Feedback Effects with Risk Aversion: An Application to Climate Risk*

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Abstract

We study the implications of the "feedback effect" when a firm's investment decision affects its exposure to a systematic risk factor. As a leading example, we consider a manager's decision to invest in a "green" project based on her firm's stock price. The price reflects both investors' information about project cash flows and its discount rate, which depends on investors' exposure to climate risk. The interaction between the firm's investment decision and its usefulness as a hedge yields novel predictions about the likelihood of investment, expected returns, and future profitability. Moreover, while feedback makes the investment decision more informationally efficient, it can reduce investor welfare.

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1 Introduction

Financial markets serve two central roles in the economy. First, markets aggregate and transmit investor information about future investment opportunities. The literature on "feedback effects" underscores the importance of this role, demonstrating that firms' stock prices convey information to managers regarding their investments' expected cash flows.[1] Second, markets enable investors to hedge and share risks. A firm's stock price reflects both investors' preferences towards these risks and the firm's exposure to them. We show that the interaction of these two roles has important consequences for firms' investment decisions and for investor welfare. Importantly, a firm's investment choices, which depend on the information in its stock price, endogenously change its risk exposures. In turn, this change drives investor demand for the stock for hedging, which affects both welfare and the stock price itself.

This interaction is particularly important for understanding firms' decisions to invest in projects that are exposed to climate risk. Investors are broadly and heterogeneously exposed to climate risk and try to mitigate and hedge their exposures, but often find this difficult due to a lack of suitable hedging instruments. For instance, Krueger, Sautner, and Starks (2020) report that 25% of the institutional investors they surveyed actively hedge climate risks, while 26% use valuation models that incorporate climate risks, 34% engage in climate-oriented portfolio diversification and 38% analyze the carbon footprint of portfolio firms. Consistent with this demand for managing climate risk, a growing empirical literature shows that firms' climate exposures influence their valuations and, in particular, their discount rates.[2] Moreover, consistent with feedback from prices, empirical evidence shows firms' investments in "green" projects respond to changes in their stock prices, even when driven by shocks to investor demand for green exposure rather than cash flow news (e.g., Li, Shan, Tang, and Yao (2020), Bai, Chu, Shen, and Wan (2021), Briere and Ramelli (2021)).

We develop a model of feedback effects in which a firm's manager decides whether to invest in a climate-exposed project. The firm's stock is traded by risk-averse investors who are informed about the project's expected cash flows and have different exposures to climate risk. The manager is uninformed about investors' cash flow information and their aggregate exposure to climate risk, which captures the market's concern about adverse climate outcomes and drives aggregate demand for hedging. The manager's objective is to maximize the

[1] See Bond, Edmans, and Goldstein (2012) for a survey on feedback effects, and Goldstein and Sapra (2014) and Goldstein and Yang (2017) for policy related discussions.

[2] See, e.g., Choi, Gao, and Jiang (2020), Engle, Giglio, Kelly, Lee, and Stroebel (2020), Alekseev, Giglio, Maingi, Selgrad, and Stroebel (2021), Bolton and Kacperczyk (2021a), Bolton and Kacperczyk (2021b), Gillan, Koch, and Starks (2021), Giglio, Kelly, and Stroebel (2021), and Pastor, Stambaugh, and Taylor (2021).

firm's stock price, and so she chooses to invest when the project has a positive net present value (NPV).[3] The project is indexed by its sensitivity to climate risk, which we refer to as its "greenness." A "green" ("brown") project pays higher (lower) cash flows when climate outcomes are worse, while a "neutral" project's cash flows are uncorrelated with climate outcomes.[4]

Our model yields novel predictions about climate-sensitive investment. First, the probability of investment in a project increases in its profitability but can be non-monotonic in its greenness. Importantly, we find that feedback from prices generates stronger incentives for managers to avoid brown projects than to take on green projects, when both types of projects are ex-ante profitable. Second, consistent with intuition, investment in greener projects is associated with higher valuations and, consequently, lower expected returns. However, this relation is weaker for brown than green investment. Third, given investment, the firm's expected return increases with the project's ex-ante profitability and decreases with managers' uncertainty about investors' climate exposures. Finally, investment in brown projects is associated with a greater increase in the firm's expected cash flows than investment in green projects.

Our analysis also has implications for investor welfare. While feedback necessarily increases the firm's market value, we show that it can decrease investor welfare when the project has a positive (ex-ante) expected NPV and investors' per-capita ownership of the firm is sufficiently small, or their exposures to climate risk shocks are sufficiently diverse. This may appear surprising given that the manager maximizes the stock price, which reflects investors' preferences towards both the project's cash flows and its risk exposures. We show that the price does not appropriately reflect the full impact of investment choices on welfare, and specifically does not account for the fact that investment in a climate-exposed project endogenously changes investors' ability to hedge climate risks. As such, our analysis suggests that price incentives can be insufficient to motivate managers to optimally invest in green projects, so that providing climate-based incentives can help improve investor welfare.

To establish these results, we first show that the stock price depends on two types of information in equilibrium. First, the price reflects investors' information about the project's future cash flows, or profitability: we refer to this as "cash flow" news. Second, the stock price reflects the discount or premium due to the project's exposure to climate risk, which

[3]Existing models of the feedback effect assume that the manager maximizes expected cash flows. This is often equivalent to maximizing expected price in these settings because the price is determined by risk-neutral traders or a risk-neutral market maker. In our model, the price also reflects a risk adjustment, and so the two objectives are different. As we discuss in Sections 2 and 3.1, we focus on price-maximization because it is empirically and theoretically important.

[4]As we discuss further in Section 3.1, this interpretation is consistent with the labels in the empirical literature (e.g., Engle et al. (2020), Bolton and Kacperczyk (2021a)).

we denote as “discount rate” news.[5] This risk adjustment is determined by the product of the project’s greenness and investors’ aggregate exposure to climate risk, which correspond to the project’s climate risk factor loading and the climate risk premium, respectively. The manager’s investment decision follows a threshold strategy: she only invests when the stock price is sufficiently high. As a result, the manager invests only if the price conveys sufficiently good “cash flow” news (i.e., investors have positive information about future cash flows) or sufficiently good “discount rate” news (i.e., the price discount is sufficiently low).

An immediate consequence is that the probability of investment rises with an increase in the project’s profitability because this makes good cash flow news more likely. The project’s greenness has a more subtle impact on investment. First, because greener projects have higher valuations on average, the manager is more likely to invest in them all else equal. Second, greater absolute exposure to climate shocks (i.e., higher “greenness” or “brownness”) also renders the project’s NPV (at the investment stage) more volatile. All else equal, this reduces the likelihood of investment in ex-ante profitable projects, but increases the likelihood of investment in ex-ante unprofitable ones. Together, these effects imply that for ex-ante profitable projects, the manager is considerably less likely to invest in a brown project than a neutral one, but that the manager is only marginally more likely to invest in a green project than a neutral one. On the other hand, for ex-ante unprofitable projects, the manager is much more likely to invest in a green than a neutral project, but may be almost as likely to invest in a brown as a neutral project.

Because green projects hedge adverse climate outcomes, they tend to have higher valuations and thus lower expected returns. In contrast, because brown projects perform poorly when climate outcomes are worse, they have lower valuations on average. Therefore, if a manager chooses to invest in a brown project, it must be because either: (i) the cash flow news that she learns from the price is sufficiently positive to overcome the discount due to its climate exposure, or (ii) its discount rate is unexpectedly low relative to the average brown project. This implies that, conditional on investment, brown projects tend to have higher future cash flows (or profitability) than green projects. Moreover, a brown project that the manager chooses to invest in is likely valued at a premium relative to the average brown project. As a result, the negative relation between expected returns and “greenness” is less prominent for brown firms than for green firms.

Having explored our model’s implications for market outcomes, we next characterize how the firm’s investment in climate-exposed projects impacts investor welfare. First, investment

[5] As in other models with CARA investors and (conditionally) normal payoffs, the risk premium (discount) is an additive adjustment and so, technically, not a discount *rate*. However, we use this terminology for expositional clarity and intuition. This terminology is consistent with, e.g., Avdis (2016) and Bond and Garcia (2021).

affects welfare through the market value of the firm's shares that investors hold – we label this the "valuation" channel. Second, and novel to our analysis, both brown and green investment help to complete the market by allowing investors to hedge their differential exposures to climate risk by trading the stock, which tends to raise welfare. We refer to this as the "hedging" channel. Finally, investment alters investors' aggregate exposure to climate shocks, which we refer to as the "climate risk" channel: green projects improve welfare by reducing investors' exposure to such shocks, and vice versa for brown projects.[6]

Recent evidence highlights the potential importance of the hedging channel: investors have different exposures to climate risk due to differences in their demographic characteristics and risk preferences (e.g., Ilhan, Krueger, Sautner, and Starks (2021)), but find this risk difficult to hedge.[7] For instance, Engle et al. (2020) argue that a dynamic equity portfolio optimized to hedge climate risk is at most 30% correlated with news on such risk. Similarly, based on their survey evidence, Krueger et al. (2020) state that "... many market participants, including institutional investors, find climate risks difficult to price and hedge, possibly because of their systematic nature, [...] and challenges in finding suitable hedging instruments." This suggests that the additional hedging opportunities brought on by a firm's climate-exposed investments can meaningfully improve investors' ability to share this risk.

While feedback necessarily increases the firm's market value, we show that it can decrease investor welfare when the project has a positive (ex-ante) expected NPV and investors' risk-sharing needs are sufficiently large, or per-capita ownership of the firm is sufficiently small. This is because the manager's objective, which is to maximize the firm's stock price, appropriately accounts for the valuation channel, but not the other channels. For instance, the climate risk premium in the stock price is driven by the *average* investor's exposure to climate risk, and so ignores the fact that investors have differential exposures to such risk. Thus, it completely ignores the hedging channel. Moreover, price also does not fully account for the climate risk channel, because it does not reflect the heterogeneity in risk exposures across investors.[8]

Intuitively, without feedback, the manager would always invest in a project with a positive

[6]Common to other settings, investment also affects the cash flow risk that investors have to bear, which we refer to as the project risk channel. Moreover, investors' private information renders the project more useful in hedging because it reduces uncertainty about the part of the project cash flow that is not related to the climate. This appears as a separate channel in our model and increases the investment's impact on investor utility.

[7]In Giglio et al. (2021)'s review of the empirical literature on climate finance, they state, "... many of the effects of climate change are sufficiently far in the future that neither financial derivatives nor specialized insurance markets are available to directly hedge those long-horizon risks. Instead, investors are largely forced to insure against realizations of climate risk by building hedging portfolios on their own."

[8]Intuitively, the welfare impact of changes to average climate risk exposure is amplified when investors have differential exposures because this leads to inequality in equilibrium allocations.

expected NPV, but with feedback, she would not invest in such a project if the equilibrium price was sufficiently low. The resulting decrease in investment leads to an increase in welfare via the valuation channel, but a decrease in welfare due to the hedging channel. When investors' exposures to climate risk are sufficiently diverse, so that risk-sharing needs are large, or per-capita ownership of the firm is sufficiently small, the latter effect dominates: in these cases, welfare is higher without feedback than with. In such settings, our analysis suggests that providing additional incentives for managers to invest in green projects (e.g., by linking their compensation to ESG scores) can lead to improvement in investor welfare, even though it may lead to informationally inefficient over-investment, lower valuations, and lower future profitability. This is consistent with both the recent popularity of, and increased skepticism about, the use of climate targets in executive compensation.[9]

The rest of the paper is organized as follows. The next section discusses the related literature and clarifies our contribution. Section 3 presents the benchmark model and discusses key assumptions. Section 4 characterizes the equilibrium in our setting. Section 5 characterizes the model's implications for market outcomes and Section 6 presents our main results regarding feedback and welfare. Section 7 summarizes the empirical predictions from our analysis, and Section 8 concludes. Proofs of our results are in Appendix A, and a general characterization of welfare in our setting is presented in the Internet Appendix.

2 Related Literature

Our paper is related to two strands of literature. First, it adds to the literature on feedback effects (see Bond et al. (2012) and Goldstein and Yang (2017) for recent surveys). Much of this existing literature focuses on economies in which (i) investors are either risk-neutral or the stock price is set by a risk-neutral market maker, (ii) the noise in prices arises due to noise traders with unmodeled utility functions, and (iii) the manager's investment choice maximizes the firm's expected terminal cash flow. As a result, such models are not well suited to study how discount rate variation affects investment decisions or how feedback affects investor welfare. Instead, we consider a setting in which investors are risk-averse, noise in prices arises due to hedging needs, and the manager's chooses investment to maximize the firm's market value (i.e., the risk-adjusted present value, or NPV, of the project).[10]

[9] Hill (2021) cites a survey by Deloitte in September 2021 which suggests that 24% of respondent firms expected to link their long-term executive compensation to climate metrics. Similarly, Edmans (2021) argues that while 51% of large U.S. companies and 45% of leading U.K. firms use ESG metrics in their incentive plans, this may be misguided because of measurement concerns, unintended consequences, and the disconnect between ESG and financial performance.

[10] See Diamond and Verrecchia (1981), Wang (1994), Schneider (2009), Ganguli and Yang (2009), Manzano and Vives (2011), and Bond and Garcia (2021) for models in which noise is driven by hedging needs. Existing

The most closely related papers in this literature are Dow and Rahi (2003), Hapnes (2020), and Gervais and Strobl (2021). Dow and Rahi (2003) explore how increases in informed trading affect investment efficiency and risk sharing in a setting where prices are set by a risk-neutral market maker. They argue that investment efficiency always improves with more informed trading, but risk sharing may either worsen due to the Hirshleifer (1971) effect, or increase when it decreases investors' uncertainty over the component of the asset's payoffs that are unrelated to the component they wish to hedge. Hapnes (2020) characterizes managerial investment behavior and investor information acquisition in a Grossman and Stiglitz (1980)-type model with feedback; however, the analysis does not study the effect of feedback on welfare. Gervais and Strobl (2021) consider the impact of informed, active money management on investment decisions in a setting with feedback. They study how the gross and net performance of the actively managed fund compares with the market portfolio and study how the presence of an informed money manager affects welfare.

