theta_H + \theta_L]$.

Suppose $g \geq \theta_H + \theta_L$. Then again $K_i(g) = \frac{\theta_i}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_i(g)$. Substituting for $V_i(g)$ and taking the derivative, we get

$$K_i'(g) = \tfrac{\theta_i}{(1-\delta p)g} + \tfrac{\delta(1-p)}{1-\delta p}\tfrac{C_i}{g} = \tfrac{\theta_i(1-\delta p)-\theta_j\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)g}$$

where $j \in \{H, L\}, j \neq i$.

For $i = H$, $\theta_H(1-\delta p) - \theta_L\delta(1-p) > 0$ and therefore $K_H'(g) > 0$. For $i = L$, since $\frac{\theta_H}{\theta_L} < (1-\delta p)/(\delta(1-p))$ in the low-polarization case, it follows that $\theta_L(1-\delta p) - \theta_H\delta(1-p) > 0$ and therefore $K_L'(g) > 0$. ■

The claim immediately implies that the responder accepts any proposal with a g' higher than the status quo g and if the responder accepts a proposal with g' lower than the status quo, then the responder must receives a positive transfer. Formally,

Corollary 1. *Consider $z' = (g', x'_H, x'_L) \in B$. For any $i \in \{H, L\}$, (i) if $g' \geq g$, then $\alpha^i(g, z') = 1$; (ii) if $g' < g$ and $\alpha^i(g, z') = 1$, then $x'_i > 0$.*

For notational convenience, let $U_i^P(z) = x_i + \theta_i \ln(g) + \delta[pV_i(g) + (1-p)W_i(g)]$ and $U_i^R(z) = x_i + \theta_i \ln(g) + \delta[(1-p)V_i(g) + pW_i(g)]$. That is, $U_i^P(z)$ ($U_i^R(z)$) denotes party i's payoff when the implemented budget is z in the current period and party i is the proposer (responder). The next claim establishes that all proposals made in equilibrium are accepted by the responder.

Claim 3. *Under our conjecture of the equilibrium strategy-payoff pair, $\alpha^j(g, \pi^i(g)) = 1$ for all g and all $i, j \in \{H, L\}$, $j \neq i$.*

Proof: Consider $j = H$ first.

If $g \leq g_L^*$, then $U_H^R(\pi^L(g)) = \theta_H \ln(g_L^*) + \delta[(1-p)V_H^* + pW_H^*] \geq K_H(g) = \theta_H \ln(g) + \delta[(1-p)V_H^* + pW_H^*]$ and therefore $\alpha^H(g, \pi^L(g)) = 1$.

If $g \in [g_L^*, \theta_H + \theta_L]$, then $\gamma^L(g) = g$ and $\chi_H^L(g) = 0$, which implies that $U_H^R(\pi^L(g)) = K_H(g)$ and therefore $\alpha^H(g, \pi^L(g)) = 1$.

If $g > \theta_H + \theta_L$, then $\gamma^L(g) = \theta_H + \theta_L$ and $\chi_H^L(g) = K_H(g) - \theta_H \ln(\theta_H + \theta_L) - \delta[pV_H(\theta_H + \theta_L) + (1-p)W_H(\theta_H + \theta_L)]$, which implies that $U_H^R(\pi^L(g)) = K_H(g)$ and therefore $\alpha^H(g, \pi^L(g)) = 1$.

Now consider $j = L$.

If $g \leq g_H^*$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L(g_H^*) + pW_L(g_H^*)]$. Since $K'_L(g) > 0$ by Claim 2 and $U_L^R(\pi^H(g)) = K_L(g_H^*)$, it follows that $U_L^R(\pi^H(g)) \geq K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$ for $g \leq g_H^*$.

If $g \geq g_H^*$, then an argument similar to the case of $j = H$ shows that $U_L^R(\pi^H(g)) = K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$. ∎

We next show that the proposer has no profitable one-shot deviation. Consider the following three cases for party L.

- $g \leq g_L^*$: Since $g_L^* = \arg\max f_L(g)$, party L has no incentive to deviate from proposing $\gamma^L(g) = g_L^*$ and $\chi_H^L(g) = 0$.
- $g_L^* < g \leq \theta_H + \theta_L$: We first show that proposing $\pi^L(g)$ is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} > g$ and then show that it is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} < g$.
  - $\hat{g} > g$: Consider $\hat{z} = (\hat{g}, 0, 1 - \hat{g})$. Then $U_L^P(\hat{z}) = f_L(\hat{g})$. As shown in the proof of Claim 1, $f_L(\hat{g})$ is decreasing in $\hat{g}$ for $\hat{g} > g_L^*$. Since $\pi^L(g) = (g, 0, 1 - g)$, this

implies that $U_L^P(\pi^L(g)) > U_L^P(\hat{z})$ for any $\hat{g} > g > g_L^*$. Since party L's payoff is decreasing in x_H, $U_L^P(\hat{z}) \geq U_L^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in B$, it follows that $U_L^P(\pi^L(g)) > U_L^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $\hat{g} > g > g_L^*$. Also, since $\alpha^H(g, \pi^L(g)) = 1$ by Claim 3, and $U_L^P(\pi^L(g))$ is higher than $U_L^P((g, 0, 0))$, the status quo payoff, it follows that proposing $\pi^L(g)$ is better than proposing any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in B$ with $\hat{g} > g$.

- $\hat{g} < g$: If $\hat{g} < g$, then by Corollary 1, $\alpha^H(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_H > 0$. Since party L's payoff is strictly decreasing in x_H, we only need to consider proposals such that the responder's acceptance constraint (7) is binding. Using (7),

$$\hat{x}_H = K_H(g) - \theta_H \ln(\hat{g}) - \delta[(1-p)V_H(\hat{g}) + pW_H(\hat{g})]. \tag{37}$$

Consider $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that (37) holds. Substituting for $\hat{x}_H$ from (37) and taking the derivative, we get

$$\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{\theta_H + \theta_L}{\hat{g}} + \delta[(1-p)V_H'(\hat{g}) + pW_H'(\hat{g})] + \delta[pV_L'(\hat{g}) + (1-p)W_L'(\hat{g})] \tag{38}$$

For $\hat{g} < g_L^*$, $\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{\theta_H + \theta_L}{\hat{g}} > 0$.

For $g_L^* < \hat{g} < g_H^*$, $\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{\theta_H + \theta_L}{\hat{g}} + \frac{\delta p}{1-\delta p}\frac{\theta_H}{\hat{g}} + \frac{\delta p}{1-\delta p}(-1 + \frac{\theta_L}{g}) = \frac{1}{1-\delta p}(-1 + \frac{\theta_H + \theta_L}{\hat{g}}) > 0$.

For $g_H^* < \hat{g} < g \leq \theta_H + \theta_L$, $\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{\hat{g}} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}V_L'(\hat{g}) + \frac{1}{1-\delta p}\frac{\theta_H}{\hat{g}} + \frac{\delta(1-p)}{1-\delta p}V_H'(\hat{g}) = \frac{1}{1-\delta}(-1 + \frac{\theta_H + \theta_L}{g}) > 0$.

So $U_L^P(\hat{z})$ is increasing in $\hat{g}$ for $\hat{g} < g$, and therefore the proposer has no incentive to make any proposal with $\hat{g} < g$.

- $g > \theta_H + \theta_L$: Consider $\hat{z} = (\hat{g}, 0, 1-\hat{g})$ with $\hat{g} > g$. By Corollary 1, $\alpha^H(g, \hat{z}) = 1$. Since $U_L^P(\hat{z}) = f_L(\hat{g})$ and $f_L(\hat{g})$ is decreasing in $\hat{g}$, it follows that $U_L^P((g, 0, 1-g)) \geq U_L^P((\hat{g}, 0, 1-\hat{g}))$ if $\hat{g} \geq g$.

  Now consider $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that $\hat{g} \leq g$ and $\alpha^H(g, \hat{z}) = 1$. By Corollary 1, $\hat{x}_H > 0$ if $\hat{g} > g$. Again we only need to consider proposals such that the responder's acceptance constraint is binding. As before, we obtain (38). Substituting for $V_L'(\hat{g}), W_L'(\hat{g}), V_H'(\hat{g}), W_H'(\hat{g})$, we get

$$\begin{aligned}\frac{\partial U_L^P}{\partial \hat{g}} &= -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{\hat{g}} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L}{\hat{g}} + \left(\frac{1}{1-\delta p}\frac{\theta_H}{\hat{g}} + \frac{\delta(1-p)}{1-\delta p}\frac{C_H}{\hat{g}}\right) \\ &= -1 + \frac{\theta_H + \theta_L}{\hat{g}}.\end{aligned}$$

  Since $\gamma^L(g) = \theta_H + \theta_L$, it follows that $U_L^P(\pi^L(g)) \geq U_L^P(\hat{z})$ for any $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that $\hat{g} < g$ and $\alpha^H(g, \hat{z}) = 1$. Combined with $U_L^P((g, 0, 1-g)) \geq U_L^P((\hat{g}, 0, 1-\hat{g}))$ if $\hat{g} \geq g$, $\pi^L(g)$ is optimal for party L to propose.

Party H also has no incentive to deviate. We omit the details of the proof because the argument is similar to that for party L. ∎

10.5 Proof of Proposition 4

To prove Proposition 4, we first establish some properties of an equilibrium strategy-payoff pair that satisfies (G1′)-(G4′) in the high-polarization case.

10.5.1 Properties of an equilibrium strategy-payoff pair that satisfies (G1′)-(G4′) in the high-polarization case

Suppose $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$ and (V_H, W_H, V_L, W_L) is an equilibrium strategy-payoff pair that satisfies (G1′)-(G4′). In what follows, we establish several properties of σ and (V_H, W_H, V_L, W_L).

Recall that $V_i^* = \max_g f_i(g)$ is proposer i's highest payoff without the responder's constraint (7). As in the low-polarization case, we denote $W_L(g_H^*)$ by W_L^* and $W_H(g_L^*)$ by W_H^*.