We view our analysis as complementary. We focus on how investment in a project affects the risk exposure of a firm's cash flows, which in turn, affects how useful the stock is for hedging. This highlights a novel channel through which feedback affects welfare: intuitively, the firm's investment decision *endogenously* affects "effective" market completeness in our setting.[11] Also, since investors are identically informed in our analysis, the traditional Hirshleifer (1971) effect is turned off, which allows us to clearly distinguish our novel channel from earlier work.

Since the stock price reflects both expected cash flows and a discount due to risk, a cash-flow-maximizing investment rule yields different choices than a market-value-maximizing investment rule. In much of the earlier literature, the two objectives are equivalent since the price is set by risk-neutral participants. Albagli, Hellwig, and Tsyvinski (2021) and Banerjee, Davis, and Gondhi (2021) also distinguish between price maximization and value maximization. Even though the price is set by a risk-neutral market maker in these models, there is a wedge between expected price and expected cash flows because the price depends non-linearly on investors' information about cash flows. In our model, the price is set by market clearing and the difference reflects a risk premium that compensates investors for bearing both cash flow risk and climate risk. This is particularly important for our analysis of investor welfare, and distinguishes our results from the earlier work.

Our focus on analytically characterizing welfare is also complementary to recent work by Bond and Garcia (2021), who study the consequences of index investing. They point out that

feedback models with risk-neutral pricing include Dow, Goldstein, and Guembel (2017), Davis and Gondhi (2019), Goldstein, Schneemeier, and Yang (2020).

[11] This also distinguishes our analysis from Marín and Rahi (1999), Marín and Rahi (2000), and Eckwert and Zilcha (2003), who consider how exogenous differences in market completeness influence investor welfare.

while indexing may reduce aggregate price efficiency, in so doing it improves retail investor welfare due to improvements in risk sharing. Tension between notions of firm profitability and welfare also appears in Goldstein and Yang (2022), who propose a model of commodity financialization. They show that improvements in price informativeness always increase producer profits due to better-informed real investment, but may ultimately harm producer welfare by destroying risk-sharing opportunities, similar to the Hirshleifer (1971) effect.

Similar to our findings, other papers studying discrete investment choice also emphasize the importance of the firm's "default" investment decision in the absence of feedback. For instance, Dow et al. (2017) show that the nature of investors' equilibrium information acquisition decisions hinges on whether the firm defaults to a risky or a riskless project. Davis and Gondhi (2019) show that complementarity in learning by investors depends on both the default investment decision and on the correlation between the investment and assets in place. Goldstein et al. (2020) study information acquisition in a feedback model with multiple sources of uncertainty. They show that investors seek to acquire the same information as management for positive NPV projects, but different information for negative NPV projects. Our analysis complements this earlier work by identifying a novel tension between managerial investment choices and welfare that is driven by how investment affects the ability of investors to use the stock to hedge risk.

Our paper is also related to the growing theoretical literature on ESG investing and climate risk.[12] Our work is most closely related to Pástor et al. (2021) and Goldstein et al. (2021). Pástor et al. (2021) show that green assets have lower costs of capital because investors enjoy holding them and they hedge climate risk. Consequently, sustainable investing can encourage firms to switch to green technology, and so can have a positive social impact. However, while their analysis has implications for investment behavior, their focus on asset-pricing implications differs from ours. Unlike in our setting, the manager does not learn about project cash flows from the price (i.e., there is no informational feedback effect), and the manager's investment choice does not affect the usefulness of the firm's stock as a hedge for climate risk. Hence, our analysis highlights the importance of accounting for managerial learning about both "cash flow" news and "discount rate" news in prices, and of the endogenous consequences of firm investment behavior for risk sharing. Furthermore, while green firms also enjoy a lower cost of capital relative to brown firms in our setting, our focus on investor welfare uncovers that this premium alone is not sufficient to induce a price-maximizing manager to make socially optimal investment decisions, even in the absence of

[12] This includes Heinkel, Kraus, and Zechner (2001), Friedman and Heinle (2016), Chowdhry, Davies, and Waters (2019), Oehmke and Opp (2020), Pedersen, Fitzgibbons, and Pomorski (2021), Pástor, Stambaugh, and Taylor (2021), Goldstein, Kopytov, Shen, and Xiang (2021).

an explicit preference for "social impact."

Goldstein et al. (2021) develop a noisy rational expectations model with traditional and green investors and noise traders. Traditional and green investors are informed about different components of a firm's payoff and have different preferences over them. This can lead to multiplicity in equilibria: in one equilibrium, the price is driven by information about monetary information, while in the other, the price is driven by non-monetary (climate impact) information. Our analysis complements this work. In our setting, investors also care about two aspects of a project – the cash flows it generates and the exposure to climate risk. Moreover, exposure to climate risk endogenously generates "noise" in prices, which affects the managers' inference about projects cash flows. Unlike Goldstein et al. (2021), however, because this noise arises from investor hedging demands, we are able to characterize investor welfare, and study how it depends on the managers' investment decisions.

3 Model

Payoffs. There are three dates $t \in \{1, 2, 3, 4\}$ and two securities. The risk-free security is normalized to the numeraire. The risky security is a claim to the terminal cash flows V generated by the firm at date four, and trades on dates one and three at prices P_1 and P_3, respectively.

Investors. There is a continuum of investors, indexed by $i \in [0, 1]$, with CARA utility over terminal wealth with risk aversion γ. Investor i has initial endowment of n shares of the risky asset and, $z_i = Z + \zeta_i$ units of exposure to a non-tradeable source of income that has payoff of $-\eta_C$, where $Z \sim N\left(\mu_Z, \tau_Z^{-1}\right)$, $\zeta_i \sim N\left(0, \tau_\zeta^{-1}\right)$ and $\eta_C \sim N\left(0, \tau_\eta^{-1}\right)$ are independent of each other and all other random variables.[13] Investor i chooses trades X_{it}, $t \in \{1, 3\}$ to maximize her expected utility over terminal wealth, which is given by

$$W_i = (n + X_{i1} + X_{i3}) V - X_{i3} P_3 - X_{i1} P_1 - z_i \eta_C. \tag{1}$$

We interpret η_C as climate risk shocks, which reduce investor wealth and, consequently, utility.[14] Furthermore, Z captures investors' aggregate exposure to climate risk shocks, and μ_Z is the average exposure to climate risk. The natural restriction for this interpretation is $\mu_Z > 0$ i.e., these climate shocks, in expectation, have a negative impact on the average investor. In our analysis, we will focus on this restriction to clearly distinguish between projects that are positively vs. negatively exposed to the climate.

[13] We let $\tau_{(\cdot)}$ denote the unconditional precision and $\sigma^2_{(\cdot)}$ the unconditional variance of all random variables.

[14] While, for concreteness, we refer to η_C as a non-tradeable payoff, it is equivalent to interpret it as a non-monetary climate shock to which investors are differentially exposed and that affects their utility directly.

We further require the parameter restriction $\left(\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}\right)^{-1}-\gamma^2\frac{1}{\tau_\eta}>0$ in order to ensure that the unconditional expected utility is finite. Intuitively, if this condition is violated, the climate payoffs $z_i\eta_C$ are sufficiently uncertain ex-ante that the expected utility diverges to $-\infty$. We summarize these restrictions in the following assumption, which is maintained throughout our analysis.

Assumption 1. *(i) The average exposure to climate risk μ_Z is positive, i.e., $\mu_Z>0$.*
(ii) Uncertainty about overall climate payoffs is sufficiently small i.e., $\left(\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}\right)^{-1}>\gamma^2\frac{1}{\tau_\eta}$.

The firm. The firm generates cash flows $A\sim N\left(\mu_A,\tau_A^{-1}\right)$ from assets in place. In addition, the firm's manager decides whether to invest in a new project. The investment decision is binary and denoted by $k\in\{0,1\}$. The total cash flow to the firm, given an investment choice k, is given by

$$V(k)=A+k\left(\theta+\alpha\eta_C+\sqrt{1-\alpha^2}\eta_I-c\right), \tag{2}$$

where $\theta\sim N\left(\mu_\theta,\tau_\theta^{-1}\right)$ and $\eta_C,\ \eta_I\sim N\left(0,\tau_\eta^{-1}\right)$ are independent of each other and other random variables, $\alpha\in[-1,1]$ and $c\geq 0$. The component θ reflects the learnable component of cash flows for the investment opportunity, η_C reflects shocks to the "climate" component of cash flows and η_I reflects shocks to the "idiosyncratic" component of cash flows. The cost of investment is c, which is assumed to be non-negative.

The parameter α captures the extent to which the project's cash flows are correlated with climate risk shocks. When $\alpha=0$, the new project's cash flows are uncorrelated with climate risk and so not useful for hedging – we refer to such projects as "neutral" projects. When $\alpha>0$, the project's cash flows are *higher* when climate outcomes are worse (η_C is higher), and so we refer to these projects as "green" projects. This increase in cash flows may be due to higher demand for the product (e.g., electric vehicles) or regulatory changes (e.g., higher taxes on greenhouse gas emissions) driven by adverse changes in the climate. Analogously, when $\alpha<0$, the project's cash flows are *lower* when climate outcomes are worse, and so we refer to these projects as "brown" projects.

Information and timing of events. Figure 1 summarizes the timing of events. At date one, all investors observe θ perfectly. Let $\mathcal{F}_{i1}=\sigma(\theta,z_i,P_1)$ and $\mathcal{F}_{i3}=\sigma(\theta,z_i,P_1,P_3,k)$ denote investor i's information set at the trading stages, with associated expectation, covariance, and variance operators, $\mathbb{E}_{it}[\cdot]$, $\mathbb{C}_{it}[\cdot]$ and $\mathbb{V}_{it}[\cdot]$, respectively. Then, investor i chooses trade X_i to maximize her expected utility i.e.,

$$\mathcal{W}_i\equiv\sup_{x\in\mathbb{R}}\mathbb{E}_{i1}\left[-e^{-\gamma W_i}\right]. \tag{3}$$

Figure 1: Timeline of events

$t=1$	$t=2$	$t=3$	$t=4$
Investor observes θ, z_i and submits trade X_{i1} Asset price is P_1	Manager chooses $k(P_1)$	Investor submits trade X_{i3} Asset price is P_3	The firm pays $V(k)$

The date one price is determined by the market clearing condition

$$\int_i X_{i1} di = 0. \tag{4}$$

At date two, the manager invests to maximize the date three price, which, in our model, is equivalent to following a positive net present value rule. The manager does not observe θ directly, but can condition on the information in the stock price P_1. Hence, her information set is the public signal $\mathcal{F}_m = \sigma(P_1)$.

The date three price is again determined by the market clearing condition (4), evaluated at the $t = 3$ trades X_{i3} that maximize trader expected utilities at that date. Note, however, that since the manager's investment decision is perfectly anticipated by investors at date one, and there are no additional shocks or information, it will follow that in equilibrium the date three price is equal to the date one price.

At date four, the firm's terminal cash flows V are realized and paid to the investors.

Equilibrium. An equilibrium consists of trades $\{X_{i1}, X_{i3}\}$, prices $\{P_1, P_3\}$, and an investment rule $k(P_1)$ such that (i) the trades X_{it} maximize investor i's expected utility, given her information $\mathcal{F}_{it}$ and the investment rule $k(P_1)$, (ii) the investment rule $k(P_1)$ maximizes the expected date three price $\mathbb{E}[P_3|\mathcal{F}_m]$, and (iii) the equilibrium prices $\{P_1, P_3\}$ are determined by market clearing at dates one and three, respectively.

3.1 Discussion of Assumptions

Green and brown projects. There is some disagreement in the literature regarding whether green or brown stocks are better hedges against climate risk (e.g., see Giglio et al. (2021)). In our model, both green and brown projects are correlated with climate risk (albeit in opposite directions), and so can be used to hedge such risks. As we shall see, in expectation, green stocks carry a price premium, while brown stocks carry a discount, as a result of their exposure to climate risk, which is consistent with Engle et al. (2020) and Bolton and Kacperczyk (2021a) . The extent to which these stocks are useful for hedging is driven by the magnitude of α (i.e., $|\alpha|$), which we will refer to as the climate-risk sensitivity of the project.

Since the investment decision is binary, one can apply our analysis to study divestment decisions. For instance, a firm with $k = 1$ and $\alpha < 0$ is a firm with an existing negative climate exposure (e.g., a traditional car manufacturer). In this case, a decision of $k = 0$ corresponds to a decision to divest brown technology, or equivalently, an investment in green technology that mitigates the firm's existing exposure (e.g., by transitioning to electric vehicle technology). However, in the analysis that follows, the leading application we have in mind is that of a neutral firm deciding whether to invest in a climate-exposed project.

Aggregate demand for hedging. In our model, the average investor's disutility from climate shocks, and, consequently, their desire for hedging (as captured by Z) is stochastic. As in prior work, this ensures that the price does not fully reveal information about θ to the manager.[15] Moreover, it is consistent with the empirical evidence that aggregate demand for climate hedges varies across time and economic conditions. For instance, Bolton and Kacperczyk (2021a) shows that the pricing of carbon-transition risk varies across countries and has risen over time. Moreover, Choi et al. (2020) show that the price premium applied to green vs. brown stocks varies with weather patterns, and Alekseev et al. (2021) shows that weather patterns influence mutual-fund demand for climate-exposed stocks. As we discuss below, this variation generates changes in the "discount rate" that the manager applies to the project when making her investment decision.

Price maximization. The manager in our model invests to maximize price, which can be thought of as her investing only in projects with positive NPV. This is consistent with what managers reportedly do in practice (e.g., see the survey evidence of Graham and Harvey (2001)), and with prior work that builds upon the investment CAPM and q theory of investment, which typically assumes that the firm invests to maximize its market capitalization (e.g., Zhang (2005), Liu, Whited, and Zhang (2009)).[16] Existing papers on feedback effects commonly assume that the firm maximizes expected future cash flows. However, in these papers, the marginal investor is typically risk neutral and so this objective is essentially equivalent to maximizing the firm's price. In our setting, the two objectives are different because of the impact of the risk premium on the price. As such, maximizing expected cash flows would mechanically differ because it fails to account for the cost of capital.

Homogeneous investor information. Since our primary focus is on *managerial* learning from prices, we shut down *investor* learning from prices by assuming that all investors share a common signal about fundamentals.[17] The assumption makes the analysis more

[15] In other settings, such aggregate noise prevents the price from being fully revealing to all investors (e.g., see Ganguli and Yang (2009), Bond and Garcia (2021)).

[16] It is also consistent with what is taught to MBA students in business schools.

[17] This is a common feature of feedback models in which there is a single informed investor (e.g., Edmans,

tractable and ensures that the financial market equilibrium does not exhibit multiplicity of the type studied by Ganguli and Yang (2009). Moreover, in our welfare analysis, this assumption ensures that the traditional Hirshleifer (1971) effect does not arise in our setting.[18] This allows us to clearly distinguish our results from the existing literature of welfare in financial markets, which typically focuses on the Hirshleifer (1971) effect (e.g., Marín and Rahi (2000), Dow and Rahi (2003)). Finally, we have confirmed that our main results are qualitatively similar when investors have private signals and learn from the price.[19]

Binary investment decision. The manager's investment decision in our model is binary and thus resembles exercising a real option. This implies that the firm is only useful in hedging when the manager invests, which makes our results stark. However, the economic forces underlying our results, including the nature of the equilibrium and our welfare analysis, carry over to more general investment decisions (subject to $k \geq 0$). For example, under a continuous investment choice, as the firm invests more, its cash flows are increasingly driven by the risk investors seek to hedge, as opposed to the firm's assets in place, which generates similar results to the ones we study.[20]

Two trading dates. The manager in our model both learns from the stock price and seeks to maximize the stock price. Hence, we require a well-defined market price prior to the investment decision, from which the manager can learn, and a well-defined market price after the investment decision, over which we can specify the manager's maximization problem. Because existing papers on feedback effects typically assume that managers maximize expected cash flows, they do not face such a problem, and consequently work in a single trading-date setting. However, our results are not an artifact of the two-date setting. As will be seen, because the trading dates are otherwise identical, in equilibrium the price is identical at both dates. One could capture similar forces in a setting with a single trading date, but in which the manager simultaneously commits to a real investment schedule $k(P)$ at the same time that traders trade. However, because of the conceptual complications and questionable practical relevance of such a model, we instead work with the two trading date setting.

Assets in place. The presence of assets in place is not qualitatively important for our results, but adds realism to our model by ensuring the firm's cash flows remain uncertain in the absence of investment. Indeed, in our welfare analysis, the impact of the manager's

Goldstein, and Jiang (2015)).

[18]The Hirshleifer (1971) effect focuses on how greater private information reduces welfare by inhibiting risk sharing.

[19]An earlier draft of the paper formally studied such a version of the model.

[20]An earlier version of the paper considered more general investment decisions and found that the key economic forces that drive our results obtain in this more general setting.

investment decision on the stock's effectiveness as a hedge grows stronger if there are no assets in place; in this case, the asset is more sensitive to climate shocks (i.e., η_C) when the firm invests. The assumption that assets in place are uncorrelated with climate risk is made for expositional clarity and can be relaxed. For instance, if A is positively correlated with η_C, one can decompose A as $A = \lambda\eta_C + \varepsilon_A$ for $\lambda > 0$ and $\mathbb{C}(\varepsilon_A, \eta_C) = 0$. In this case, the investment decision still changes the overall exposure of the firm to climate risk (i.e., λ with no investment vs. $\lambda + \alpha$ with investment), and the economic forces underlying our analysis continue to operate.[21]

Market incompleteness. Our welfare analysis reflects the fact that in our model, the firm's investment decision endogenously affects the effective completeness of the market with respect to climate risk. Specifically, when the manager invests in a green or brown project, all else equal, investors benefit because they can use the stock to hedge their exposure to climate risk. In principle, if markets are complete and investors have access to securities that allow them to perfectly hedge such risks (i.e., trade a security that provides exposure only to η_C), then this force would be absent from our setting. Nevertheless, our results on probability of investment, expected returns and firm performance would remain unchanged.

However, in practice, there is ample evidence that climate risk is difficult to hedge (e.g., Giglio et al. (2021), Engle et al. (2020)). As Pástor et al. (2021) point out "[u]nanticipated climate changes present investors with an additional source of risk, which is non-traded and only partially insurable." Indeed, Engle et al. (2020) find that a dynamic equity portfolio optimized to hedge climate risk is at most 30% correlated with news on such risk. This suggests that investment choices by firms are likely to affect the ability of investors to hedge climate risk, especially when these investments are aggregated across sectors or industries.

4 Equilibrium

In general, solving for an equilibrium with feedback effects is complicated by the fact that the asset price must simultaneously clear the market, be consistent with manager and trader beliefs, and be consistent with the anticipated real investment decision. We focus on equilibria of the following form.

Definition 1. *A* **threshold equilibrium** *is one in which:*
(i) the price at both dates depends on the underlying random variables through a linear statistic, $s_p = \theta + \frac{1}{\beta}Z$, where β is an endogenous constant,

[21] Gao and Liang (2013) and Davis and Gondhi (2019) consider settings in which assets in place and project payoffs are positively correlated. However, these papers do not focus on how investment in the project affects investor welfare through its impact on risk sharing.

(ii) the manager invests in the project if and only if $P_1(s_p) > P_1(\bar{s})$ for an endogenous threshold $\bar{s}$, and
(iii) the price is continuous and weakly increasing in s_p and takes an identical piecewise linear form at both dates

$$P_3 = P_1 = \begin{cases} A_1 + B_1 s_p & s_p > \bar{s} \\ A_0 & s_p \leq \bar{s} \end{cases} \tag{5}$$

where the price coefficients A_0, A_1 and B_1 are endogenous.

This type of equilibrium has an intuitive structure and a number of desirable properties. First, the equilibrium price is a generalized linear function of fundamentals, i.e., it is monotonic in $s_p = \theta + \frac{1}{\beta} Z$. Second, the manager invests when the signal s_p is sufficiently positive, i.e., when it exceeds a cutoff $\bar{s}$. Thus, the price naturally is piecewise linear in s_p, increasing in s_p when the manager invests, and constant when she does not. These properties ensure the analysis is tractable and facilitate comparison to existing work.

In the appendix, we formally solve the model by working backwards. We sketch the approach here. Given an investment decision $k \in \{0, 1\}$ at date 2, investor i's beliefs about the asset payoff at $t = 3$ are conditionally normal, with

$$\mathbb{E}_{i3}\left[V\left(k\right)\right] = \mu_A + k\left(\theta - c\right), \; \mathbb{C}_{i3}\left(V\left(k\right), \eta_C\right) = \frac{k\alpha}{\tau_\eta}, \text{ and } \mathbb{V}_{i3}\left(V\left(k\right)\right) = \frac{1}{\tau_A} + \frac{k^2}{\tau_\eta}, \tag{6}$$

and hence her optimal demand is[22]

$$X_{i1} + X_{i3} = \frac{\mathbb{E}_{i3}\left[V\left(k\right)\right] + \gamma \mathbb{C}_{i3}\left(V\left(k\right), \eta_C\right) z_i - P_3}{\gamma \mathbb{V}_{i3}\left(V\left(k\right)\right)} - n. \tag{7}$$

Market clearing implies that

$$P_3 = \begin{cases} \mu_A - \frac{\gamma n}{\tau_A} & k = 0 \\ \mu_A - \frac{\gamma n}{\tau_A} + (\theta - c) - \frac{\gamma}{\tau_\eta}\left(n - \alpha Z\right) & k = 1 \end{cases}. \tag{8}$$

At date two, the manager optimally chooses whether to invest and aims to maximize the date three price, i.e.,

$$k^* = \underset{k \in \{0,1\}}{\arg\max}\, \mathbb{E}\left[P_3 | \mathcal{F}_m\right]. \tag{9}$$

The expression for P_3 above implies that the manager should invest if and only if $s_p > \bar{s}$,

[22] Recall that X_{it} is the trade at date t and so the optimal demand for the asset at date t is $\sum_{s \leq t} X_{is}$.

where

$$s_p \equiv \theta + \frac{\gamma}{\tau_\eta}\alpha Z \quad \text{and} \quad \bar{s} \equiv c + \frac{\gamma}{\tau_\eta}n. \tag{10}$$

Finally, stepping back to $t = 1$, observe from the conjectured form of price in a threshold equilibrium that, when $s_p > \bar{s}$, P_1 precisely reveals s_p to investors, while when $s_p \leq \bar{s}$, P_1 reveals only that $s_p \leq \bar{s}$. Importantly, this knowledge is sufficient to perfectly anticipate the manager's investment decision. As a result, investors can perfectly anticipate P_3 and therefore the equilibrium price at $t = 1$ must satisfy $P_1 = P_3$ in order to preclude arbitrage.

The following proposition establishes the existence and uniqueness of a threshold equilibrium.

Proposition 1. *There exists a unique threshold equilibrium in which the equilibrium prices are*

$$P_1 = P_3 = \mu_A - \frac{\gamma n}{\tau_A} + k\left(s_p - c - \frac{\gamma}{\tau_\eta}n\right), \tag{11}$$

and the manager's investment decision is

$$k = \mathbf{1}\left\{P_1 > \mu_A - \tfrac{\gamma n}{\tau_A}\right\} = \mathbf{1}\left\{s_p > \bar{s}\right\}, \tag{12}$$

where $s_p \equiv \theta + \frac{\gamma}{\tau_\eta}\alpha Z$ *and* $\bar{s} \equiv c + \frac{\gamma}{\tau_\eta}n$.

A couple of observations about the equilibrium are in order. First, and key for the tractability of our setting, in this equilibrium the traders' beliefs about the asset payoff remain *conditionally* normal in all states of the world since the manager's investment decision is P_1-measurable.

Second, the manager's optimal investment takes the form of a net present value (NPV) rule, whereby she invests if and only if the NPV of the project

$$NPV \equiv s_p - \bar{s} = \underbrace{\theta - c}_{\text{cash flows}} \underbrace{- \tfrac{\gamma}{\tau_\eta}(n - \alpha Z)}_{\text{discount rate}} \tag{13}$$

is greater than zero. The first term, $\theta - c$, reflects the expected cash flows from the project, net of investment costs – this captures the "cash flow" news contained in the price. The second term, $-\frac{\gamma}{\tau_\eta}(n - \alpha Z)$, reflects a discount due to the risk premium investors demand for holding shares of the stock. This measures the impact of "discount rate" news in the price. Consistent with intuition, the discount is higher (the NPV is lower) when the firm is larger (i.e., n is higher) because investors have to bear more aggregate risk. Moreover, the discount is lower (higher) for green (brown, respectively) projects when $Z > 0$.[23] This is

[23]It is possible that $Z < 0$ in our model so that brown projects are priced at a premium. However, the

because green projects reduce investors' exposure to (negative) climate shocks, while brown projects exacerbate it.

4.1 Price versus value maximization

When investors anticipate that the manager will invest, the price reveals s_p to the manager, which is a sufficient statistic for the NPV of the project. However, it does not separately reveal the cash flow, θ, or discount rate, $\frac{\gamma\alpha}{\tau_\eta}Z$, components in s_p, which serve as noise for one another. As discussed in Section 3.1, earlier work on feedback effects assumes that managers maximize expected value, but typically considers settings in which this coincides with price maximization.[24] However, in our model, the two objectives lead to different investment behavior. To see this, observe that the manager's conditional expectation of cash flows, given s_p, is

$$\mathbb{E}\left[V\left(k\right)|s_p\right] = \mu_A + k\left(\mathbb{E}\left[\theta|s_p\right] - c\right), \text{ where} \tag{14}$$

$$\mathbb{E}\left[\theta|s_p\right] = \frac{\tau_\theta\mu_\theta + \tau_p\left(s_p - \frac{\gamma}{\tau_\eta}\alpha\mu_Z\right)}{\tau_\theta + \tau_p}, \text{ and } \tau_p = \tau_Z\left(\frac{\tau_\eta}{\gamma\alpha}\right)^2. \tag{15}$$

This implies the manager would invest if and only if $s_p \geq \bar{s}_V$, where

$$\bar{s}_V = c - \frac{\tau_\theta\left(\mu_\theta - c\right)}{\tau_p} + \frac{\gamma}{\tau_\eta}\alpha\mu_Z. \tag{16}$$

Thus, value maximization leads to more investment than price maximization (i.e., $\bar{s} > \bar{s}_V$) if and only if:

$$\frac{\gamma}{\tau_\eta}(n - \alpha\mu_Z) > -\frac{\tau_\theta\left(\mu_\theta - c\right)}{\tau_p}.$$

In particular, there is "over-investment" with value maximization when projects are sufficiently profitable ex-ante (i.e., $\mu_\theta - c$ is sufficiently high), investors' expected climate exposures are small (i.e., μ_Z is low), or the project is sufficiently brown (i.e., α is small or negative).[25] This reflects the fact that the value maximizing rule over-weights the "cash flow" channel and neglects the "discount rate" channel.

In practice, managerial investment decisions appear to be consistent with price maximization: Graham and Harvey (2001) document that around 75% of CFOs they survey use NPV analysis when evaluating projects (see their Figure 2). The above comparison implies

probability of this outcome can be made arbitrarily small setting μ_Z and τ_Z appropriately.

[24] In much of this work, the price is set by risk-neutral investors or market makers, and so there is no risk premium.

[25] Note that when investors are risk-neutral (i.e., $\gamma = 0$), $\bar{s} = \bar{s}_V = c$ and so the investment rules coincide.

that many models which ignore the discount rate channel (i.e., assume value maximization) predict that investment in profitable, climate-neutral (or brown) projects should be higher than it actually is. Moreover, as we discuss further in the next section, appropriately accounting for the impact of discount rates on investment decisions has important implications for market outcomes.

5 Implications

In this section, we present the model's implications for observable quantities. Section 5.1 discusses how the equilibrium probability of investment depends on the model's parameters, and in particular, how it varies across green and brown projects. Section 5.2 describes how average returns conditional on investment depend on the climate-risk sensitivity of the project and other model parameters, while Section 5.3 characterizes the firm's profitability conditional on investment.

5.1 Probability of investment

We begin by characterizing how the probability of investment depends on model parameters.