Lemma 6. *Under (G1′) and (G2′), if $g \le g_L^*$, then $V_L(g) = V_L^*$, $\chi_L^L(g) = 1 - g_L^*$, $\chi_H^L(g) = 0$, and $W_H(g) = W_H^*$. Under (G3′), if $g \le \underline{g}_H$ or $g \ge g_H^*$, then $V_H(g) = V_H^*$, $\chi_H^H(g) = 1 - g_H^*$, $\chi_L^H(g) = 0$, and $W_L(g) = W_L^*$.*

We omit the proof since it is similar to that of Lemma 2.

Lemma 7. *Under (G1′)-(G3′), (i) if $g \in [g_L^*, \underline{g}_H]$, then $V_L(g) = \frac{1}{1-\delta p}[1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*]$, (ii) if $g \in [\underline{g}_H, \theta_H + \theta_L]$, then*

$$V_L(g) = \tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g), \tag{39}$$

and (iii) if $g \in [\theta_H + \theta_L, g_H^*]$, then

$$V_H(g) = \tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(g). \tag{40}$$

We omit the proof since it is similar to that of Lemma 3.

Lemma 8. *Under (G1′)-(G3′), $g_L^* = \theta_L$ and $g_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$.*

Proof: We omit the proof for g_L^* since it is the same as that of Lemma 4.

Now consider g_H^*. If $g > g_H^*$, then $V_H'(g) = 0$ by Lemma 6 and $W_H'(g) = \frac{\theta_H}{(1-\delta p)g}$ by Lemma 1, and therefore

$$f_H'(g) = -1 + \tfrac{\theta_H}{g} + \delta(1-p)W_H'(g) = -1 + \tfrac{(1+\delta-2\delta p)\theta_H}{(1-\delta p)g}. \tag{41}$$

If $g_H^* < \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, then (41) implies that $f_H'(g) > 0$ for $g \in (g_H^*, \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p})$, contradicting that $g_H^* \in \arg\max f_H(g)$. Hence $g_H^* \ge \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$. If $g \in (\theta_H + \theta_L, g_H^*)$, then by (G3′), $f_H(g) = V_H(g)$, and by (40)

$$f_H'(g) = -\tfrac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{(1-\delta)g}. \tag{42}$$

If $g_H^* > \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, then (42) implies that $f_H'(g) < 0$ for $g \in (\frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}, g_H^*)$, contradicting that $g_H^* \in \arg\max f_H(g)$. Hence $g_H^* \leq \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$.

Since $g_H^* \leq \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$ and $g_H^* \geq \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, it follows that $g_H^* = \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$. ∎

Recall that we guess in (G4′) that V_i is piecewise linear in g and $\ln(g)$ if $\gamma^i(g) = \theta_H + \theta_L$. Specifically, suppose that for $g \in [\theta_H+\theta_L, g_H^*]$, $V_L(g)$ takes the form $V_L(g) = B_L^1 g + C_L^1 \ln(g) + D_L^1$; for $g \geq g_H^*$ such that $\gamma_L(g) = \theta_H+\theta_L$, $V_L(g)$ takes the form $V_L(g) = B_L^2 g + C_L^2 \ln(g) + D_L^2$; for $g \in [\tilde{g}_H, \theta_H + \theta_L]$, $V_H(g)$ takes the form $V_H(g) = B_H^1 g + C_H^1 \ln(g) + D_H^1$.

Lemma 9. *Under (G1′)-(G4′), $B_i^1 = \frac{\delta(1-p)(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)}$ and $C_i^1 = -\frac{\theta_j}{1-\delta}$ for $i,j \in \{H, L\}$ with $j \neq i$, and $B_L^2 = 0$, $C_L^2 = -\frac{\theta_H}{1-\delta p}$.*

Proof: Similar to the proof of Lemma 5, we can write

$$V_i(g) = \chi_i^i(g) + \tfrac{1+\delta-2\delta p}{1-\delta p}\theta_i \ln(\theta_H + \theta_L) + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p} V_i(\theta_H + \theta_L),$$

where

$$\chi_j^i(g) = K_j(g) - \tfrac{\theta_j}{1-\delta p}\ln(\theta_H + \theta_L) - \tfrac{\delta(1-p)}{1-\delta p} V_j(\theta_H + \theta_L),$$
$$K_j(g) = \tfrac{\theta_j}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p} V_j(g).$$

If $g \in [\theta_H + \theta_L, g_H^*]$, then $V_H(g) = \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(g)$ by Lemma 7. Substituting in $K_H(g)$, we get

$$K_H(g) = \tfrac{\delta(1-p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(g).$$

Substituting in $V_L(g)$ and matching coefficients, we get $B_L^1 = \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}$ and $C_L^1 = -\frac{\theta_H}{1-\delta}$. A similar argument shows that $B_H^1 = \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}$ and $C_H^1 = -\frac{\theta_L}{1-\delta}$.

To find B_L^2 and C_L^2, note that if $g \geq g_H^*$, then by Lemma 6, $V_H(g) = V_H^*$. By Lemma 1, $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H^*$. Matching coefficients gives $B_L^2 = 0$ and $C_L^2 = -\frac{\theta_H}{1-\delta p}$. ∎

We next find the thresholds $\underline{g}_H$ and $\tilde{g}_H$ that are consistent with (G1′)-(G4′). Recall that $\theta_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$.

Lemma 10. *Under (G1′)-(G4′), the threshold $\underline{g}_H \in (0, \theta_H + \theta_L)$ is given by $\underline{g}_H = \psi$ where*

$$\begin{aligned}\psi = \min\{g :& \tfrac{\delta(1-p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g) \\ =& \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{1}{1-\delta p}\left[\theta_L - \tfrac{\delta(1-p)}{1-\delta}\theta_H\right]\ln(\theta_H^*) \\ &+ \tfrac{\delta(1-p)}{(1-\delta p)(1-\delta)}\left[(\theta_H + \theta_L)[\ln(\theta_H + \theta_L) - 1] + \tfrac{\delta(1-p)}{1+\delta-2\delta p}\theta_H^*\right].\}\end{aligned} \tag{43}$$

Proof: By (G3′) (ii) and (iv), the threshold $\underline{g}_H$ satisfies

$$\theta_L \ln(g_H^*) + \delta[(1-p)V_L(g_H^*) + pW_L(g_H^*)] = W_L(\underline{g}_H) = K_L(\underline{g}_H). \tag{44}$$

By(G3′)(ii), $W_L(g_H^*) = K_L(g_H^*)$. Hence by Lemma 1, we can rewrite the left-hand side of the above equation as

$$\tfrac{\theta_L}{1-\delta p}\ln(g_H^*) + \tfrac{\delta(1-p)}{1-\delta p}V_L(g_H^*). \tag{45}$$

By (G1′), $g_H^* > \theta_H + \theta_L$. Hence $\gamma^L(g_H^*) = \theta_H + \theta_L$ by (G2′). So $V_L(g_H^*)$ can be written as

$$V_L(g_H^*) = \chi_L^L(g_H^*) + \tfrac{1+\delta-2\delta p}{1-\delta p}\theta_L \ln(\theta_H + \theta_L) + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p}V_L(\theta_H + \theta_L),$$

where $\chi_L^L(g_H^*) = 1 - \chi_H^L(g_H^*) - \gamma^L(g_H^*) = 1 - \chi_H^L(g_H^*) - \theta_H - \theta_L$, and

$$\begin{aligned}\chi_H^L(g_H^*) =& K_H(g_H^*) - \tfrac{\theta_H}{1-\delta p}\ln(\theta_H + \theta_L) - \tfrac{\delta(1-p)}{1-\delta p}V_H(\theta_H + \theta_L),\\ K_H(g_H^*) =& \tfrac{\theta_H}{1-\delta p}\ln(g_H^*) + \tfrac{\delta(1-p)}{1-\delta p}V_H(g_H^*).\end{aligned}$$

By Lemma 7,

$$\begin{aligned}V_L(\theta_H + \theta_L) &= \tfrac{(1-\delta p)(1-\theta_H-\theta_L)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(\theta_H + \theta_L),\\ V_H(\theta_H + \theta_L) &= \tfrac{(1-\delta p)(1-\theta_H-\theta_L)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(\theta_H + \theta_L),\\ V_H(g_H^*) &= \tfrac{(1-\delta p)(1-g_H^*)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(g_H^*).\end{aligned}$$

Substituting in all expressions, (45) becomes

$$\begin{aligned}&\tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{1}{1-\delta p}\left[\theta_L - \tfrac{\delta(1-p)}{1-\delta}\theta_H\right]\ln(g_H^*)\\ &+\tfrac{\delta(1-p)}{(1-\delta p)(1-\delta)}\left[(\theta_H + \theta_L)[\ln(\theta_H + \theta_L) - 1] + \tfrac{\delta(1-p)}{1+\delta-2\delta p}g_H^*\right].\end{aligned}$$

By (G3′)(ii) and Lemma 1, we can write $K_L(\underline{g}_H)$ as

$$K_L(\underline{g}_H) = \tfrac{\theta_L}{1-\delta p}\ln(\underline{g}_H) + \tfrac{\delta(1-p)}{1-\delta p}V_L(\underline{g}_H).$$

By Lemma 7 this becomes

$$K_L(\underline{g}_H) = \tfrac{\theta_L}{1-\delta}\ln(\underline{g}_H) + \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}(1 - \underline{g}_H).$$

Hence $\underline{g}_H$ is given by

$$\begin{aligned}\tfrac{\theta_L}{1-\delta}\ln(\underline{g}_H) \quad + \quad & \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}(1 - \underline{g}_H) =\\ & \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{1}{1-\delta p}\left[\theta_L - \tfrac{\delta(1-p)}{1-\delta}\theta_H\right]\ln(g_H^*)\\ & +\tfrac{\delta(1-p)}{(1-\delta p)(1-\delta)}\left[(\theta_H + \theta_L)[\ln(\theta_H + \theta_L) - 1] + \tfrac{\delta(1-p)}{1+\delta-2\delta p}g_H^*\right].\end{aligned} \tag{46}$$

The above condition gives at most two values for $\underline{g}_H$ since the left-hand side is strictly concave in $\underline{g}_H$. We show below only one solution is lower than $\theta_H + \theta_L$, and hence is a candidate for $\underline{g}_H$ by (G3′). We show that at $\theta_H + \theta_L$, the left-hand side is strictly greater than the right-hand side of (46). Substituting $\underline{g}_H = \theta_H + \theta_L$ the condition simplifies to

$$\begin{aligned}&\tfrac{\theta_L(1-\delta)-\theta_H\delta(1-p)}{(1-\delta p)(1-\delta)}\ln(\theta_H + \theta_L) + \tfrac{\delta^2(1-p)^2}{(1-\delta)(1+\delta-2\delta p)(1-\delta p)}(\theta_H + \theta_L) >\\ &\tfrac{\theta_L(1-\delta)-\theta_H\delta(1-p)}{(1-\delta p)(1-\delta))}\ln(g_H^*) + \tfrac{\delta^2(1-p)^2}{(1-\delta)(1+\delta-2\delta p)(1-\delta p)}(g_H^*).\end{aligned} \tag{47}$$

The left-hand side and right-hand side are the same function, $h(g) = \frac{\theta_L(1-\delta)-\theta_H\delta(1-p)}{(1-\delta p)(1-\delta)}\ln(g) + \frac{\delta^2(1-p)^2}{(1-\delta)(1+\delta-2\delta p)(1-\delta p)}g$, evaluated at $\theta_H+\theta_L$ and g_H^*, respectively. It is straightforward to show $h'(g) < 0$. Since $\theta_H+\theta_L < g_H^*$, it follows (47) is true. Given (47) is true, the value that satisfies (46) such that $\underline{g}_H < \theta_H+\theta_L$ must be the minimum of the two solutions. ∎

Lemma 11. *Under (G1′)-(G4′), the threshold $\tilde{g}_H \in (0, \theta_H+\theta_L)$ is given by*

$$\frac{\delta(1-p)(1-\tilde{g}_H)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(\tilde{g}_H) = \frac{\delta(1-p)(1-\theta_L-\theta_H)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(\theta_H+\theta_L). \tag{48}$$

Proof: By (G3′) (ii) and (iv), the threshold $\tilde{g}_H$ satisfies

$$\theta_L\ln(\theta_H+\theta_L) + \delta[(1-p)V_L(\theta_H+\theta_L) + pW_L(\theta_H+\theta_L)] = K_L(\tilde{g}_H). \tag{49}$$

By Lemma 7, $V_L(g) = \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(g)$ for $g \in [\tilde{g}_H, \theta_H+\theta_L]$. Substituting this in (49) and using Lemma 1, we get (48). ∎

10.5.2 Derivation of condition $(**)$

Note that $\alpha^H(g, (\theta_H+\theta_L, x_H, x_L)) = 1$ with $x_H = 1-\theta_L-\theta_H$, $x_L = 0$ for all $g \geq \theta_H+\theta_L$ is satisfied if

$$1-(\theta_H+\theta_L) + \theta_H\ln(\theta_H+\theta_L) + \delta[(1-p)V_H(\theta_H+\theta_L) + pW_H(\theta_H+\theta_L)] \geq K_H(g).$$

Substituting for $K_H(g)$ and $W_H(g)$ using Lemma 1 and substituting for $V_H(g) = V_H^*$ for $g \geq g_H^*$ using Lemma 6, the inequality becomes

$$1-(\theta_H+\theta_L) + \frac{\theta_H}{1-\delta p}\ln(\theta_H+\theta_L) + \frac{\delta(1-p)}{1-\delta p}V_H(\theta_H+\theta_L) \geq \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H^*. \tag{50}$$

Note that the right-hand side of (50) is increasing in g, implying that if the inequality holds for $g=1$, then it holds for all $g \geq \theta_H+\theta_L$. Substituting for $V_H(\theta_H+\theta_L)$ and $V_H(g_H^*)$ using Lemma 7 and letting $g=1$, we can rewrite inequality (50) to be

$$1-(\theta_H+\theta_L) + \frac{\theta_H}{1-\delta}\ln(\theta_H+\theta_L) \geq \frac{\delta(1-p)(\theta_H+\theta_L-\theta_H^*)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\delta(1-p)\theta_H}{(1-\delta p)(1-\delta)}\ln(\theta_H^*). \tag{$**$}$$

10.5.3 Proof of Proposition 4

We proceed by first conjecturing an equilibrium strategy-payoff pair and then verifying that it satisfies guesses (G1′)-(G4′), equilibrium conditions (E1)-(E3), and our assumption on α^i that all proposals made on the equilibrium path are satisfied.

We conjecture an equilibrium strategy-payoff pair such that for any $i, j \in \{H, L\}$ with $j \neq i$, the acceptance strategy $\alpha^i(g, z)$ satisfies (E1), the proposal strategies are

$$\gamma^L(g) = \begin{cases} g_L^* & \text{for } g \leq g_L^*, \\ g & \text{for } g_L^* \leq g \leq \theta_H+\theta_L, \\ \theta_H+\theta_L & \text{for } \theta_H+\theta_L \leq g, \end{cases}$$

$$\chi_H^L(g) = \begin{cases} 0 & \text{for } g \le \theta_H + \theta_L, \\ K_H(g) - \theta_H \ln(\theta_H + \theta_L) - \delta[(1-p)V_H(\theta_H + \theta_L) + pW_H(\theta_H + \theta_L)] & \text{for } \theta_H + \theta_L \le g, \end{cases}$$

$$\gamma^H(g) = \begin{cases} g_H^* & \text{for } g \le \underline{g}_H, \\ g' \in [\theta_H + \theta_L, g_H^*] & \text{for } \underline{g}_H \le g \le \tilde{g}_H, \\ \theta_H + \theta_L & \text{for } \tilde{g}_H \le g \le \theta_H + \theta_L, \\ g & \text{for } \theta_H + \theta_L \le g \le g_H^*, \\ g_H^* & \text{for } g_H^* \le g, \end{cases}$$

$$\chi_L^H(g) = \begin{cases} 0 & \text{for } g \le \tilde{g}_H, \\ K_L(g) - \theta_L \ln(\theta_H + \theta_L) - \delta[(1-p)V_L(\theta_H + \theta_L) + pW_L(\theta_H + \theta_L)] & \text{for } g \in [\tilde{g}_H, \theta_H + \theta_L], \\ 0 & \text{for } g \ge \theta_H + \theta_L, \end{cases}$$

and $\chi_i^i(g) = 1 - \gamma^i(g) - \chi_j^i(g)$, where $g_L^* = \theta_L$, $g_H^* = \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, $\underline{g}_H$ satisfies (43), $\tilde{g}_H$ satisfies (49), g' satisfies

$$\theta_L \ln(g') + \delta[(1-p)V_L(g') + pW_L(g')] = K_L(g), \tag{51}$$

and the associated payoff functions are

$$V_L(g) = \begin{cases} V_L^* & \text{for } g \le g_L^*, \\ \frac{1}{1-\delta p}(1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*) & \text{for } g_L^* \le g \le \underline{g}_H, \\ \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(g) & \text{for } \underline{g}_H \le g \le \theta_H + \theta_L, \\ B_L^1 g + C_L^1 \ln(g) + D_L^1 & \text{for } \theta_H + \theta_L \le g \le g_H^*, \\ C_L^2 \ln(g) + D_L^2 & \text{for } g_H^* \le g, \end{cases}$$

$$W_L(g) = \begin{cases} W_L^* & \text{for } g \le \underline{g}_H \text{ and } g \ge g_H^*, \\ \frac{1}{1-\delta p}[\theta_L \ln(g) + \delta(1-p)V_L(g)] & \text{for } \underline{g}_H \le g \le g_H^*, \end{cases}$$

$$V_H(g) = \begin{cases} V_H^* & \text{for } g \le \underline{g}_H, \\ \frac{(1-\delta p)(1-\gamma^H(g))}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(\gamma^H(g)) & \text{for } \underline{g}_H \le g \le \tilde{g}_H, \\ B_H^1 g + C_H^1 \ln(g) + D_H^1 & \text{for } \tilde{g}_H \le g \le \theta_H + \theta_L, \\ \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(g) & \text{for } \theta_H + \theta_L \le g \le g_H^*, \\ V_H^* & \text{for } g_H^* \le g, \end{cases}$$

$$W_H(g) = \begin{cases} W_H^* & \text{for } g \le g_L^*, \\ \frac{1}{1-\delta p}[\theta_H \ln(g) + \delta(1-p)V_H(g)] & \text{for } g_L^* \le g, \end{cases}$$

where $B_i^1 = \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}$, $C_i^1 = -\frac{\theta_j}{1-\delta}$, $D_i^1 = \frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{(\theta_H+\theta_L)[\ln(\theta_H+\theta_L)-1]}{1-\delta}$, $C_L^2 = -\frac{\theta_H}{1-\delta p}$, and

$$W_L^* = \tfrac{\delta(1-p)}{(1+\delta-2\delta p)(1-\delta)}(1-g_H^*) + \tfrac{\theta_L}{1-\delta}\ln(g_H^*), \tag{52}$$

$$V_L^* = \tfrac{1}{1-\delta p}[1-\theta_L+\theta_L\ln(\theta_L)+\delta(1-p)W_L^*], \tag{53}$$

$$V_H^* = \tfrac{(1-\delta p)(1-g_H^*)}{(1+\delta-2\delta p)(1-\delta)} + \tfrac{\theta_H}{1-\delta}\ln(g_H^*), \tag{54}$$

$$W_H^* = \tfrac{1}{1-\delta p}[\theta_H\ln(g_L^*)+\delta(1-p)V_H^*]. \tag{55}$$

We next verify that this conjecture satisfies (G1′)-(G4′).

For (G1′), since $g_L^* = \theta_L$ and $g_H^* = \frac{(1+\delta-2\delta p)\theta_H}{1-\delta p}$, clearly $g_L^* < \theta_H+\theta_L < g_H^*$ in the high-polarization case, and it only remains to show that $g_i^* \in \arg\max f_i(g)$. In Claim 4 below, we show that (i) $g_H^* \in \arg\max f_H(g)$, and (ii) $g_L^* \in \arg\max f_L(g)$ when $\psi \geq \theta_L^*$, where ψ is defined in (43).