Proposition 2. *In equilibrium, the unconditional probability of investment is given by*

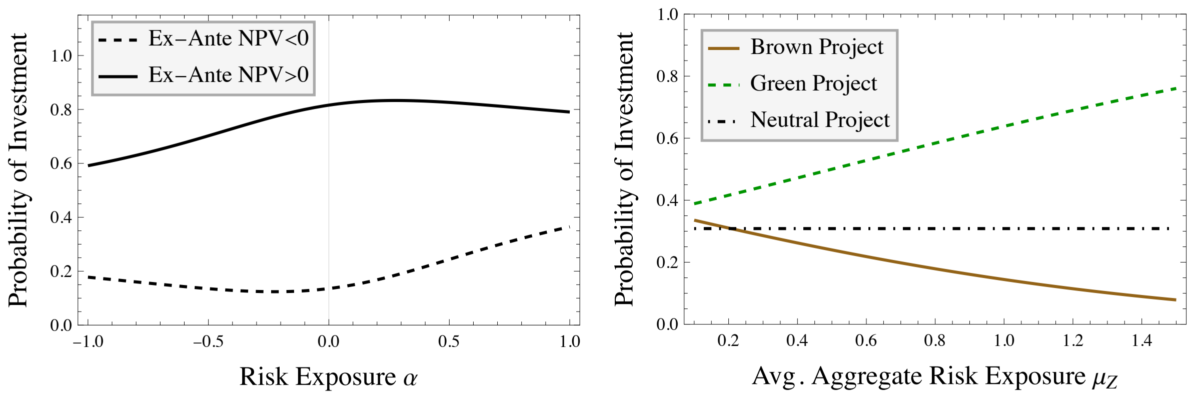
$$\Pr\left(s_p > \bar{s}\right) = \Phi\left(\frac{\mathbb{E}\left[s_p\right] - \bar{s}}{\sqrt{\mathbb{V}\left[s_p\right]}}\right), \tag{17}$$

where $\mathbb{E}\left[s_p\right] = \mu_\theta + \frac{\gamma}{\tau_\eta}\alpha\mu_Z$, $\mathbb{V}\left[s_p\right] = \frac{1}{\tau_\theta} + \frac{1}{\tau_Z}\left(\frac{\gamma\alpha}{\tau_\eta}\right)^2$, *and* $\Phi\left(\cdot\right)$ *denotes the CDF of a standard normal random variable. The probability of investment:*
(i) increases with $\mu_\theta - c$;
(ii) decreases with n;
(iii) increases with μ_Z *for green firms (i.e.,* $\alpha > 0$*), but decreases with* μ_Z *for brown firms (i.e.,* $\alpha < 0$*);*
(iv) increases with τ_θ *and* τ_Z *if and only if* $\mathbb{E}[s_p] - \bar{s} = \mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta} > 0$; *and*
(v) decreases with α *if and only if* $\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} - \frac{\tau_\eta\tau_Z}{\gamma\alpha}\frac{1}{\tau_\theta}\mu_Z\right)\operatorname{sgn}(\alpha) > 0$.

Consistent with intuition, the proposition establishes that the probability of investment increases in the expected NPV of the project $\mathbb{E}[s_p] - \bar{s}$ and decreases (increases) with the variance of the price signal $\mathbb{V}[s_p]$ when $\mathbb{E}[s_p] - \bar{s} > 0$ ($\mathbb{E}[s_p] - \bar{s} < 0$, respectively). This directly implies parts (i)-(iv) of the proposition. From equation (13), we know that the expected NPV increases with expected profitability $\mu_\theta - c$, decreases with size n, and increases with μ_Z if and only if $\alpha > 0$, which implies (i)-(iii). Similarly, an increase in τ_θ or τ_Z leads to a reduction in the variance of the price signal $\mathbb{V}[s_p]$, which leads to more investment when the expected NPV is positive (i.e., $\mathbb{E}[s_p] - \bar{s} > 0$), but less investment when it is negative.

Figure 2: Probability of Investment

This figure plots the probability that the firm invests investment as a function of α and μ_Z. Unless otherwise mentioned, the parameters employed are: $\tau_\theta = \tau_\eta = \tau_A = \gamma = 1$; $\tau_Z = 0.5$; $\mu_Z = 0.5$; $n = 0.1$. The left-hand plot depicts results for both projects that have positive and negative ex-ante NPV. In the ex-ante positive (negative) NPV project case, we set $\mu_\theta - c = -1$ ($\mu_\theta - c = 1$), which implies that, $\forall \alpha \in [-1, 1]$, $\mathbb{E}[s_p] - \bar{s} > 0$ ($\mathbb{E}[s_p] - \bar{s} < 0$).



(a) Prob(Invest) vs. α

(b) Prob(Invest) vs. μ_Z

Part (v) of the proposition shows that the project's sensitivity to the risk factor, α, has a nuanced impact on the likelihood that the manager invests. An increase in α has two, potentially offsetting, effects. First, an increase in α increases the expected NPV $\mathbb{E}[s_p] - \bar{s}$ because it reduces the discount due to climate risk. Second, an increase in the magnitude $|\alpha|$ increases the variance of the price signal $\mathbb{V}[s_p]$. This tends to increase (decrease) the likelihood of an investment that has a negative (positive) NPV conditional on the price signal. The overall effect of α depends on the relative magnitude of these effects.

We next derive the implications of this result for the likelihood that a manager invests in a green (brown) project relative to a climate-neutral project.

Corollary 1. *(i) The firm is more likely to invest in a green than a neutral project, i.e.,* $\Pr(s_p > \bar{s}; \alpha = 1) > \Pr(s_p > \bar{s}; \alpha = 0)$ *if and only if*

$$\frac{\gamma \mu_Z}{\tau_\eta} > \left(\sqrt{1 + \left(\frac{\gamma}{\tau_\eta}\right)^2 \frac{\tau_\theta}{\tau_Z}} - 1 \right) \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} \right).$$

(ii) The firm is more likely to invest in a neutral than a brown project, i.e., $\Pr(s_p > \bar{s}; \alpha = 0) > \Pr(s_p > \bar{s}; \alpha = -1)$ *if and only if*

$$\frac{\gamma\mu_Z}{\tau_\eta} > -\left(\sqrt{1+\left(\frac{\gamma}{\tau_\eta}\right)^2\frac{\tau_\theta}{\tau_Z}}-1\right)\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta}\right).$$

The corollary shows that for ex-ante profitable projects, the manager may be less likely to invest in a green than a neutral project despite the fact that it has, in expectation, a reduced (and potentially negative) risk premium. The manager is more likely to invest in a green project only when the average exposure to climate risk μ_Z is sufficiently large relative to the project's ex-ante profitability $\mu_\theta - c$. Again, this arises because an increase in α (from zero to one) has two, potentially offsetting effects: (i) it increases the expected NPV of the project, and (ii) it increases the variance of the price signal. This second effect reduces the likelihood of investment for profitable projects because the manager's investment decision in such projects is effectively a real option that is "in the money" ex ante. When the project's ex-ante profitability is sufficiently large, this effect dominates and so the firm is more likely to invest in a neutral project.

In contrast, for sufficiently profitable projects (i.e., when $\mu_\theta - c > \frac{\gamma n}{\tau_\eta}$), the firm is always more likely to invest in a neutral project than a brown one. This is because the two effects of an change in α (from zero to negative one) reinforce each other. Relative to a neutral project, a brown project has a lower expected NPV *and* gives rise to a more variable price signal. For profitable projects, both effects tend to lower the likelihood of investment.

5.2 Expected return conditional on investment

The impact of investment on expected returns is widely studied empirically, and existing evidence typically focuses on the average relationship between these variables in the cross-section. Our results highlight that the relation can depend critically on the nature of projects that firms invest in and on the information environment.

The next proposition establishes our main results on expected returns.

Proposition 3. *In equilibrium, the expected return conditional on no investment is*

$$\mathbb{E}\left[V - P_3 | k = 0\right] = \frac{\gamma n}{\tau_A}, \tag{18}$$

and the expected return conditional on investment is given by

$$\mathbb{E}\left[V - P_3 | k=1\right] = \gamma n \left(\frac{1}{\tau_A} + \frac{1}{\tau_\eta}\right) - \frac{\gamma\alpha}{\tau_\eta}\mathbb{E}\left[Z | s_p > \bar{s}\right]. \tag{19}$$

Conditional on investment, the expected return:
(i) increases in $\mu_\theta - c$, τ_Z, and n,
(ii) decreases in μ_Z for $\alpha > 0$ and increases in μ_Z for $\alpha < 0$, and,
(iii) decreases in α if $\mu_Z > 0$ and $\alpha \geq 0$.

When the manager does not invest, the expected return reflects the standard risk premium that investors demand for owning the stock i.e., $\frac{\gamma n}{\tau_A}$. Conditional on investment, the expected return is driven by two components: (i) the standard risk premium $\gamma n \left(\frac{1}{\tau_A} + \frac{1}{\tau_\eta}\right)$, and (ii) the investors' expected aggregate exposure to climate risk conditional on investment: $-\frac{\gamma\alpha}{\tau_\eta}\mathbb{E}\left[Z | s_p > \bar{s}\right]$. Intuitively, as previously discussed, $-\frac{\gamma\alpha}{\tau_\eta}Z$ captures the portion of the project's discount rate that is driven by its exposure to climate risk, and so $-\frac{\gamma\alpha}{\tau_\eta}\mathbb{E}\left[Z | s_p > \bar{s}\right]$ captures the expectation of this discount rate conditional on investment.

To understand the determinants of this expectation, note that we can write:

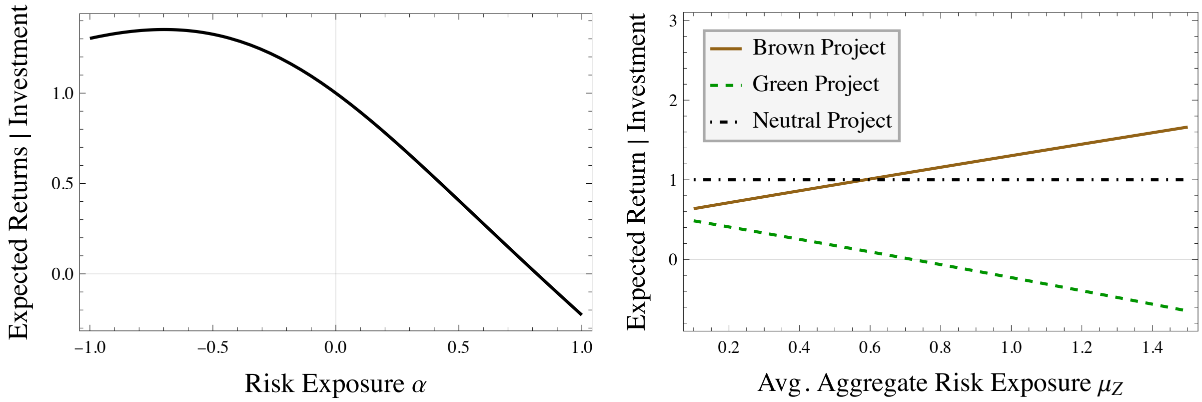
$$-\frac{\gamma\alpha}{\tau_\eta}\mathbb{E}\left[Z | s_p > \bar{s}\right] = \underbrace{-\frac{\gamma\alpha}{\tau_\eta}\mu_Z}_{\text{avg. risk exposure}} - \overbrace{\left(\frac{\gamma\alpha}{\tau_\eta}\right)^2 \frac{\frac{1}{\tau_Z}}{\sqrt{\mathbb{V}\left[s_p\right]}} H\left(\frac{\bar{s} - \mathbb{E}[s_p]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right)}^{\text{adjustment for manager's investment strategy}}, \tag{20}$$

where $H(\cdot) = \frac{\phi(\cdot)}{1-\Phi(\cdot)}$ is the hazard ratio of the standard normal distribution. This reveals two channels through which investment relates to expected returns. First, investment unconditionally raises the magnitude of the covariance between the firm's cash flows and the climate risk shock, which leads to the portion of the risk premium driven by the average risk exposure term μ_Z. Second, and novel to our model, the manager chooses to invest only when price is high, and thus their expectation of the discount rate, $\mathbb{E}\left[Z | s_p\right]$, is low. Consequently, conditional on investment, the discount rate is, in expectation, below its unconditional mean; this channel is captured by the hazard-rate term above.

This decomposition provides intuition for the above proposition. An increase in profitability $\mu_\theta - c$ increases the likelihood of investment and, therefore, increases the expected return via the second term in (20). Similarly, when the risk factor generates less variation in the discount rate (i.e., τ_Z is greater), $\mathbb{V}\left[s_p\right]$ tends to fall, which raises expected returns. While an increase in firm size n tends to lower the likelihood of investment and so decreases expected return via the second term in (20), this channel is dominated by the direct effect

Figure 3: Expected Returns Given Investment

This figure plots the firm's expected returns given investment as a function of the model's parameters. Unless otherwise mentioned, the parameters employed are: $\tau_\theta = \tau_\eta = \tau_A = \mu_Z = \mu_\theta = c = \gamma = 1; \tau_Z = 2; n = 0.5$.



(a) Expected Returns vs. α (b) Expected Returns vs. μ_Z

of the standard risk premium channel (i.e., $\gamma n\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)$), and so expected returns increase with n.

Moreover, it is immediate that expression (20) decreases in μ_Z when $\alpha > 0$, but increases otherwise. This is intuitive: when the average exposure to climate risk increases, expected returns for green projects decrease while those for brown projects increase. Similarly, for green projects ($\alpha > 0$), an increase in α leads to a direct decrease in the risk premium term (since $\mu_Z > 0$), and an increase in the variance of the price signal $\mathbb{V}[s_p]$. Both these effects tend to lower the expected return.

We next characterize how expected returns, conditional on investment, vary with the type of project the firm is investing in.

Corollary 2. *Conditional on investment, a green firm always has a lower expected return than a neutral firm i.e.,*

$$\mathbb{E}\left[V - P_3 | k = 1; \alpha = 1\right] \leq \mathbb{E}\left[V - P_3 | k = 1; \alpha = 0\right].$$

However, a neutral firm has a higher expected return than a brown firm i.e.,

$$\mathbb{E}\left[V - P_3 | k = 1; \alpha = -1\right] \leq \mathbb{E}\left[V - P_3 | k = 1; \alpha = 0\right]$$

if and only if μ_Z is sufficiently small.