Since V_i and W_i are continuous under our conjecture of the equilibrium strategy-payoff pair, f_i is continuous. It is also piecewise differentiable. Specifically,

$$f_L'(g) = \begin{cases} -1+\frac{\theta_L}{g} & \text{for } g < g_L^*, \\ \frac{1}{1-\delta p}[-1+\frac{\theta_L}{g}] & \text{for } g \in (g_L^*, \underline{g}_H), \\ -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} & \text{for } g \in (\underline{g}_H, \theta_H+\theta_L), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_L^1+\frac{C_L^1}{g}) & \text{for } g \in (\theta_H+\theta_L, g_H^*), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L^2}{g} & \text{for } g \geq g_H^*, \end{cases}$$

$$f_H'(g) = \begin{cases} -1+\frac{\theta_H}{g} & \text{for } g < g_L^*, \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} & \text{for } g \in (g_L^*, \underline{g}_H), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)\gamma^H(g)})\frac{d\gamma^H(g)}{dg} & \text{for } g \in (\underline{g}_H, \tilde{g}_H), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_H^1+\frac{C_H^1}{g}) & \text{for } g \in (\tilde{g}_H, \theta_H+\theta_L), \\ -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g} & \text{for } g \in (\theta_H+\theta_L, g_H^*), \\ -1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} & \text{for } g > g_H^*. \end{cases}$$

Claim 4. *Under our conjecture of the equilibrium strategy-payoff pair, (i) $g_H^* \in \arg\max f_H(g)$, and (ii) if $\psi \geq \theta_L^*$, then $g_L^* \in \arg\max f_L(g)$.*

Proof: Consider part (i) first.

- $g < g_L^*$: $f_H'(g)$ is decreasing in g. At $g_L^* = \theta_L$, $f_H'(g_L^*) > 0$, hence for $g < g_L^*$, $f_H'(g) > 0$.
- $g \in (g_L^*, \underline{g}_H)$: $f_H'(g)$ is decreasing in g. Since $\underline{g}_H < g_H^*$ and $f_H'(g) = -1 + \frac{g_H^*}{g}$, it follows that $f_H'(g) > 0$ for $g \in (g_L^*, \underline{g}_H)$.

- $g \in (\underline{g}_H, \tilde{g})$: We compare $f_H(g)$ in this range to $f_H(g_H^*)$. First define the functions

$$\begin{aligned} n(x) &= 1 - x + \tfrac{\theta_H(1+\delta-2\delta p)}{1-\delta p}\ln(x), \text{ and} \\ m(y) &= \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p}\left[\tfrac{(1-\delta p)(1-y)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(y)\right]. \end{aligned}$$

  By these definitions $f_H(g_H^*) = n(g_H^*) + m(g_H^*)$, and for $g \in (\underline{g}_H, \tilde{g})$, $f_H(g) = n(g) + m(\gamma^H(g))$. Further note that $g_H^* = \arg\max n(x)$, and $g_H^* = \arg\max m(y)$, hence $n(g_H^*) \geq n(g)$ for all g, and $m(g_H^*) \geq m(\gamma^H(g))$ for all $\gamma^H(g)$, so $f_H(g_H^*)$ is greater than $f_H(g)$ for $g \in (\underline{g}_H, \tilde{g})$.

- $g \in (\tilde{g}_H, \theta_H+\theta_L)$: $f_H'(g)$ strictly decreasing in g. Since $f_H'(\theta_H+\theta_L) = \frac{\theta_H\delta(1-p)-\theta_L(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)(\theta_H+\theta_L)} > 0$, $f_H'(g) > 0$ everywhere in this interval.

- $g \in (\theta_H + \theta_L, g_H^*)$: $f_H'(g)$ strictly decreasing in g. Since $-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g_H^*} = 0$, it follows that for $g \in (\theta_H + \theta_L, g_H^*)$, $f_H'(g) > 0$.

- $g > g_H^*$: $f_H'(g) = -1 + \frac{g_H^*}{g} < 0$.

  To summarize, $f_H(g)$ is strictly increasing for $g \in (\tilde{g}_H, g_H^*)$, and strictly decreasing for $g > g_H^*$ hence $g_H^* \in \arg\max f_H(g)$ for $g > \tilde{g}_H$. Further, we showed for $g \in (\underline{g}_H, \tilde{g}_H)$, $f_H(g) \leq f_H(g_H^*)$ and for $g < \underline{g}_H$, $f_H(g)$ is increasing. Hence, by continuity of $f_H(g)$, $g_H^* \in \arg\max f_H(g)$ for all g.

Now consider Part (ii).

- $g < g_L^*$: then $f_L'(g) > 0$.

- $g \in (g_L^*, \underline{g}_H)$: $f_L'(g)$ is strictly decreasing in g. Since $f_L'(g) = \frac{1}{1-\delta p}[-1 + \frac{\theta_L}{g}]$, it follows that $f_L'(g) < 0$ for $g \in (g_L^*, \underline{g}_H)$.

- $g \in (\underline{g}_H, \theta_H + \theta_L)$: $f_L'(g)$ is strictly decreasing in g. Since $\underline{g}_H = \psi$ by Lemma 10, we have $\underline{g}_H = \psi \geq \theta_L^*$. Since $-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} = 0$ if $g = \frac{\theta_L(1+\delta-2\delta p)}{1-\delta p}$, it follows that $f_L'(g) < 0$ for all $g \in (\underline{g}_H, \theta_H + \theta_L)$.

- $g \in (\theta_H+\theta_L, g_H^*)$: The monotonicity of $f_L'(g)$ is determined by $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^1}{1-\delta p}$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^1}{1-\delta p} > 0$, then $f_L'(g)$ is strictly decreasing in g. Since $-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_L^1 + \frac{C_L^1}{g}) = \frac{\theta_L\delta(1-p)-\theta_H(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)(\theta_H+\theta_L)} \leq 0$ if $g = \theta_H+\theta_L$, it follows that $f_L'(g) < 0$ for $g \in (\theta_H + \theta_L, g_H^*)$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^1}{1-\delta p} \leq 0$, then $f_L'(g)$ is weakly increasing in g. Since $-1+\frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g}+\frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_L^1+\frac{C_L^1}{g}) = -1+\frac{\theta_L}{\theta_H}-\frac{\delta(p+\delta-2\delta p)}{(1+\delta-2\delta p)(1-\delta p)} < 0$ when $g = g_H^*$, it follows that $f_L'(g) < 0$ for $g \in (\theta_H + \theta_L, g_H^*)$.

- $g > g_H^*$: The monotonicity of $f_L'(g)$ is determined by $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p}$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p} > 0$, then $f_L'(g)$ is strictly decreasing in g. Since $-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L^2}{g} = -1 + \frac{\theta_L}{\theta_H} - \frac{\delta(p+\delta-2\delta p)}{(1-\delta p)(1+\delta-2\delta p)} < 0$ if $g = g_H^*$, it follows that $f_L'(g) < 0$ for $g > g_H^*$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p} \leq 0$, then $f_L'(g)$ is weakly increasing in g. In this case, $f_L'(g) = -1 + \frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p} < 0$ when $g = 1$ and therefore $f_L'(g) < 0$ for $g > g_H^*$.

To summarize, $f_L(g)$ is strictly increasing for $g < g_L^*$ and strictly decreasing for $g > g_L^*$ and therefore $g_L^* \in \arg\max f_L(g)$. ∎

The conjectured equilibrium strategy-payoff pair clearly satisfies (G2′)-(G4′) with the exception of $g_L^* \leq \underline{g}_H < \tilde{g}_H < \theta_H + \theta_L$. When $\psi \geq \theta_L^*$, we have $g_L^* = \theta_L < \theta_L^* \leq \psi = \underline{g}_H$. To verify that $\underline{g}_H < \tilde{g}_H < \theta_H + \theta_L$, we next establish some monotonicity properties of K_L.

Claim 5. *Under our conjecture of the equilibrium strategy-payoff pair, $K_L(g)$ is strictly increasing for $g \in [0, \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)})$ and strictly decreasing for $g \in (\frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}, 1]$.*

Proof: Consider the following cases:

- $g \leq g_L^*$: Then $K_L(g) = \theta_L \ln(g) + \delta[(1-p)V_L^* + pW_L^*]$, which is increasing in g.

- $g \in [g_L^*, \underline{g}_H]$: Then

$$K_L(g) = \theta_L \ln(g) + \tfrac{\delta(1-p)}{1-\delta p}(1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*) + \delta p W_L^*,$$

  Taking the derivative, we get

$$K_L'(g) = \tfrac{1+\delta-2\delta p}{1-\delta p}\tfrac{\theta_L}{g} - \tfrac{\delta(1-p)}{1-\delta p},$$

  and $K_L'(g) > 0$ if and only if $g < \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L$.

- $g \in [\underline{g}_H, \theta_H + \theta_L]$: Then

$$\begin{aligned} K_L(g) &= \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}V_L(g) \\ &= \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}\left[\tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g)\right]. \end{aligned}$$

  Taking the derivative, we get

$$K_L'(g) = \tfrac{1}{1-\delta}\left[\tfrac{-\delta(1-p)}{1+\delta-2\delta p} + \tfrac{\theta_L}{g}\right],$$

  and $K_L'(g) > 0$ if and only if $g < \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L$. Note that since $\theta_H + \theta_L > \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L$ in the high-polarization case, $K_L'(g) < 0$ for $g \in (\frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L, \theta_H + \theta_L)$.

- $g \in [\theta_H + \theta_L, g_H^*]$: Then
$$\begin{aligned} K_L(g) &= \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}V_L(g) \\ &= \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}(B_L^1 g + C_L^1\ln(g) + D_L^1). \end{aligned}$$
Taking the derivative, we get
$$K_L'(g) = \tfrac{1}{(1-\delta p)(1-\delta)}\left[\tfrac{(1-\delta)\theta_L - \delta(1-p)\theta_H}{g} + \tfrac{\delta^2(1-p)^2}{1+\delta-2\delta p}\right],$$
which is increasing in g since $(1-\delta)\theta_L - \delta(1-p)\theta_H < 0$ in the high-polarization case. Straightforward calculation shows that $K_L'(g) < 0$ for $g = g_H^*$. Hence, $K_L(g)$ is strictly decreasing for $g \in [\theta_H + \theta_L, g_H^*]$.