As discussed above, in the case of green firms ($\alpha > 0$), the two components in expression

(20) both serve to decrease expected returns as $|\alpha|$ increases. More interestingly, expression (20) also implies that for brown firms ($\alpha < 0$), the two components can move in opposite directions as α changes. As a result, when the aggregate expected exposure to climate risk μ_Z is sufficiently small, brown firms may have lower expected returns than neutral firms. This is a consequence of the manager's reliance on price when investing. All else equal, brown projects are ex-ante less attractive than green projects due to a higher average discount due to climate risk. Hence, in order for a manager to find a brown project sufficiently attractive to invest in, the first-period price must be particularly high, reflecting, in part, a lower than expected discount rate.

Panel (a) in Figure 3 provides an illustration of this result: it plots the expected return, conditional on investment, as a function of α. Moreover, note that this non-monotonicity arises when μ_Z is sufficiently small, as illustrated in panel (b). This suggests that the extent of the non-monotonicity depends on the aggregate expected exposure to climate risk, and so, in general, can vary over time and across markets – which may be captured by measures of climate risk news such as those proposed by Engle et al. (2020), Choi et al. (2020), and Alekseev et al. (2021).

5.3 Future profitability

Standard models of feedback effects typically imply that more informative prices lead to more profitable investment decisions. A number of empirical papers find evidence consistent with this prediction (e.g., Chen, Goldstein, and Jiang (2007)). In our model, however, investment need not always raise the firm's expected cash flows, as summarized by the following proposition.

Proposition 4. *Let* $\Delta V \equiv \mathbb{E}[V|k=1] - \mathbb{E}[V|k=0]$ *denote the change in expected cash flows due to investment. Then,*

(i) ΔV *is always positive when* $\alpha = 0$.

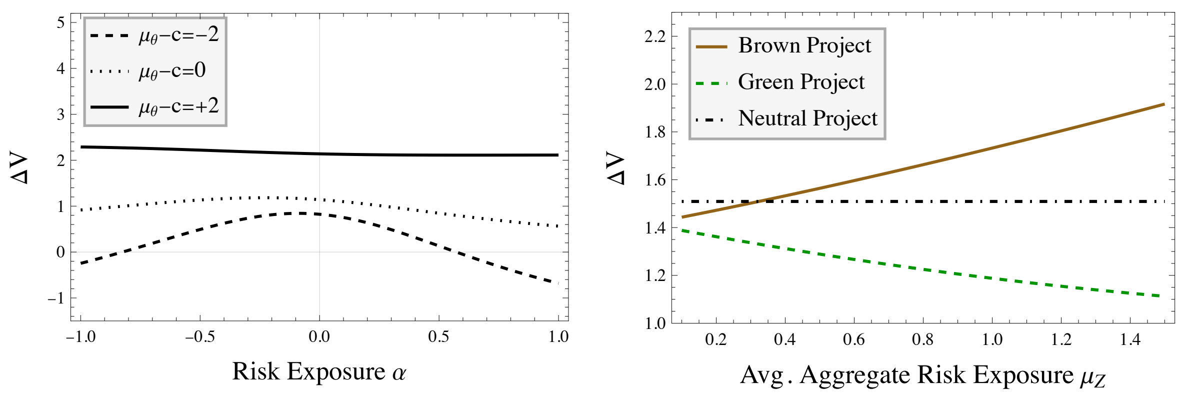
(ii) ΔV *is negative when* $\alpha \neq 0$ *and* $\mu_\theta - c$ *is sufficiently negative.*

(iii) ΔV *is increasing in* $\mu_\theta - c$ *and* n, *decreasing in* μ_Z *when* $\alpha > 0$, *and increasing in* μ_Z *when* $\alpha < 0$.

Figure 4 provides an illustration of this result. As before, the result follows from the observation that the manager's investment decision depends not only on the project's profitability, but also its discount rate. Part (i) of the above proposition corresponds to the standard intuition from the existing literature – when the project does not have a climate risk exposure, feedback-based investment increases the firm's expected profitability. However, part (ii) implies that when the project has a climate risk exposure, the manager may still invest even when it has low or even negative expected profitability because it is sufficiently valuable to investors as a climate risk hedge. This occurs when the project's ex-ante profitability is low. In this case, the manager invests in the project because of good discount rate news, as opposed to cash flow news. That is, the price signal is more likely to be sufficiently high for investment ($s_p > \bar{s}$) because of high hedging benefits (driven by a realization of Z) as opposed to high future cash flows (θ).

Figure 4: Change in Expected Cash Flows Due to Investment

This figure plots the impact of the investment on the firm's expected cash flows as a function of α and μ_Z. Unless otherwise mentioned, the parameters employed are: $\tau_\theta = \tau_\eta = c = \tau_A = \tau_Z = \mu_Z = \gamma = 1$; $n = 0.5$. Panel (b) focuses on the case in which $\mu_\theta - c = 1$.



(a) ΔV vs. α (b) ΔV vs. μ_Z

Consistent with intuition, the expected change in future cash flows as a result of investment increases with the project's expected profitability. Moreover, because the threshold for investment increases with n, conditional on investment, expected future cash flows increase with n. Finally, the dependence on μ_Z follows from the fact that holding cash flows fixed, the expected pricing of the project increases with μ_Z for green projects, but decreases with μ_Z for brown projects. As a result, conditional on investment, expected cash flows are decreasing in μ_Z for green projects, but increasing in μ_Z for brown projects.

6 Welfare

In this section, we explore the relationship between feedback, investment, and investor welfare. We characterize the channels through which investment affects investor welfare and show that the firm's equilibrium price does not fully capture the investment's impact on welfare. That is, even though investing may raise the firm's price, it can nevertheless lead to lower welfare. Consequently, the manager's use of the information in price to guide her investment decision need not enhance investor welfare.

Existing models of feedback effects focus on the impact that feedback has on a firm's expected cash flows. In many such models, investors are risk neutral so that maximizing expected cash flows aligns with welfare maximization.[26] However, in our model, investor risk aversion implies that investment has multiple, potentially offsetting effects on investor welfare, due to the riskiness of the project and the stock's usefulness as a hedge.

Note that because traders are ex-ante symmetric, the ex-ante expected utility of an arbitrary trader is an unambiguous measure of welfare:

$$\mathcal{W} \equiv \mathbb{E}\left[-e^{-\gamma W_i(k(s_p))}\right] \tag{21}$$

$$= \Pr\left(k=1\right)\mathbb{E}\left[-e^{-\gamma W_i(1)}|k=1\right] + \Pr\left(k=0\right)\mathbb{E}\left[-e^{-\gamma W_i(0)}|k=0\right], \tag{22}$$

where

$$W_i\left(k\right) = \begin{cases} X_i\left(V\left(1\right)-P\right)+nV\left(1\right)-z_i\eta_C & k=1 \\ nV\left(0\right)-z_i\eta_C & k=0 \end{cases}. \tag{23}$$

Proposition 8 in Appendix B characterizes this expression in closed form. However, to understand the relevant economic forces of investment, it is helpful to study the simpler special case in which investment is *fixed* at arbitrary level k, in which case the model reduces to a standard unconditionally linear-normal form. In this case, we have

$$\mathbb{E}\left[-e^{-\gamma W_i(k)}\right] = -D(k)\exp\left\{Q(k)\right\} \tag{24}$$

where the function Q can be expressed, after grouping terms, as

$$Q\left(k\right) = -\gamma \underbrace{\mathbb{E}[P(k)]n}_{\text{Valuation channel}} + \tfrac{1}{2}\gamma^2 \underbrace{k^2\left(\tfrac{1}{\tau_\theta}-\tfrac{1}{\tau_\eta}\right)n^2}_{\text{Project risk channel}} - \tfrac{1}{2}\gamma^2\tfrac{1}{\tau_A}n^2 + \tfrac{1}{2}\gamma^2\tfrac{1}{\tau_\eta}\mu_Z^2$$

$$+\tfrac{1}{2}\gamma^2 \frac{\gamma^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2 \underbrace{\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\left(\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^2}_{\text{Hedging channel}}} \underbrace{(\mu_Z-k\alpha n)^2}_{\text{Climate risk channel}}$$

and D can be expressed as

[26] Section 2 discusses notable exceptions, like Dow and Rahi (2003).

$$D(k) = \underbrace{\sqrt{\frac{\frac{1}{\tau_A}+k^2\frac{1}{\tau_\eta}}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\eta}+\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}\right)}}}_{\text{Value of information}} \sqrt{\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}} \left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2 \frac{1}{\tau_\eta} + \gamma^2 \underbrace{\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)} \left(\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^2}_{\text{Hedging channel}} \right)^{-1/2}. \tag{25}$$

We have explicitly labeled all terms in these expressions that depend on the investment level k and will discuss them in turn.

There are five components that capture different channels through which investment affects welfare.

- The **valuation channel** reflects the fact that investment affects the investor's expected wealth via their ownership of n shares. This channel increases (decreases) welfare if the project's NPV is positive (negative), and consequently, investment increases (decreases, respectively) the expected stock price. Note this channel fully captures the investment's impact on the firm's expected cash flows, since these cash flows are picked up by the price.

- The **project risk channel** reflects the fact that the project changes the investors' firm-specific risk exposure, and depends on the relative importance of the learnable and unlearnable components (θ and η, respectively).[27]

- The **climate risk channel** captures the fact that, holding the amount of risk sharing constant, the investment affects investors' aggregate exposure to climate shocks. The average investor's total climate exposure is $(\mu_Z - k\alpha n) * \eta_C$. Thus, when $\alpha > 0$ ($\alpha < 0$), the investment mitigates (amplifies) aggregate climate exposure, which raises (lowers) investor welfare.

- The **value of information channel** captures the fact that investors' information about the project's cash flows renders the stock more useful in hedging; specifically, observing θ increases the conditional correlation between the stock's payoffs and η_C. Moreover, this effect is only relevant when the project is undertaken, and so disappears when $k = 0$. This takes a familiar form of the ratio of investors' conditional variance

[27]The learnable portion of the project payoff, θ, is risky ex-ante, which tends to make investment reduce welfare. On the other hand, welfare reflects the total risk of the project as opposed to the price, which reflects the marginal riskiness – this makes investment more attractive from a welfare perspective.

of the asset return with and without conditioning on θ.[28]

- Finally, the **hedging channel** in both Q and D reflects that the project enables investors to use the firm's stock to hedge their idiosyncratic exposures to climate risk, ζ_i, and consequently improves welfare, all else equal. This term has a clear interpretation as reflecting both the proportion of climate exposures that are hedgeable (i.e., the idiosyncratic components ζ_i), and the effectiveness of the asset as a hedging instrument for climate risk, η_C:

$$\text{Hedging channel} = \underbrace{\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}}_{\substack{\text{Risk-reduction through hedging}\\ \equiv\mathbb{V}(\eta_C)-\mathbb{V}(\eta_C|V-P,z_i)}} \times \underbrace{\left(\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^2}_{\text{\% hedgeable climate exposure}} . \tag{26}$$

Importantly, the manager's objective function – i.e., the firm's price – captures the valuation channel but does not appropriately capture the other channels. To gain some intuition, note that the share price $P(k)$ can be expressed as

$$P(k) = \mathbb{E}_i\left[V\right] + \gamma Z \mathbb{C}_i\left[V,\eta_C\right] - \gamma n \mathbb{V}_i\left[V\right]. \tag{27}$$

This expression reveals that the price does not reflect the stock's usefulness for sharing climate risk, i.e., the hedging channel. The welfare impact of this channel depends the diversity in investors' climate exposures (i.e., τ_ζ^{-1}), since this determines investors' overall gains from sharing climate risk. However, the price is independent of τ_ζ^{-1} and depends upon on investors' climate exposures only through the *aggregate* climate exposure Z. Similarly, the price does not reflect the value of information channel, because it does not capture the additional hedging benefit that investors gain from observing θ when the manager invests (i.e., when $k = 1$). Because both these channels improve welfare, this implies that a price-maximizing manager tends to under-invest relative to a welfare-maximizing rule.

The price also does not fully account for the climate risk and project risk channels. Investor welfare reflects the *total risk* from these sources, as well as how this risk varies

[28] In our model, while investors are endowed with information, this term still captures the improvement in utility as a result of observing θ relative to being uninformed. Specifically, given fixed k, this ratio can be represented as $\frac{\mathbb{V}(V|\theta,z_i,P)}{\mathbb{V}(V-P|z_i)}$, which reflects the proportional improvement in expected utility for a trader who trades conditional on θ, z_i, and P vs. z_i alone. The welfare expressions in Bond and Garcia (2021) include a similar term, which they further decompose into a product of the classic value of cash-flow information, $\frac{\mathbb{V}(V|\theta,z_i,P)}{\mathbb{V}(V|z_i,P)}$, and the value of providing vs. demanding liquidity (i.e., using a price-contingent schedule vs. not), $\frac{\mathbb{V}(V|z_i,P)}{\mathbb{V}(V-P|z_i)}$. Because these effects are not a primary focus of our analysis we choose to concisely represent them in a single term.

across investors. However, the price reflects the *marginal risk* of project cash flows and climate shocks from buying an additional share to the average investor. Whether these differences lead to relative over-investment or under-investment depends on features of the investment project.

First, the climate risk channel implies that for green projects ($\alpha > 0$), the manager tends to under-invest, while the effect is reversed for brown projects ($\alpha < 0$). This is because the manager does not fully internalize the reduction (increase) in aggregate climate risk due to her investment in green (brown) projects. Interestingly, this occurs despite the fact that the price includes a climate risk premium – this risk premium is simply too small relative to how the investment's impact on climate risk influences welfare. Intuitively, the fact that there is inequality in investors' climate risk exposures increases the negative impact that this risk has on welfare relative to the case in which investors share the same climate exposure. This amplifies the positive effect that green investment has on welfare by mitigating aggregate climate risk (and vice versa for brown investment). Critically, the price does not account for this because, again, it depends only on the average exposure of investors to climate risk and does not account for the heterogeneity in their exposures.

Second, the project risk channel implies that the manager tends to over-invest when the learnable component of cash flows is more volatile (i.e., $\frac{1}{\tau_\theta}$ is higher), while she tends to under-invest when the unlearnable component of cash flows is more volatile (i.e., $\frac{1}{\tau_\eta}$ is higher). Intuitively, the price tends to over-weight project risk since the marginal risk (reflected in the price) is higher than the average risk – this tends to lead to under-investment. However, since the learnable component of project value θ is known by all traders at the trading stage, it does not carry a risk-premium in the asset price, and so this component leads to relative over-investment.

Overall, the misalignment between the manager's objective and investor welfare implies that feedback from prices need not necessarily improve welfare. To formalize this intuition, we next compare investor welfare in our setting to a benchmark in which the manager ignores the information in price and instead chooses k to maximize the *ex-ante expectation* of the firm's market value. Recall from Proposition 1 that the investment's effect on price is $s_p - \bar{s}$, and so, in this benchmark, the manager invests if and only if $\mathbb{E}[s_p] - \bar{s} > 0$.

The next proposition characterizes sufficient conditions under which feedback reduces welfare.

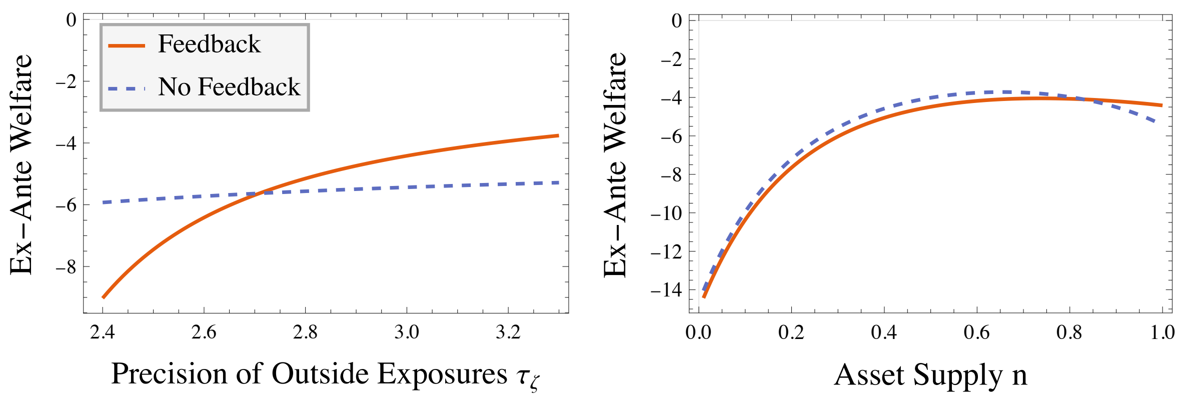
Proposition 5. *Feedback reduces welfare if the expected NPV (i.e., $\mathbb{E}[s_p] - \bar{s}$) is positive, and*

(i) n is sufficiently small, or

(ii) risk-sharing needs are sufficiently large (i.e., τ_ζ is sufficiently small).

Figure 5: Ex-ante welfare: Feedback vs. No feedback

This figure plots the ex-ante welfare (i.e., ex-ante expected utility) as a function of α and μ_Z for a project with positive expected NPV. Unless otherwise mentioned, the parameters employed are: $\tau_\theta = 0.5$; $\tau_\zeta = 3$; $\tau_Z = 2$; $\mu_A = 0$; $\tau_A = 5$; $\mu_\theta = c = \tau_\eta = \gamma = \mu_Z = n = 1$. These parameters ensure the expected NPV of the project is positive.



(a) Welfare vs. τ_ζ (b) Welfare vs. n

Figure 5 provides an illustration of these results, and the intuition is as follows. When the ex-ante NPV of the project is positive, in the no-feedback benchmark, the manager always invests. In contrast, given feedback, the firm does not invest for $s_p < \bar{s}$. On the one hand, feedback improves the expected value of the firm and so increases welfare via the valuation channel. On the other hand, because it leads to no-investment in some states, feedback reduces the ability of investors to use the risky security as a hedge, and so reduces welfare via the hedging channel. It also affects aggregate exposure to climate risk (with the direction depending on the sign of α).

When the per-capita endowment of shares n is small, the valuation and project risk channels are relatively small. Moreover, the firm's investment decision has a small effect on the aggregate climate exposure, and so the climate risk channel is muted. However, the hedging channel remains important since it is unaffected by n: regardless of the firm's size, its stock remains a useful hedge in the event of investment. Consequently, the hedging channel is dominant in the limit, and investors are better off with a rule that always invests, and so yields hedging benefits in all states of the world. Analogously, when τ_ζ is small, investors' exposures to the climate are highly diverse, so that the ability to share risk provides them with large welfare gains. Hence, the hedging channel dominates in the limit, once again leading investors to prefer an investment rule that ensures that the asset is always useful for hedging.

As an aside, note that the manager's use of price information *always* enhances the firm's expected price: because the manager's objective is to maximize price, any additional information that she conditions on can only increase the price, in expectation. Hence, Proposition 5 implies that an increase in the firm's market value need not align with an increase in investor welfare.

7 Empirical Predictions

Our model offers predictions that are relevant to several streams of empirical work, which we summarize in this section.

Climate investment and divestment. Proposition 2 sheds light on how firm-level investment in green projects varies with market conditions. Consistent with intuition, when aggregate climate exposures (μ_Z) are high, the amount of green (brown) investment increases (decreases). Importantly, this is not due to any explicit preference for green projects by investors but instead works through the fact that green projects enjoy a lower cost of capital in the event of high climate exposures. Aggregate climate exposures in our model may be proxied for using news on climate change, as measured in, e.g., Engle et al. (2020) and Choi et al. (2020). Intuitively, negative climate news likely raises investors' expectations of their exposure to the climate.

Result (iv) in Proposition 2 speaks to how the informativeness of price affects the prevalence of green investment. The model predicts that for projects that are already ex-ante attractive (i.e., have positive expected NPV $\mathbb{E}[s_p] - \overline{s} > 0$), investment is less likely when the price contains more information about cash flows or discount rates (i.e., τ_θ or τ_Z are lower). However, if the project is marginal (i.e., $\mathbb{E}[s_p] - \overline{s} = 0$), then additional information has no effect. Importantly, the result also implies that more information in the price can reduce investment in green projects if they have a positive expected NPV.

Corollary 1 and panel (a) of Figure 2 also highlight that, in our setting, the financial market may be more effective at discouraging brown investment than encouraging green investment, or vice versa. This may be surprising because the risk premium is linear in the project's risk exposure, so that the magnitude of the price discount that accompanies a brown project (with $\alpha = -1$) is precisely equal to the magnitude of the price premium that accompanies a green project (with $\alpha = 1$). The reason is that both types of projects also raise variation in the discount rate, relative to a climate-neutral project. Given that the manager's investment decision is a real option, this tends to reduce the likelihood of investment for ex-ante profitable projects, and increase it for ex-ante unprofitable projects. Hence, the market is more likely to discourage brown projects than encourage green projects that are ex-ante profitable, and vice versa for projects that are ex-ante unprofitable.

These results imply that the impact of policy changes that target a firm's climate investments are likely to be asymmetric. For ex-ante profitable projects, policies that encourage firms to divest from brown technologies are likely to have a larger impact than policies that encourage neutral firms to adopt green technologies. On the other hand, for ex-ante unprofitable technologies, encouraging firms to divest from brown technologies might require stronger incentives.

Climate exposure and expected returns. Significant prior literature studies how firms' climate exposures affect their expected returns (i.e., whether there is a "climate risk premium"; see Gillan et al. (2021) for a review). Our model predicts that the impact of a project's climate exposure on expected future returns depends upon whether the project is green or brown. Specifically, while green projects earn unambiguously lower expected returns the greener they are, the relation between expected returns and "brownness" is relatively flat. This implies one must exercise caution in interpreting cross-sectional evidence about how average returns vary with measures of climate risk exposure (e.g., environmental scores, carbon emission intensity), and may help reconcile the mixed empirical evidence in the literature. For instance, while Chava (2014) and Bolton and Kacperczyk (2021a) find that greener firms earn lower returns (or have lower costs of equity and debt capital), Larcker and Watts (2020) and Berk and van Binsbergen (2021) argue there does not appear to be a large difference in returns across green vs. non-green bonds and stocks, respectively.

Proposition 3 also predicts that all else equal, expected returns conditional on investment are *higher* when there is (i) *less uncertainty* about the aggregate climate risk exposure (i.e., τ_Z is higher) and when (ii) the ex-ante profitability of the project is higher (i.e., $\mu_\theta - c$ is higher). These predictions are a consequence of feedback from prices. The manager invests in a climate-exposed project only when she observes a price signal that implies the project has positive NPV. Projects with higher ex-ante profitability $\mu_\theta - c$ have positive NPV even for relatively high discount rates. Consequently, compared to less profitable projects, highly profitable projects in which the manager finds it optimal to invest have, on average, higher expected returns.[29] Similarly, a decrease in uncertainty about climate risk exposures implies that the manager's investment decision is driven to a greater extent by the information in price on expected cash flows rather than the discount rate. As a result, conditioning on investment is a better indicator that cash flows are high than that the discount rate is low.

Climate investment and future profitability. Considerable prior work studies whether ESG-related investments are, in general, more profitable according to metrics that

[29] These findings are related but distinct from Fama and French (2015)'s argument for why more profitable firms will experience higher returns. Fama and French (2015)'s explanation is of an econometric nature: controlling for firms' valuations, in the cross-section, more profitable firms must earn lower returns because they ultimately pay higher dividends. In our model, this relationship instead arises due to the manager's optimal investment choice.

capture expected cash flows such as ROA, ROE, and revenue growth (Gillan et al. (2021)). Following Proposition 4, our model suggests that managers may invest in speculative, unprofitable green projects when investors' expected exposure to climate risk is sufficiently large (i.e., μ_Z is large) and thereby the cost of capital for such projects is especially low. Conversely, managers invest only in highly profitable brown opportunities because, for these projects to have a positive NPV, their profitability must overcome their higher discount rates. This suggests the surprising prediction that, when climate risk exposure is high, one should expect to see a negative cross-sectional relation between project "greenness" and standard measures of ex-post profitability. This is consistent with the evidence documented by Di Giuli and Kostovetsky (2014), who show that an improvement in CSR policies is associated with future under-performance and decline in future ROA, and Cheema-Fox, Serafeim, and Wang (2022), who show that pure-play green stocks experience high returns but low profitability. Notably, this is a novel consequence of managerial learning from prices and not of, e.g., agency problems, which have been argued to generate such a relation (Cheng, Hong, and Shue (2013), Buchanan, Cao, and Chen (2018)).

Executive compensation and climate-risk metrics. Our welfare results speak to the recent debate on the effectiveness of the use of climate-risk metrics in executive compensation. On the one hand, there has been a rapid increase in the use of such measures. Edmans (2021) cites that "51% of large U.S. companies and 45% of leading U.K. firms use ESG metrics in their incentive plans," and Hill (2021) cites a survey conducted by Deloitte in September 2021, which suggests that "24 per cent of companies polled expected to link their long-term incentive plans for executives to net zero or climate measures over the next two years."[30]

On the other hand, there is ample skepticism about the effectiveness of such incentives. In addition to issues around measurement and monitoring of such objectives and the possibility of unintended consequences, Edmans (2021) argues that incentivizing ESG performance may not necessarily lead to better financial performance. Instead, he advocates for the use of long-term stock-based compensation, arguing that "[s]ince material ESG factors ultimately improve the long-term stock price, this holds CEOs accountable for material ESG issues – even if they aren't directly measurable."

Our analysis suggests that this may not be true because the stock price (even in the long term) does not fully account for the benefit of investing in climate-exposed projects. As such, providing additional incentives based on climate metrics (e.g., bonuses linked to climate targets) can improve overall investor welfare. This is despite the fact that such incentives

[30]More broadly, Edmans, Gosling, and Jenter (2021) find that over 50% of surveyed directors and investors report that offering variable pay to CEO is in part useful to "motivate the CEO to improve outcomes other than long-term shareholder value."

may decrease stock prices and future profitability on average by leading to inefficient over-investment in green projects. Yet, when investors have diverse climate risk exposures and find it difficult to hedge these exposures, such incentives improve their ability to hedge risks and, consequently, can improve investor welfare.

8 Conclusions

In this paper, we study a model of informational feedback effects in which a firm's investment alters its exposure to an aggregate risk. Our leading example is investment in a technology that changes the sensitivity of the firm's performance to the climate. When a firm invests in a project that is exposed to the climate, it affects how useful the asset is as a hedge of climate risk. As a result, the firm's stock price reflects information about investors' climate exposures and the project's expected cash flows, which are both relevant to the manager's investment choice. We show that this leads to several results that are novel to the growing climate finance literature.

We show that the likelihood of investment increases with a project's ex-ante profitability, but can be non-monotonic in its greenness. Moreover, the greenness of a firm's investment is negatively related to its expected returns, though this effect is muted for firms investing in brown projects. Conditional on investment, brown projects tend to have higher future profitability than comparable green projects. Finally, we characterize conditions under which feedback from prices to investment can lead to lower welfare, even though it always leads to higher firm value. Thus, our analysis suggests that providing additional climate-based incentives for managers can play an important role in improving investor welfare.

A notable contribution of our analysis is to provide a tractable feedback effects framework with risk aversion. Given its tractability, this framework can be extended naturally along various dimensions. Immediate extensions include generalizations to the structure of cash flows and information. For instance, allowing for both public and private information signals would enable future research to assess the merits of disclosure regarding firms' climate risk exposures. Similarly, introducing multiple dimensions of fundamentals as in Goldstein and Yang (2019), Goldstein et al. (2020) could enable future work to assess how climate-exposed investments interact with the other risks that firms face. Finally, it may be interesting to consider how information acquisition by investors and managers interact in our setting.