- $g \geq g_H^*$: Then
$$K_L(g) = \theta_L\ln(g) + \delta[(1-p)V_L(g) + pW_L(g)] \tag{56}$$
$$= \theta_L\ln(g) + \delta(1-p)(C_L^2\ln(g) + D_L^2) + \delta p W_L^*. \tag{57}$$
Since $\frac{\theta_H}{\theta_L} > (1-\delta p)/(\delta(1-p))$ in the high-polarization case, this implies that
$$K_L'(g) = \tfrac{\theta_L}{g} - \tfrac{\delta(1-p)\theta_H}{(1-\delta p)g} < 0. \tag{58}$$

Hence, $K_L(g)$ is strictly increasing in g for $g \in [0, \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)})$ and strictly decreasing in g for $g \in (\frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}, 1]$. ∎

Recall that in our conjectured equilibrium strategy-payoff pair, $\underline{g}_H$ satisfies $K_L(\underline{g}_H) = K_L(g_H^*)$ and $\tilde{g}_H$ satisfies $K_L(\tilde{g}_H) = K_L(\theta_H + \theta_L)$. Since K_L is continuous, $K_L(g) = -\infty$ when $g = 0$, and $\frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)} < \theta_H + \theta_L < g_H^*$ in the high-polarization case, it follows from Claim 5 that there exist $\underline{g}_H < \tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)} < \theta_H + \theta_L < g_H^*$ such that $K_L(\underline{g}_H) = K_L(g_H^*)$ and $K_L(\tilde{g}_H) = K_L(\theta_H + \theta_L)$.

Corollary 2. *There exist $\underline{g}_H$ and $\tilde{g}_H$ where $\underline{g}_H < \tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)} < \theta_H + \theta_L < g_H^*$ such that $K_L(\underline{g}_H) = K_L(g_H^*)$ and $K_L(\tilde{g}_H) = K_L(\theta_H + \theta_L)$.*

We next verify that (E1)-(E3) in the definition of equilibrium strategy-payoff pair are satisfied. Condition (E1) is satisfied by construction.

The values V_L^*, W_L^*, V_H^* and W_H^* satisfy
$$\begin{aligned} V_L^* &= 1 - g_L^* + \theta_L\ln(g_L^*) + \delta[pV_L^* + (1-p)W_L^*], \\ W_L^* &= \theta_L\ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*], \\ V_H^* &= 1 - g_H^* + \theta_H\ln(g_H^*) + \delta[pV_H^* + (1-p)W_H(g_H^*)], \\ W_H^* &= \theta_H\ln(g_L^*) + \delta[(1-p)V_H^* + pW_H^*]. \end{aligned}$$

These together with Lemmas 6, 7 and 9 show that (E3) is satisfied, that is, these payoff functions are consistent with the strategy profile.

The next claim establishes that all proposals made in equilibrium are accepted.

Claim 6. *Under our conjecture of the equilibrium strategy-payoff pair, $\alpha^j(g, \pi^i(g)) = 1$ for all g and all $i, j \in \{H, L\}$, $j \neq i$.*

Proof: We omit the proof for $j = H$ since it is similar to that for Claim 3.

Now consider $j = L$.

If $g \leq g_L^*$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*] \geq K_L(g) = \theta_L \ln(g_H) + \delta[(1-p)V_L^* + pW_L^*]$ and therefore $\alpha^L(g, \pi^H(g)) = 1$.

If $g \in [g_L^*, \underline{g}_H]$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*] = K_L(g_H^*)$. Since $K_L(\underline{g}_H) = K_L(g_H^*)$ and K_L is increasing in g on $[g_L^*, \underline{g}]$ by Claim 5, it follows that $U_L^R(\pi^H(g)) \geq K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$ for $g \in [g_L^*, \underline{g}_H]$.

If $g \in [\underline{g}_H, g_H^*]$, then $U_L^R(\pi^H(g)) = K_L(g)$ and $\alpha^L(g, \pi^H(g)) = 1$.

If $g \in [g_H^*, 1]$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*] = K_L(g_H^*)$. Since $K_L(g)$ is decreasing in g on $[g_H^*, 1]$ by Claim 5, it follows that it follows that $U_L^R(\pi^H(g)) \geq K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$ for $g \in [g_H^*, 1]$. ∎

The remainder of the proof shows that (E2) is satisfied. The next claim establishes that $K_H(g)$ is increasing, which is useful later in the proof.

Claim 7. *Under our conjecture of the equilibrium strategy-payoff pair, if $\psi \geq \theta_L^*$, then $K_H(g)$ is strictly increasing.*

Proof:

- $g \leq g_L^*$: Then $K_H(g) = \theta_H \ln(g) + \delta[(1-p)V_H^* + pW_H^*]$ and therefore it is strictly increasing.

- $g \in [g_L^*, \underline{g}_H]$: Then $K_H(g) = \theta_H \ln(g) + \delta(1-p)V_H^* + \frac{\delta p}{1-\delta p}[\theta_H \ln(g) + \delta(1-p)V_H^*]$ and therefore it is strictly increasing.

- $g \in [\underline{g}_H, \tilde{g}_H]$: Then $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H(g)$, and $K'_H(g) = \frac{\theta_H}{(1-\delta p)g} + \frac{\delta(1-p)}{1-\delta p}V'_H(g)$. The function $V_H(g)$ is
$$V_H(g) = \tfrac{(1-\delta p)(1-\gamma^H(g))}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(\gamma^H(g)),$$
and $\gamma^H(g)$ is given by (51), which implies
$$\begin{aligned}&\tfrac{\theta_L}{1-\delta p}\ln(\gamma^H(g)) + \tfrac{\delta(1-p)}{1-\delta p}\left[\tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}\gamma^H(g) - \tfrac{\theta_H}{1-\delta}\ln(\gamma^H(g)) + D_L^1\right]\\ =&\tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}\left[\tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g)\right].\end{aligned} \tag{59}$$
Rearranging (59) gives
$$\begin{aligned}\ln(\gamma^H(g)) = \left[\tfrac{1-\delta p}{\theta_L(1-\delta)-\theta_H\delta(1-p)}\right][\theta_L \ln(g) + \tfrac{\delta(1-p)(1-g)}{1+\delta-2\delta p} - \tfrac{\delta^2(1-p)^2\gamma^H(g)}{(1-\delta p)(1+\delta-2\delta p)}\\ - \tfrac{\delta(1-p)(1-\delta)}{1-\delta p}D_L^1].\end{aligned}$$

Substituting $\ln(\gamma^H(g))$ into $V_H(g)$ and taking the derivative, we have

$$V'_H(g) = \frac{\theta_H\theta_L(1-\delta p)}{(1-\delta)[\theta_L(1-\delta)-\theta_H\delta(1-p)]g} - \frac{\theta_H\delta(1-p)(1-\delta p)}{(1-\delta)[\theta_L(1-\delta)-\theta_H\delta(1-p)](1+\delta-2\delta p)}$$
$$- \frac{d\gamma^H(g)}{dg}\frac{\theta_L(1-\delta p)-\theta_H\delta(1-p)}{(1+\delta-2\delta p)[\theta_L(1-\delta)-\theta_H\delta(1-p)]},$$

and $K'_H(g) = A(g) + B(g)$ where

$$A(g) = \frac{\theta_H}{(1-\delta p)g} + \frac{\theta_H\delta(1-p)}{(1-\delta)[\theta_L(1-\delta)-\theta_H\delta(1-p)]}\left[\frac{\theta_L}{g} - \frac{\delta(1-p)}{1+\delta-2\delta p}\right],$$
$$B(g) = -\frac{\delta(1-p)[\theta_L(1-\delta p)-\theta_H\delta(1-p)]}{(1-\delta p)(1+\delta-2\delta p)[\theta_L(1-\delta)-\theta_H\delta(1-p)]}\frac{d\gamma^H(g)}{dg}.$$

Consider $A(g)$ first. The coefficient on $1/g$ can be either positive or negative.

Suppose first the coefficient on $1/g$ is positive. Then $A(g)$ is strictly decreasing in g and is minimized at $g = \tilde{g}_H$. By Corollary 2, $\tilde{g}_H < [\theta_L(1+\delta-2\delta p)]/[\delta(1-p)]$. Since $A(g) = \theta_H\delta(1-p)/[\theta_L(1-\delta p)(1+\delta-2\delta p)] > 0$ when $g = [\theta_L(1+\delta-2\delta p)]/[\delta(1-p)]$, it follows that $A(g) > 0$ for $g \in [\underline{g}_H, \tilde{g}_H]$ in this case.

Now suppose the coefficient on $1/g$ is negative, then $A(g)$ is strictly increasing in g and is minimized at $g = \underline{g}_H$. We have $\underline{g}_H = \psi \geq \theta_L^*$. When $g = \theta_L^*$, $A(g) = \theta_H[\theta_H\delta(1-p) - \theta_L(1-\delta p)]/[\theta_L(1+\delta-2\delta p)[\theta_H\delta(1-p) - \theta_L(1-\delta)]]$, which is strictly positive in the high-polarization case. It follows that $A(g) > 0$ for $g \in [\underline{g}_H, \tilde{g}_H]$.

Now consider $B(g)$. Since $\gamma^H(g)$ satisfies (59), by the implicit function theorem,

$$\frac{d\gamma^H(g)}{dg} = \frac{\gamma^H(g)(1-\delta p)[\theta_L(1+\delta-2\delta p)-g\delta(1-p)]}{g[(1+\delta-2\delta p)(\theta_L(1-\delta)-\theta_H\delta(1-p))+\gamma^H(g)\delta^2(1-p)^2]}. \tag{60}$$

At $\gamma^H(g) = g_H^*$ the denominator of $d\gamma^H(g)/dg$ is negative. Since the denominator is increasing in $\gamma^H(g)$ and $\gamma^H(g) \leq g_H^*$, the denominator is negative. Since $g \leq \tilde{g}_H < [\theta_L(1+\delta-2\delta p)]/[\delta(1-p)]$, the numerator is positive, and therefore $d\gamma^H(g)/dg < 0$. Since this is the high-polarization case and $d\gamma^H(g)/dg < 0$, it follows that $B(g) > 0$.