References

Albagli, E., C. Hellwig, and A. Tsyvinski (2021). Dispersed information and asset prices. *TSE Working Paper*. 2

Alekseev, G., S. Giglio, Q. Maingi, J. Selgrad, and J. Stroebel (2021). A quantity-based approach to constructing climate risk hedge portfolios. Technical report, Working Paper. 2, 3.1, 5.2

Avdis, E. (2016). Information tradeoffs in dynamic financial markets. *Journal of Financial Economics 122*(3), 568–584. 5

Bai, J. J., Y. Chu, C. Shen, and C. Wan (2021). Managing climate change risks: Sea level rise and mergers and acquisitions. *Unpublished working paper*. 1

Banerjee, S., J. Davis, and N. Gondhi (2021). Incentivizing effort and informing investment: The dual role of stock prices. *Available at SSRN 3955129*. 2

Berk, J. and J. H. van Binsbergen (2021). The impact of impact investing. *Available at SSRN 3909166*. 7

Bolton, P. and M. Kacperczyk (2021a). Do investors care about carbon risk? *Journal of Financial Economics 142*(2), 517–549. 2, 4, 3.1, 7

Bolton, P. and M. Kacperczyk (2021b). Global pricing of carbon-transition risk. Technical report, National Bureau of Economic Research. 2

Bond, P., A. Edmans, and I. Goldstein (2012). The real effects of financial markets. *The Annual Review of Financial Economics is 4*, 339–60. 1, 2

Bond, P. and D. Garcia (2021). The equilibrium consequences of indexing. *Forthcoming in Review of Financial Studies*. 5, 10, 2, 15, 28

Briere, M. and S. Ramelli (2021). Green sentiment, stock returns, and corporate behavior. *Available at SSRN 3850923*. 1

Buchanan, B., C. X. Cao, and C. Chen (2018). Corporate social responsibility, firm value, and influential institutional ownership. *Journal of Corporate Finance 52*, 73–95. 7

Chava, S. (2014). Environmental externalities and cost of capital. *Management science 60*(9), 2223–2247. 7

Cheema-Fox, A., G. Serafeim, and H. S. Wang (2022). Climate solutions investments. *Available at SSRN*. 7

Chen, Q., I. Goldstein, and W. Jiang (2007). Price informativeness and investment sensitivity to stock price. *Review of Financial Studies 20*(3), 619–650. 5.3

Cheng, I.-H., H. Hong, and K. Shue (2013). Do managers do good with other people's money? *NBER Working paper 19432*. 7

Choi, D., Z. Gao, and W. Jiang (2020). Attention to global warming. *The Review of Financial Studies 33*(3), 1112–1145. 2, 3.1, 5.2, 7

Chowdhry, B., S. W. Davies, and B. Waters (2019). Investing for impact. *The Review of Financial Studies 32*(3), 864–904. 12

Davis, J. and N. Gondhi (2019). Learning in financial markets: Implications for debt-equity conflicts. 10, 2, 21

Di Giuli, A. and L. Kostovetsky (2014). Are red or blue companies more likely to go green? politics and corporate social responsibility. *Journal of Financial Economics 111*(1), 158–180. 7

Diamond, D. W. and R. E. Verrecchia (1981). Information aggregation in a noisy rational expectations economy. *Journal of Financial Economics 9*(3), 221–235. 10

Dow, J., I. Goldstein, and A. Guembel (2017). Incentives for information production in markets where prices affect real investment. *Journal of the European Economic Association 15*(4), 877–909. 10, 2

Dow, J. and R. Rahi (2003). Informed trading, investment, and welfare. *Journal of Business 76*(3), 439–454. 2, 3.1, 26

Eckwert, B. and I. Zilcha (2003). Incomplete risk sharing arrangements and the value of information. *Economic Theory 21*(1), 43–58. 11

Edmans, A. (June 27, 2021). Why companies shouldn't tie ceo pay to esg metrics. *The Wall Street Journal*. 9, 7

Edmans, A., I. Goldstein, and W. Jiang (2015). Feedback effects, asymmetric trading, and the limits to arbitrage. *American Economic Review 105*(12), 3766–97. 17

Edmans, A., T. Gosling, and D. Jenter (2021). Ceo compensation: Evidence from the field. *CEPR Discussion Paper No. DP16315*. 30

Engle, R. F., S. Giglio, B. Kelly, H. Lee, and J. Stroebel (2020). Hedging climate change news. *Review of Financial Studies 33*(3), 1184–1216. 2, 4, 1, 3.1, 5.2, 7

Fama, E. F. and K. R. French (2015). A five-factor asset pricing model. *Journal of Financial Economics 116*(1), 1–22. 29

Friedman, H. L. and M. S. Heinle (2016). Taste, information, and asset prices: Implications for the valuation of csr. *Review of Accounting Studies 21*(3), 740–767. 12

Ganguli, J. V. and L. Yang (2009). Complementarities, multiplicity, and supply information. *Journal of the European Economic Association 7*(1), 90–115. 10, 15, 3.1

Gao, P. and P. J. Liang (2013). Informational feedback, adverse selection, and optimal disclosure policy. *Journal of Accounting Research 51*(5), 1133–1158. 21

Gervais, S. and G. Strobl (2021). Money management and real investment. *Working Paper*. 2

Giglio, S., B. Kelly, and J. Stroebel (2021). Climate finance. *Annual Review of Financial Economics 13*, 15–36. 2, 7, 3.1

Gillan, S. L., A. Koch, and L. T. Starks (2021). Firms and social responsibility: A review of esg and csr research in corporate finance. *Journal of Corporate Finance 66*, 101889. 2, 7

Goldstein, I., A. Kopytov, L. Shen, and H. Xiang (2021). On esg investing: Heterogeneous preferences, information, and asset prices. *Working paper*. 2, 12

Goldstein, I. and H. Sapra (2014). Should banks' stress test results be disclosed? an analysis of the costs and benefits. *Foundations and Trends in Finance 8*(1), 1–54. 1

Goldstein, I., J. Schneemeier, and L. Yang (2020). Market feedback: Who learns what? *Unpublished working paper*. 10, 2, 8

Goldstein, I. and L. Yang (2017). Information disclosure in financial markets. *Annual Review of Financial Economics 9*, 101–125. 1, 2

Goldstein, I. and L. Yang (2019). Good disclosure, bad disclosure. *Journal of Financial Economics 131*(1), 118–138. 8

Goldstein, I. and L. Yang (2022). Commodity financialization and information transmission. *Journal of Finance, Forthcoming*. 2

Graham, J. R. and C. R. Harvey (2001). The theory and practice of corporate finance: Evidence from the field. *Journal of Financial Economics 60*(2-3), 187–243. 3.1, 4.1

Grossman, S. and J. Stiglitz (1980). On the impossibility of informationally efficient markets. *American Economic Review 70*(3), 393–408. 2

Hapnes, E. (2020). Endogenous information acquisition with feedback effects and risk-aversion. Unpublished working paper, EPFL and Swiss Finance Institute. 2

Heinkel, R., A. Kraus, and J. Zechner (2001). The effect of green investment on corporate behavior. *Journal of Financial and Quantitative Analysis 36*(4), 431–449. 12

Henderson, H. and S. Searle (1981). On deriving the inverse of a sum of matrices. *SIAM Review 23*(1), 53–60. B.1, B.1, B.1, B.2, B.2

Hill, A. (November 14, 2021). Executive pay and climate: can bonuses be used to reduce emissions? *The Financial Times*. 9, 7

Hirshleifer, J. (1971). The private and social value of information and the reward to inventive activity. *American Economic Review*, 561–574. 2, 3.1, 18

Ilhan, E., P. Krueger, Z. Sautner, and L. T. Starks (2021). Climate risk disclosure and institutional investors. *Swiss Finance Institute Research Paper* (19-66). 1

Krueger, P., Z. Sautner, and L. T. Starks (2020). The importance of climate risks for institutional investors. *The Review of Financial Studies 33*(3), 1067–1111. 1

Larcker, D. F. and E. M. Watts (2020). Where's the greenium? *Journal of Accounting and Economics 69*(2-3), 101312. 7

Li, Q., H. Shan, Y. Tang, and V. Yao (2020). Corporate climate risk: Measurements and responses. *Available at SSRN 3508497*. 1

Liu, L. X., T. M. Whited, and L. Zhang (2009). Investment-based expected stock returns. *Journal of Political Economy 117*(6), 1105–1139. 3.1

Manzano, C. and X. Vives (2011). Public and private learning from prices, strategic substitutability and complementarity, and equilibrium multiplicity. *Journal of Mathematical Economics 47*(3), 346–369. 10

Marín, J. M. and R. Rahi (1999). Speculative securities. *Economic Theory 14*(3), 653–668. 11

Marín, J. M. and R. Rahi (2000). Information revelation and market incompleteness. *The Review of Economic Studies 67*(3), 563–579. 11, 3.1

Oehmke, M. and M. M. Opp (2020). A theory of socially responsible investment. *Swedish House of Finance Research Paper* (20-2). 12

Pastor, L., R. F. Stambaugh, and L. A. Taylor (2021). Dissecting green returns. Technical report, National Bureau of Economic Research. 2

Pástor, L., R. F. Stambaugh, and L. A. Taylor (2021). Sustainable investing in equilibrium. *Journal of Financial Economics 142*(2), 550–571. 2, 12, 3.1

Pedersen, L. H., S. Fitzgibbons, and L. Pomorski (2021). Responsible investing: The esg-efficient frontier. *Journal of Financial Economics 142*(2), 572–597. 12

Schneider, J. (2009, December). A rational expectations equilibrium with informative trading volume. *Journal of Finance 64*(6), 2783–2805. 10

Wang, J. (1994). A model of competitive stock trading volume. *Journal of Political Economy 102*(1), 127–168. 10

Zhang, L. (2005). The value premium. *The Journal of Finance 60*(1), 67–103. 3.1

A Proofs

A.1 Proof of Proposition 1

Begin by conjecturing an equilibrium of the form posited in the text. Suppose that there is a random variable of the form $s_p = \theta + \frac{1}{\beta}Z$ and threshold $\overline{s} \in \mathbb{R}$ such that the asset price at the two trading dates is identical and takes the form

$$P_3 = P_1 = \begin{cases} A_1 + B_1 s_p & s_p > \overline{s} \\ A_0 & s_p \leq \overline{s} \end{cases} \tag{28}$$

for constants A_t and B_t such that this function is strictly increasing in the investment region, $B_1 > 0$ weakly increasing overall, $A_1 + B_1\overline{s}_p \geq A_0$.

We can now derive the equilibrium, and confirm the above conjecture, by working backwards. At date $t = 3$, traders can observe the actual investment decision made at $t = 2$. Hence, they perceive the asset payoff as conditionally normally distributed with conditional moments

$$\mathbb{E}_{i3}[V(k)] = \mathbb{E}_{i3}[A + k(\theta + \alpha\eta_C + \sqrt{1-\alpha^2}\eta_I - c)] = \mu_A + k(\theta - c) \tag{29}$$

$$\mathbb{C}_{i3}(V(k), \eta_C) = k\alpha\frac{1}{\tau_\eta} \tag{30}$$

$$\mathbb{V}_{i3}(V(k)) = \frac{1}{\tau_A} + k^2\frac{1}{\tau_\eta} \tag{31}$$

The problem of an arbitrary trader at this date is a static one,

$$\max_{x\in\mathbb{R}} \mathbb{E}_{i3}[-e^{-\gamma W_{i4}}] \tag{32}$$

where the terminal wealth is

$$W_{i4} = (n + X_{i1} + x)\,V - xP_3 - X_{i1}P_1 - z_i\eta_C. \tag{33}$$

where X_{i1}, the trade from the $t = 1$ trading round, is taken as given.

It is immediate that this problem leads to a standard mean-variance demand function

$$X_{i3} = \frac{\mathbb{E}_{i3}[V(k)] + \gamma\mathbb{C}_{i3}(V(k), \eta_C)z_i - P_3}{\gamma\mathbb{V}_{i3}(V(k))} - (n + X_{i1}). \tag{34}$$

Plugging in for the conditional moments from above and enforcing market clearing yields equilibrium price

$$P_3 = \mu_A + k\,(\theta - c) + \gamma k\alpha\frac{1}{\tau_\eta}Z - \gamma\left(\frac{1}{\tau_A} + k^2\frac{1}{\tau_\eta}\right)n \tag{35}$$

$$= \mu_A - \gamma\frac{1}{\tau_A}n + k\left(\theta + \gamma\alpha\frac{1}{\tau_\eta}Z - c - \gamma\frac{1}{\tau_\eta}n\right) \tag{36}$$

where the second line collects terms and uses the fact that $k \in \{0, 1\}$ implies $k = k^2$ to simplify. Hence, to be consistent with our initial conjecture, the endogenous signal s_p has coefficient $\frac{1}{\beta} = \frac{\gamma\alpha}{\tau_\eta}$ on Z.

Stepping back to $t = 2$, the manager's problem is to solve

$$\max_{k \in \{0,1\}} \mathbb{E}[P_3 | \mathcal{F}_m] \tag{37}$$

where she can condition on the first period asset price, $\mathcal{F}_m = \sigma(P_1)$. Using the expression for P_3 derived above, the manager's problem reduces to

$$\max_{k \in \{0,1\}} k\mathbb{E}\left[s_p - c - \gamma \frac{1}{\tau_\eta} n \middle| \mathcal{F}_m \right] \tag{38}$$

The optimal investment is therefore

$$k = \begin{cases} 1 & \mathbb{E}[s_p | \mathcal{F}_m] > c + \gamma \frac{1}{\tau_\eta} n \\ 0 & \mathbb{E}[s_p | \mathcal{F}_m] \leq c + \gamma \frac{1}{\tau_\eta} n \end{cases} \tag{39}$$

Given the conjectured price function, if the manager observes $P_1 = A_0$, she infers $s_p \leq \overline{s}$, while if she observes any $P_1 > A_0$, she infers the realized value of s_p, necessarily strictly greater than $\overline{s}$. Hence, a threshold value $\overline{s} = c + \gamma \frac{1}{\tau_\eta} n$ is consistent with the initial conjecture.