To summarize, $K'_H(g) = A(g) + B(g) > 0$ for $g \in [\underline{g}_H, \tilde{g}_H]$.

- $g \in [\tilde{g}_H, \theta_H + \theta_L]$: Then $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H(g)$. Substituting for $V_H(g)$ and taking the derivative, we get

$$K'_H(g) = \frac{(1-\delta)\theta_H - \delta(1-p)\theta_L}{(1-\delta p)(1-\delta)g} + \frac{\delta(1-p)}{1-\delta p}B_H^1. \tag{61}$$

  If $(1-\delta)\theta_H - \delta(1-p)\theta_L > 0$, then, $K'_H(g) > 0$ since $B_H^1 > 0$.

  If $(1-\delta)\theta_H - \delta(1-p)\theta_L < 0$, then $K'_H(g)$ is increasing in g. We have $\tilde{g}_H > \underline{g}_H = \psi \geq \theta_L^*$. Plugging $g = \theta_L^*$ in (61), we get $K'_H(g) = \frac{\theta_H(1-\delta p)-\theta_L\delta(1-p)}{(1-\delta p)(1+\delta-2\delta p)\theta_L} > 0$, and therefore $K_H(g)$ is strictly increasing for $g \in [\tilde{g}_H, \theta_H + \theta_L]$.

- $g \in [\theta_H + \theta_L, g_H^*]$: Then $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H(g)$. Substituting for $V_H(g)$ and taking the derivative, we get

$$K'_H(g) = \frac{\theta_H}{(1-\delta)g} - \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)},$$

which is strictly higher than 0 for $g \leq g^*_H$.

- $g > g^*_H$: Then $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V^*_H$, which is clearly strictly increasing in g.

Hence, $K_H(g)$ is strictly increasing. ∎

We next show that the proposer has no profitable one-shot deviation. We omit the proof for party L since it is similar to that in the proof of Proposition 3.

Recall that $U^P_H(z)$ denotes party H's payoff when the implemented budget is z in the current period and party H is the proposer. We next establish monotonicity properties of $U^P_H(z)$, which is useful for later part of the proof.

For any status quo g, consider proposals $z' = (g', x'_H, x'_L)$ such that the responder's acceptance constraint (7) is binding. That is,

$$x'_L = K_L(g) - \theta_L \ln(g') - \delta[(1-p)V_L(g') + pW_L(g')] = K_L(g) - K_L(g'). \tag{62}$$

Substituting in the proposer's payoff function, we get

$$U^P_H(z') = 1 - g' - x'_L + \theta_H \ln(g') + \delta[pV_H(g') + (1-p)W_H(g')] \Rightarrow \tag{63}$$

$$\frac{\partial U^P_H}{\partial g'} = -1 + \frac{\theta_H+\theta_L}{g'} + \delta[(1-p)V'_L(g') + pW'_L(g')] + \delta[pV'_H(g') + (1-p)W'_H(g')] \tag{64}$$

Substituting for V'_L, W'_L, V'_H, W'_H, we get closed-form solution for $\frac{\partial U^P_H}{\partial g'}$ except when $g \in (\underline{g}_H, \tilde{g}_H)$. Specifically, if $g' < \underline{g}^*_L$, then $\frac{\partial U^P_H}{\partial g'} = \frac{\theta_H+\theta_L}{g'} - 1 > 0$; if $g' \in (\underline{g}^*_L, \underline{g}_H)$, then $\frac{\partial U^P_H}{\partial g'} = \frac{1+\delta-2\delta p}{1-\delta p}(\frac{\theta_H+\theta_L}{g'} - 1) > 0$; if $g' \in (\tilde{g}_H, \theta_H + \theta_L)$, then $\frac{\partial U^P_H}{\partial g'} = \frac{1+\delta-2\delta p}{1-\delta p}(\frac{\theta_H+\theta_L}{g'} - 1) > 0$; if $g' \in (\theta_H + \theta_L, g^*_H)$, then $\frac{\partial U^P_H}{\partial g'} = \frac{1}{1-\delta p}(\frac{\theta_H+\theta_L}{g'} - 1) < 0$; if $g' > g^*_H$, then $\frac{\partial U^P_H}{\partial g'} = \frac{\theta_H+\theta_L}{g'} - 1 < 0$.

Note that $\frac{\partial U^P_H}{\partial g'} = f'_H(g') + K'_L(g')$. Also, if $g' \in (\underline{g}_H, \tilde{g}_H)$, then $\frac{d\gamma^H(g')}{dg'} = \frac{K'_L(g')}{K'_L(\gamma^H(g'))}$. Hence, for $g' \in (\underline{g}_H, \tilde{g}_H)$,

$$\frac{\partial U^P_H}{\partial g'} = -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g'} + K'_L(g')C(g') \tag{65}$$

where

$$C(g') = 1 + \frac{\delta(p+\delta-2\delta p)[-(1-\delta p)\gamma^H(g')+(1+\delta-2\delta p)\theta_H]}{[(1-\delta)\theta_L - \delta(1-p)\theta_H](1+\delta-2\delta p)+\gamma^H(g')\delta^2(1-p)^2} \tag{66}$$

Straightforward calculation shows that $C(g') > 0$ in the high-polarization case where $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$. Since $K'_L(g') > 0$ for $g' < \tilde{g}$ by Claim 5 and Corollary 2, it follows that $\frac{\partial U^P_H}{\partial g'} > 0$ for $g' \in (\underline{g}_H, \tilde{g}_H)$.

To show that proposer H has no profitable one-shot deviation, consider the following cases.

- $g \leq \underline{g}_H$ or $g \geq g^*_H$: In this case, $\gamma^H(g) = g^*_H$ and $\chi^H_L(g) = 0$.

  Since $g^*_H \in \arg\max f_H(g)$, party H has no incentive to deviate from proposing $\gamma^H(g) = g^*_H$ and $\chi^H_L(g) = 0$.

- $\underline{g}_H \leq g \leq \tilde{g}_H$: In this case, $\gamma^H(g) \in [\theta_H + \theta_L, g_H^*]$ and $\chi_L^H(g) = 0$.

  We first show that proposing $\pi^H(g)$ is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} > \gamma^H(g)$ and then show that it is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} < \gamma^H(g)$.

  - $\hat{g} > \gamma^H(g)$: Since $\gamma^H(g) > \theta_H + \theta_L > \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$, by Claim 5, for $\hat{g} > \gamma^H(g)$, $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Since party L's payoff is strictly decreasing in x_L, we only need to consider proposals such that the responder's acceptance constraint (7) is binding.

    Since $U_H^P(\hat{z})$ is decreasing in $\hat{g}$ for $\hat{g} > \gamma^H(g) \geq \theta_H + \theta_L$ as shown before, the proposer has no incentive to make any proposal with $\hat{g} > \gamma^H(g)$.

  - $\tilde{g}_H \leq \hat{g} < \gamma^H(g)$: Consider $\hat{z} = (\hat{g}, 1 - \hat{g}, 0)$. Then $U_H^P(\hat{z}) = f_H(\hat{g})$. As shown in the proof of Claim 4, $f_H(\hat{g})$ is increasing in $\hat{g}$ for $\tilde{g}_H < \hat{g} < g_H^*$. Since $\pi^H(g) = (\gamma^H(g), 1 - \gamma^H(g), 0)$ where $\gamma^H(g) < g_H^*$, it follows that $U_H^P(\pi^H(g)) > U_H^P(\hat{z})$ for any $\hat{g} < \gamma^H(g) \leq g_H^*$. Since party H's payoff is decreasing in x_L, $U_H^P(\hat{z}) \geq U_H^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in B$, it follows that $U_H^P(\pi^H(g)) > U_H^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $\hat{g} < \gamma^H(g) \leq g_H^*$. Hence the proposer has no incentive to deviate and make a proposal with $\tilde{g}_H \leq \hat{g} < \gamma^H(g)$.

  - $g \leq \hat{g} \leq \tilde{g}_H$. Consider $\hat{z} = (\hat{g}, 1 - \hat{g}, 0)$. Then $U_H^P(\hat{z}) = f_H(\hat{g})$. Recall that for $g \geq g_L^*$, $f_H(g) = 1 - g + \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}\ln(g) + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}V_H(g)$. Also, for $\underline{g}_H \leq \hat{g} \leq \tilde{g}$, $V_H(\hat{g}) = V_H(\gamma^H(\hat{g}))$. Hence, $f_H(\gamma^H(\hat{g})) - f_H(\hat{g}) = -\gamma^H(\hat{g}) + \hat{g} + \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}(\ln(\gamma^H(\hat{g})) - \ln(\hat{g}) > 0$ since $\hat{g} \leq \gamma^H(\hat{g}) \leq \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}$. Since $\gamma^H(\hat{g}) < \gamma^H(g)$ and $f_H(g)$ is increasing in $(\theta_H + \theta_L, g_H^*)$ as shown in the proof of Claim 4, it follows that $f_H(\hat{g}) \leq f_H(\gamma^H(\hat{g})) \leq f_H(\gamma^H(g))$ and therefore $U_H^P(\pi^H(g)) \geq U_H^P(\hat{z})$ for any $\hat{g} \in [g, \tilde{g}_H]$. Hence proposing $\pi^H(g)$ is better than proposing any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in B$ with $g \leq \hat{g} \leq \tilde{g}_H$.

  - $\hat{g} < g$: By Corollary 2, $g < \tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$. Hence, for $\hat{g} < g$, $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$ by Claim 5.

    Consider $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that (62) holds. Since $U_H^P(\hat{z})$ is increasing in $\hat{g}$ for $\hat{g} < g$ as shown before, the proposer has no incentive to deviate and make a proposal with $\hat{g} < g$.

- $\tilde{g}_H \leq g \leq \theta_H + \theta_L$: In this case, $\gamma^H(g) = \theta_H + \theta_L$ and $\chi_L^H(g) \geq 0$.

  Let $h(g) = \max\{g' \in [0,1] : K_L(g') = K_L(g)\}$ and $l(g) = \min\{g' \in [0,1] : K_L(g') = K_L(g)\}$. By Claim 5, $h(g) \in [\frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L, \theta_H + \theta_L]$ and $l(g) \in [\tilde{g}_H, \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L]$.

  - $\hat{g} \geq h(g)$: For $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim 5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{z}$ such that (62) holds. As shown before, $U_H^P(\hat{g})$ is increasing for

$\hat{g} \in [h(g), \theta_H + \theta_L)$ and decreasing for $\hat{g} > \theta_H + \theta_L$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \geq h(g)$ and $\hat{g} \neq \theta_H + \theta_L$.

- $\hat{g} \in [l(g), h(g)]$: Consider $\hat{z} = (\hat{g}, 1 - \hat{g}, 0)$. Since $U_H^P(\hat{z}) = f_H(\hat{g})$ and $f_H(\hat{z})$ is increasing for $\hat{g} \in [l(g), h(g)]$, it follows that $U_H^P((h(g), 1 - h(g), 0)) > U_H^P(\hat{z})$ for any $\hat{g} \in (l(g), h(g)$ and therefore the proposer has no incentive to deviate and make a proposal with $\hat{g} \in [l(g), h(g)]$.
- $\hat{g} < l(g)$: For $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim 5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{z}$ such that (62) holds. As shown before, $U_H^P(\hat{g})$ is increasing for $\hat{g} < l(g)$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \geq l(g)$.

- $g \in [\theta_H + \theta_L, g_H^*]$: In this case, $\gamma^H(g) = g$ and $\chi_L^H(g) = 0$. Recall that $l(g) = \min\{g' \in [0, 1] : K_L(g') = K_L(g)\}$. In this case, $l(g) \in [\underline{g}_H, \tilde{g}_H]$.

  - $\hat{g} > g$: For $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim 5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{z}$ such that (62) holds. As shown before, $U_H^P(\hat{g})$ is decreasing for $\hat{g} > \theta_H + \theta_L$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \geq g$.
  - $\tilde{g}_H \leq \hat{g} < g$: Consider $\hat{z} = (\hat{g}, 1 - \hat{g}, 0)$. Since $U_H^P(\hat{z}) = f_H(\hat{g})$ and $f_H(\hat{g})$ is increasing if $\tilde{g}_H \leq \hat{g} < g$, it follows that the proposer has no incentive to deviate and make a proposal with $\hat{g} \in [\tilde{g}_H, g)$.
  - $l(g) \leq \hat{g} \leq \tilde{g}_H$. Consider $\hat{z} = (\hat{g}, 1 - \hat{g}, 0)$. Note that for $\hat{g} \in [l(g), \tilde{g}_H]$, $f_H(\hat{g}) < f_H(\gamma^H(\hat{g}))$. Also, since $\gamma^H(\hat{g}) < g$ and therefore $f_H(\gamma^H(\hat{g}) < f_H(g)$, it follows that $f_H(\hat{g}) < f_H(g)$. Hence the proposer has no incentive to deviate and make a proposal with $\hat{g} \in [l(g), \tilde{g}_H]$.
  - $\hat{g} \leq l(g)$: For $\hat{z} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim 5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{z}$ such that (62) holds. As shown before, $U_H^P(\hat{g})$ is increasing for $\hat{g} \leq l(g)$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \leq l(g)$.

To summarize, party H has no incentive to deviate from $\pi^H(g)$ for any $g \in [0, 1]$. ∎

10.6 Proof of Proposition 6

Fix $g \in G^s$. First we show that $g \in G$, that is, the responder's acceptance constraint binds when the status quo is in G^s. This follows immediately from the following claim:

Claim 8. *For any $g \in G^s$ and $i, j \in \{H, L\}$ with $i \neq j$, $\chi_j^i(g) = 0$.*

Proof: Fix $g \in G^s$. By definition of G^s, $\gamma^i(g) = g$. Suppose to the contrary that $\chi^i_j(g) > 0$ for $j \neq i$. Let $\tilde{\pi}^i = (\tilde{\gamma}^i, \tilde{\chi}^i_H, \tilde{\chi}^i_L)$ be an alternative proposal strategy for player i such that $\tilde{\pi}^i(g') = \pi^i(g')$ for $g' \neq g$, $\tilde{\gamma}^i(g) = \gamma^i(g)$, $\tilde{\chi}^i_j(g) = 0$ and $\tilde{\chi}^i_i(g) = \chi^i_i(g) + \chi^i_j(g) > \chi^i_i(g)$. Note that $\tilde{\pi}^i$ satisfies the responder's acceptance constraint (7) when i is the proposer. Then $\tilde{\pi}^i$ yields the same payoff to player i for any $g' \neq g$, and strictly higher payoff when the status quo is g, contradicting that π^i is an equilibrium proposal strategy. ∎

Since $g \in G$, we can simplify the proposer i's maximization problem by using Lemma 1 to substitute for W_i and W_j. Define the function $h_i : B \to \mathbb{R}$ as

$$h_i(g, x_H, x_L) = x_i + \tfrac{\theta_i}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}V_i(g).$$

Claim 9. *For any $g \in G^s$ and $i \in \{H, L\}$,*

$$\begin{aligned} V_i(g) \quad = \quad & \max_{z=(g',x'_H,x'_L)\in B} x'_i + \tfrac{1-2\delta p+\delta}{1-\delta p}\theta_i \ln(g') + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p}V_i(g') \\ & s.t. \;\; h_j(z) \geq K_j(g), g' \in G \end{aligned} \tag{67}$$

where $K_j(g) = \frac{\theta_i}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_i(g)$.

Proof: By definition of G^s, the proposal $(g, \chi^i_H(g), \chi^i_L(g))$ is a solution to the maximization problem given in (6) and (7). By Claim 8, $G^s \subseteq G$, and so the proposal $(g, \chi^i_H(g), \chi^i_L(g))$ is also a solution to (6) and (7) when the maximization is over $z = (g', x'_H, x'_L) \in B$ with $g' \in G$. Since the acceptance constraint binds for any $g \in G$, we use Lemma 1 to substitute for W_i and W_j, resulting in the maximization problem given in Claim 9. ∎

We are now ready to prove Proposition 6. Suppose h_H and h_L satisfy Kuhn-Tucker Constraint Qualification. The Lagrangian for party i's problem, for $i \in \{H, L\}$, is

$$\begin{aligned} L_i \quad = \quad & x'_i + \tfrac{1-2\delta p+\delta}{1-\delta p}\theta_i \ln(g') + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p}V_i(g') \\ & +\lambda_{1i}[1 - x'_i - x'_j - g'] + \lambda_{2i}[x'_j + \tfrac{\theta_j}{1-\delta p}\ln(g') + \tfrac{\delta(1-p)}{1-\delta p}V_j(g') - K_j(g)] \end{aligned}$$

where $j \in \{H, L\}$, $j \neq i$.

By the Kuhn-Tucker Theorem (see Takayama (1985), Theorem 1.D.3), the first order necessary conditions for (g', x'_H, x'_L) to be a a solution to (67) are $\lambda_{1i} \geq 0$, $\lambda_{2i} \geq 0$, $g' \geq 0$, $x'_H \geq 0$, $x'_L \geq 0$, and

$$1 - \lambda_{1i} \leq 0, \qquad [1 - \lambda_{1i}]x'_i = 0, \tag{68}$$

$$-\lambda_{1i} + \lambda_{2i} \leq 0, \qquad [-\lambda_{1i} + \lambda_{2i}]x'_j = 0, \tag{69}$$

$$\tfrac{\theta_i(1-2\delta p+\delta)}{g'(1-\delta p)} + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p}\tfrac{\partial V_i}{\partial g'} - \lambda_{1i} + \lambda_{2i}\left[\tfrac{\theta_j}{g'(1-\delta p)} + \tfrac{\delta(1-p)}{1-\delta p}\tfrac{\partial V_j}{\partial g'}\right] \;\; \leq \;\; 0, \tag{70}$$

$$\left[\tfrac{\theta_i(1-2\delta p+\delta)}{g'(1-\delta p)} + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p}\tfrac{\partial V_i}{\partial g'} - \lambda_{1i} + \lambda_{2i}\left[\tfrac{\theta_j}{g'(1-\delta p)} + \tfrac{\delta(1-p)}{1-\delta p}\tfrac{\partial V_j}{\partial g'}\right]\right] g' \;\; = \;\; 0, \tag{71}$$

$$1 - x'_i - x'_j - g' \geq 0 \qquad [1 - x'_i - x'_j - g']\lambda_{1i} = 0, \tag{72}$$

$$x'_j + \frac{\theta_j}{1-\delta p}\ln(g') + \frac{\delta(1-p)}{1-\delta p}V_j(g') - K_j(g) \geq 0, \tag{73}$$

$$\left[x'_j + \frac{\theta_j}{1-\delta p}\ln(g') + \frac{\delta(1-p)}{1-\delta p}V_j(g') - K_j(g)\right]\lambda_{2i} = 0. \tag{74}$$

By Claim 8, $x'_j = 0$. By (68) $\lambda_{1i} > 0$, and so the feasibility constraint (72) holds with equality. By the envelope theorem (see Takayama (1985), Theorem 1.F.1), for $i \in \{H, L\}$, we have

$$\frac{\partial V_i}{\partial g} = -\lambda_{i2}\frac{\partial K_j}{\partial g} = -\lambda_{i2}\left[\frac{\theta_j}{g(1-\delta p)} + \frac{\delta(1-p)}{1-\delta p}\frac{\partial V_j}{\partial g}\right]. \tag{75}$$

Since this holds for $i \in \{H, L\}$, we have a system of two equations in two unknowns. Solving gives

$$\frac{\partial V_i}{\partial g} = \frac{\lambda_{2i}[\lambda_{2j}\theta_i\delta(1-p) - \theta_j(1-\delta p)]}{g[(1-\delta p)^2 - \lambda_{2i}\lambda_{2j}\delta^2(1-p)^2]}, \tag{76}$$

for $i, j \in \{H, L\}$ with $j \neq i$.

Since V_H and V_L are differentiable in an open set containing g, it must be the case that $g \in (0, 1)$. Since $g \in G^s$, this in turn implies that $g' = g \in (0, 1)$. From $g' > 0$, it follows that (70) must hold with equality for $i, j \in \{H, L\}$ and $j \neq i$. From $g' < 1$, it follows that $x'_i > 0$, and hence $\lambda_{1i} = 1$ for $i \in \{H, L\}$. Substituting λ_{1i} and (76) into (70), and solving the two equations (given by (70) for $i \in \{H, L\}$) for g' and λ_{2H} in terms of λ_{2L}, we obtain

$$g' = \frac{(\lambda_{2L}\theta_H + \theta_L)(1+\delta-2\delta p)}{1-\delta p+\lambda_{2L}\delta(1-p)}, \tag{77}$$

and

$$\lambda_{2H} = \frac{(\theta_H-\theta_L)(1-\delta p) - \lambda_{2L}\theta_H(1-\delta)}{\lambda_{2L}\delta(\theta_H-\theta_L)(1-p) - \theta_L(1-\delta)}. \tag{78}$$

Consider the low-polarization case in which $\frac{\theta_H}{\theta_L} \leq \frac{1-\delta p}{\delta(1-p)}$. Note that $\delta(\theta_H - \theta_L)(1-p) - \theta_L(1-\delta) \leq 0$. Since $\lambda_{2L} \leq 1$ by (69), it follows that the denominator of (78) is nonpositive. Together with the necessary condition that $\lambda_{2H} \geq 0$, this implies

$$\lambda_{2L} \geq \frac{(\theta_H-\theta_L)(1-\delta p)}{\theta_H(1-\delta)}.$$

Thus, if $\frac{\theta_H}{\theta_L} \leq \frac{1-\delta p}{\delta(1-p)}$, we have $\lambda_{2L} \in [\frac{(\theta_H-\theta_L)(1-\delta p)}{\theta_H(1-\delta)}, 1]$. Since the right-hand side of (77) is increasing in λ_{2L}, the bounds on λ_{2L} we just found implies that $g = g' \in [\theta^*_H, \theta_H + \theta_L]$.

Next consider the high-polarization case in which $\frac{\theta_H}{\theta_L} \geq \frac{1-\delta p}{\delta(1-p)}$. Note that $\frac{(\theta_H-\theta_L)(1-\delta p)}{\theta_H(1-\delta)} \geq 1$. Since $\lambda_{2H} \geq 0$, the numerator and the denominator of (78) have the same sign. If they are both nonpositive, then

$$\frac{(\theta_H-\theta_L)(1-\delta p)}{\theta_H(1-\delta)} \leq \lambda_{2L}.$$

Since $\lambda_{2L} \leq 1$ by (69), this is only possible when $\lambda_{2L} = 1$. If instead both the numerator and the denominator of (78) are nonnegative, then $\lambda_{2H} \leq 1$ implies that

$$(\theta_H - \theta_L)(1-\delta p) - \lambda_{2L}\theta_H(1-\delta) \leq \lambda_{2L}\delta(\theta_H - \theta_L)(1-p) - \theta_L(1-\delta).$$

Since $\theta_H \geq \theta_L$, $\delta < 1$ and $\lambda_{2L} \leq 1$, this is only possible if $\lambda_{2L} = 1$. Thus, in the high-polarization case, $\lambda_{2L} = 1$. Substituting in (77), we obtain $g' = g = \theta_H + \theta_L$. ∎

10.7 Proof of Proposition 7

The derivative of θ_H^* with respect to p is

$$\frac{\partial\theta_H^*}{\partial p} = -\frac{\theta_H\delta(1-\delta)}{(1-\delta p)^2} \leq 0.$$

The derivative of θ_H^* with respect to δ is

$$\frac{\partial\theta_H^*}{\partial\delta} = \frac{\theta_H\delta(1-p)}{(1-\delta p)^2} \geq 0.$$

∎

10.8 Proof of Proposition 8

If public good spending is discretionary, then party i's expected steady state payoff is

$$\frac{1}{2(1-\delta)}[(1-\theta_i)+\theta_i\ln(\theta_i)] + \frac{1}{2(1-\delta)}[\theta_i\ln(\theta_j)] \tag{79}$$

If public good spending is mandatory, then party i's expected steady state payoff is

$$\frac{1}{2(1-\delta)}[(1-g^s)+\theta_i\ln(g^s)] + \frac{1}{2(1-\delta)}[\theta_i\ln(g^s)] \tag{80}$$

where $g^s \in [g_H^*, \theta_H+\theta_L]$.

To show that party i is better off when public spending is mandatory, we only need to show that (80) is higher than (79). After rearranging terms, it becomes

$$2\theta_i\ln(g^s) - g^s \geq \theta_i\ln(\theta_i\theta_j) - \theta_i \tag{81}$$

Consider first $i = H$. Let $k(x) = 2\theta_H\ln(x) - x$. Since $k'(x) = \frac{2\theta_H}{x} - 1 > 0$ if $x < 2\theta_H$, and $g^s \in \max\{\theta_H^*, \theta_H+\theta_L\} < 2\theta_H$ by Proposition 5, it follows that $k(g^s) > k(\theta_H)$. That is, $2\theta_H\ln(g^s) - g^s > 2\theta_H\ln(\theta_H) - \theta_H$. Since $\ln(\theta_H)^2 > \ln(\theta_L\theta_H)$, it follows that $2\theta_H\ln(g^s) - g^s > \theta_H\ln(\theta_L\theta_H) - \theta_H$.

Next consider $i = L$ in the low-polarization case. Since the left-hand side of inequality (81) is concave in g^s, it follows that (81) holds for any $g^s \in [\theta_H^*, \theta_H+\theta_L]$ if it holds for $g^s = \theta_H^*$ and for $g^s = \theta_H + \theta_L$.

If $g^s = \theta_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$, then

$$2\theta_L\ln(g^s) - g^s - \theta_L\ln(\theta_H\theta_L) + \theta_L = 2\theta_L\ln(\theta_H^*) - \theta_H^* - \theta_L\ln(\theta_H\theta_L) + \theta_L,$$

which is increasing in θ_L. Let $\theta_L = \frac{\delta(1-p)}{1-\delta p}\theta_H$. Then

$$2\theta_L\ln(\theta_H^*) - \theta_H^* - \theta_L\ln(\theta_H\theta_L) + \theta_L = \ln\left(\frac{(1+\delta-2\delta p)^2}{\delta(1-p)(1-\delta p)}\right)\frac{\delta(1-p)}{1-\delta p}\theta_H - \theta_H,$$

and it is positive if $\ln\left(\frac{(1+\delta-2\delta p)^2}{\delta(1-p)(1-\delta p)}\right) \geq \frac{1-\delta p}{\delta(1-p)}$.

Similarly, if $g^s = \theta_H + \theta_L$, then

$$\begin{aligned} 2\theta_L\ln(g^s) - g^s - \theta_L\ln(\theta_H\theta_L) + \theta_L &= 2\theta_L\ln(\theta_H+\theta_L) - \theta_L - \theta_H - \theta_L\ln(\theta_H\theta_L) + \theta_L \\ &= 2\theta_L\ln(\theta_H+\theta_L) - \theta_H - \theta_L\ln(\theta_H\theta_L), \end{aligned}$$

which is increasing in θ_L. Let $\theta_L = \frac{\delta(1-p)}{1-\delta p}\theta_H$. Then

$$2\theta_L \ln(\theta_H + \theta_L) - \theta_H - \theta_L \ln(\theta_H \theta_L) = \ln\left(\frac{(1+\delta-2\delta p)^2}{\delta(1-p)(1-\delta p)}\right) \frac{\delta(1-p)}{1-\delta p}\theta_H - \theta_H,$$

and it is positive if $\ln(\frac{(1+\delta-2\delta p)^2}{\delta(1-p)(1-\delta p)}) \geq \frac{1-\delta p}{\delta(1-p)}$. To summarize, inequality (81) holds for $i = L$ if $\ln(\frac{(1+\delta-2\delta p)^2}{\delta(1-p)(1-\delta p)}) > \frac{1-\delta p}{\delta(1-p)}$. ∎

10.9 Illustration of parties' proposal strategies for transfers

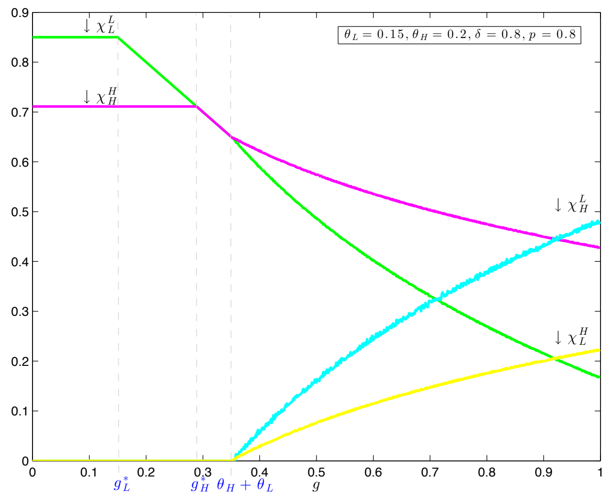


Figure 8: $\chi_j^i(g)$ in low-polarization case

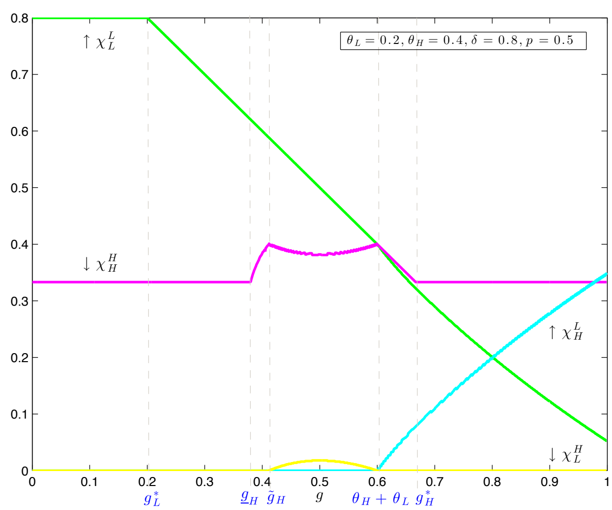


Figure 9: $\chi_j^i(g)$ in high-polarization case

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