In principle, any threshold $\overline{s}'$ such that (i) $\overline{s}' \geq c + \gamma \frac{1}{\tau_\eta} n$ and (ii) $\mathbb{E}[s_p | s_p \leq \overline{s}'] \leq c + \gamma \frac{1}{\tau_\eta} n$ is consistent with the conjecture. This is because (i) ensures that if the manager infers the realized value of s_p, necessarily strictly greater than $\overline{s}'$, her optimal action is $k = 1$, and (ii) ensures that if the manager infers $s_p \leq \overline{s}'$, her optimal action is $k = 0$. Our particular choice of $\overline{s} = c + \gamma \frac{1}{\tau_\eta} n$ is the unique such threshold that leads to a price function that is continuous at the threshold. Furthermore, with the equilibrium investment function pinned down, returning to the expression for P_3 derived above, to be consistent with our conjecture, the price coefficients must satisfy

$$A_0 = \mu_A - \gamma \frac{1}{\tau_A} n \tag{40}$$

$$A_1 = \mu_A - \gamma \frac{1}{\tau_A} n - c - \gamma \frac{1}{\tau_\eta} n \tag{41}$$

$$B_1 = 1. \tag{42}$$

Finally, stepping back to $t = 1$, the problem of an arbitrary trader is

$$\max_{x \in \mathbb{R}} \mathbb{E}_{i1}[-e^{-\gamma W_{i4}}] \tag{43}$$

where the terminal wealth is

$$W_{i4} = (n + x + X_{i3}) V - X_{i3} P_3 - x P_1 - z_i \eta_C. \tag{44}$$

and where the optimal $t = 3$ demand X_{i3} was derived above. Given the functional form for P_3, the realization of P_3 is perfectly anticipated under the trader information set $\mathcal{F}_{i1} = \sigma(\theta, z_i, P_1)$. Hence, to rule out arbitrage, the price must satisfy $P_1 = P_3$, and consequently all traders are indifferent to trading at $t = 1$ at this equilibrium price. This completes the construction of equilibrium. $\square$

A.2 Proof of Proposition 2

The probability of investment is given by

$$\Pr(s_p > \bar{s}) = 1 - \Phi\left(\frac{\bar{s} - \mathbb{E}[s_p]}{\mathbb{V}[s_p]}\right) = \Phi\left(\frac{\mathbb{E}[s_p] - \bar{s}}{\sqrt{\mathbb{V}[s_p]}}\right) \tag{45}$$

$$= \Phi\left(\frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}}\right) \tag{46}$$

This immediately implies probability of investment is increasing in $\mu_\theta - c$, decreasing in n, increasing in μ_Z. Moreover, given that Φ is strictly increasing, for any arbitrary parameter b we have, after applying the monotonic transformation $\Phi^{-1}(\cdot)$ and using the definition $NPV = \theta - c - \gamma\frac{1}{\tau_\eta}(n - \alpha Z)$ from the text to condense notation

$$\frac{\partial}{\partial b}\Pr(s_p > \bar{s}) \propto \frac{\partial}{\partial b}\frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \tag{47}$$

$$= \frac{\partial}{\partial b}\frac{\mathbb{E}[NPV]}{\sqrt{\mathbb{V}(NPV)}} \tag{48}$$

$$= \frac{\sqrt{\mathbb{V}(NPV)}\frac{\partial}{\partial b}\mathbb{E}[NPV] - \mathbb{E}[NPV]\frac{\partial}{\partial b}\sqrt{\mathbb{V}(NPV)}}{\mathbb{V}(NPV)} \tag{49}$$

$$= \frac{|\mathbb{E}[NPV]|}{\sqrt{\mathbb{V}(NPV)}}\left(\frac{\frac{\partial}{\partial b}\mathbb{E}[NPV]}{|\mathbb{E}[NPV]|} - \operatorname{sgn}(\mathbb{E}[NPV])\frac{\frac{\partial}{\partial b}\sqrt{\mathbb{V}(NPV)}}{\sqrt{\mathbb{V}(NPV)}}\right) \tag{50}$$

$$= \frac{|\mathbb{E}[NPV]|}{\sqrt{\mathbb{V}(NPV)}}\left(\frac{\frac{\partial}{\partial b}\mathbb{E}[NPV]}{|\mathbb{E}[NPV]|} - \frac{1}{2}\operatorname{sgn}(\mathbb{E}[NPV])\frac{\frac{\partial}{\partial b}\mathbb{V}(NPV)}{\mathbb{V}(NPV)}\right). \tag{51}$$

For α we have

$$\frac{\partial}{\partial \alpha}\Pr(s_p > \bar{s}) \propto \left(\frac{\partial}{\partial \alpha}\mathbb{E}[NPV] - \frac{1}{2}\mathbb{E}[NPV]\frac{\frac{\partial}{\partial \alpha}\mathbb{V}(NPV)}{\mathbb{V}(NPV)}\right) \tag{52}$$

$$= \frac{\gamma\mu_Z}{\tau_\eta} - \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta}\right)\frac{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}} \tag{53}$$

$$= \frac{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\left(\frac{\gamma\mu_Z}{\tau_\eta}\frac{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}} - \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta}\right)\right) \tag{54}$$

$$= \frac{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\left(\frac{\alpha\gamma\mu_Z}{\tau_\eta}\frac{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}} - \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta}\right)\right) \tag{55}$$

$$= \frac{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\left(\frac{\alpha\gamma\mu_Z}{\tau_\eta}\left(1 + \frac{\frac{1}{\tau_\theta}}{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\right) - \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\alpha\gamma\mu_Z}{\tau_\eta}\right)\right) \tag{56}$$

$$= \frac{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}} \left(\left(\frac{\frac{1}{\tau_\theta}}{\left(\frac{\alpha\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}\right)\mu_Z - \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta}\right)\right) \tag{57}$$

$$= -\frac{\frac{1}{\alpha}\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}} \left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} - \frac{\tau_\eta \tau_Z}{\alpha\gamma}\frac{1}{\tau_\theta}\mu_Z\right) \tag{58}$$

which implies

$$\frac{\partial}{\partial\alpha}\Pr\left(s_p > \bar{s}\right) < 0 \Leftrightarrow \operatorname{sgn}(\alpha)\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta} - \frac{\tau_\eta \tau_Z}{\alpha\gamma}\frac{1}{\tau_\theta}\mu_Z\right) > 0. \tag{59}$$

Moreover, note that because the parameters $\tau \in \{\tau_Z, \tau_\theta\}$ do not enter the expected NPV and increases in these τ strictly decrease the variance of the NPV, we have

$$\frac{\partial}{\partial\tau_Z}\Pr\left(s_p > \bar{s}\right), \frac{\partial}{\partial\tau_\theta}\Pr\left(s_p \geq \bar{s}\right) \propto -\tfrac{1}{2}\operatorname{sgn}\left(\mathbb{E}\left[NPV\right]\right)\frac{\frac{\partial}{\partial\tau}\mathbb{V}(NPV)}{\mathbb{V}(NPV)} \tag{60}$$

$$\propto \operatorname{sgn}\left(\mathbb{E}\left[NPV\right]\right) \tag{61}$$

so that the dependence is pinned down by the sign of the expected NPV, which immediately establishes the claimed result. □

A.3 Proof of Corollary 1

The first inequality follows from comparing the probabilities from Proposition 1, evaluated at $\alpha = 1$ vs. $\alpha = 0$:

$$\Phi\left(\frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\gamma\mu_Z}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right) > \Phi\left(\frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}}}\right) \tag{62}$$

$$\Leftrightarrow \frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\gamma\mu_Z}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}} > \frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}}} \tag{63}$$

$$\Leftrightarrow \mu_\theta - c - \frac{\gamma n}{\tau_\eta} + \frac{\gamma\mu_Z}{\tau_\eta} > \sqrt{1 + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{\tau_\theta}{\tau_Z}}\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta}\right) \tag{64}$$

$$\Leftrightarrow \frac{\gamma\mu_Z}{\tau_\eta} > \left(\sqrt{1 + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{\tau_\theta}{\tau_Z}} - 1\right)\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta}\right) \tag{65}$$

Similarly, the second inequality follows from comparing the probabilities from Proposition

1, evaluated at $\alpha = 0$ vs. $\alpha = -1$:

$$\Phi\left(\frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}}}\right) > \Phi\left(\frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta} - \frac{\gamma \mu_Z}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}}\right) \tag{66}$$

$$\Leftrightarrow \frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}}} > \frac{\mu_\theta - c - \frac{\gamma n}{\tau_\eta} - \frac{\gamma \mu_Z}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \tag{67}$$

$$\Leftrightarrow \sqrt{1 + \left(\frac{\gamma}{\tau_\eta}\right)^2 \frac{\tau_\theta}{\tau_Z}}\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta}\right) > \mu_\theta - c - \frac{\gamma n}{\tau_\eta} - \frac{\gamma \mu_Z}{\tau_\eta} \tag{68}$$

$$\Leftrightarrow \frac{\gamma \mu_Z}{\tau_\eta} > -\left(\sqrt{1 + \left(\frac{\gamma}{\tau_\eta}\right)^2 \frac{\tau_\theta}{\tau_Z}} - 1\right)\left(\mu_\theta - c - \frac{\gamma n}{\tau_\eta}\right). \tag{69}$$

A.4 Statement and Proof of Lemma 1

The following Lemma will be useful for proving some of the results from the body of the paper.

Lemma 1. *Define*

$$\Gamma = \mathbb{E}[s_p | s_p > \overline{s}]. \tag{70}$$

We have

$$\Gamma = \mathbb{E}\left[s_p\right] + \sqrt{\mathbb{V}\left[s_p\right]} H\left(\frac{\bar{s} - \mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right), \tag{71}$$

where $H(x) = \frac{\phi(x)}{1 - \Phi(x)}$ is the hazard ratio for the standard normal distribution. Moreover, Γ is increasing with μ_θ, c, μ_Z and n, and is increasing in α for $\alpha \geq 0$ if $\mu_Z > 0$.

The expression for Γ follows from standard results for the expectation of a truncated normal random variable. To derive the comparative statics results, note that by plugging in the explicit expressions for the threshold $\overline{s}$ and the moments of s_p, we can express Γ as

$$\Gamma = \mu_\theta + \frac{\alpha \gamma \mu_Z}{\tau_\eta} + \sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}\, H\left(-\frac{\mu_\theta - c + \frac{\alpha \gamma \mu_Z}{\tau_\eta} - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}}\right). \tag{72}$$

Note that $H(x) > 0$ and $H'(x) \in (0, 1)$. This immediately implies that Γ is increasing in μ_θ, c, μ_Z, n. To prove the claim for α, let $A \equiv \mathbb{E}\left[s_p\right] = \mu_\theta + \frac{\alpha\gamma\mu_Z}{\tau_\eta}$ and $B \equiv \sqrt{\mathbb{V}\left[s_p\right]} =$

$\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}$. Then,

$$\Gamma = A + BH\left(\frac{\bar{s}-A}{B}\right) \tag{73}$$

$$\Leftrightarrow \tfrac{\partial}{\partial\alpha}\Gamma = \tfrac{\partial}{\partial\alpha}A + \left(\tfrac{\partial}{\partial\alpha}B\right)H\left(\frac{\bar{s}-A}{B}\right) + BH'\left(\frac{\bar{s}-A}{B}\right)\times\left(\frac{-B\tfrac{\partial}{\partial\alpha}A-(\bar{s}-A)\tfrac{\partial}{\partial\alpha}B}{B^2}\right) \tag{74}$$

$$= \left(\tfrac{\partial}{\partial\alpha}A\right)\left(1-H'\left(\frac{\bar{s}-A}{B}\right)\right) + \left(\tfrac{\partial}{\partial\alpha}B\right)\left(H\left(\frac{\bar{s}-A}{B}\right)-\left(\frac{\bar{s}-A}{B}\right)H'\left(\frac{\bar{s}-A}{B}\right)\right). \tag{75}$$

Note that $H'(x)\in(0,1)$ and $H(x)-xH'(x)>0$, and that

$$\tfrac{\partial}{\partial\alpha}A = \tfrac{\gamma\mu_Z}{\tau_\eta} \tag{76}$$

$$\tfrac{\partial}{\partial\alpha}B = \tfrac{1}{2}\frac{2\alpha\left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}, \tag{77}$$

which are (weakly) positive if $\mu_Z\geq 0$ and $\alpha\geq 0$, respectively. We conclude that $\frac{\partial}{\partial\alpha}\Gamma>0$ for $\mu_Z>0$ and $\alpha\geq 0$. □

A.5 Proof of Proposition 3

Using the expression for the equilibrium asset price in eq. (11), the expressions for conditional expected return given no investment and investment are straightforward.

To derive the comparative statics results for the expected return conditional on investment, note that we can write:

$$\mathbb{E}\left[V-P_3|k=1\right] = \mathbb{E}\left[V-P_3|s_p>\bar{s}\right] \tag{78}$$

$$= \gamma n\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)-\frac{\gamma\alpha}{\tau_\eta}\mathbb{E}\left[Z|s_p>\bar{s}\right] \tag{79}$$

$$= \gamma n\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)-\mathbb{E}\left[s_p-\theta|s_p>\bar{s}\right] \tag{80}$$

$$= \gamma n\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)-\mathbb{E}\left[s_p-\left(\mu_\theta+\frac{\frac{1}{\tau_\theta}}{\frac{1}{\tau_\theta}+\frac{1}{\tau_p}}\left(s_p-\mu_\theta-\frac{\gamma\alpha}{\tau_\eta}\mu_Z\right)\right)\middle| s_p>\bar{s}\right] \tag{81}$$

$$= \gamma n\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)+\frac{\frac{1}{\tau_p}}{\frac{1}{\tau_\theta}+\frac{1}{\tau_p}}\mu_\theta-\frac{\frac{1}{\tau_\theta}}{\frac{1}{\tau_\theta}+\frac{1}{\tau_p}}\frac{\gamma\alpha}{\tau_\eta}\mu_Z-\frac{\frac{1}{\tau_p}}{\frac{1}{\tau_\theta}+\frac{1}{\tau_p}}\Gamma, \tag{82}$$

where the third equality uses the law of iterated expectations to write $\mathbb{E}[\theta|s_p>\bar{s}]=\mathbb{E}[\mathbb{E}[\theta|s_p]|s_p>\bar{s}]$, and where the last line collects terms and uses where $\Gamma=\mathbb{E}\left[s_p|s_p>\bar{s}\right]$ as in Lemma 1.

Now, plugging in for Γ from Lemma 1 and grouping terms further yields

$$\mathbb{E}\left[V-P_3|k=1\right]=\gamma n\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)-\frac{\gamma\alpha}{\tau_\eta}\left(\mu_Z+\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}{\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}H\left(-\frac{\mu_\theta-c+\frac{\alpha\gamma\mu_Z}{\tau_\eta}-\frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right). \tag{83}$$

This immediately implies $\mathbb{E}\left[V-P_3|k=1\right]$ is increasing in $\mu_\theta-c$. Further, it is decreasing in μ_Z for $\alpha>0$ and increasing for $\alpha<0$ since

$$\frac{\partial}{\partial\mu_Z}\mathbb{E}\left[V-P_3|k=1\right]=-\frac{\gamma\alpha}{\tau_\eta}\left(1-\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}H'\left(-\frac{\mu_\theta-c+\frac{\alpha\gamma\mu_Z}{\tau_\eta}-\frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right)<0 \tag{84}$$

because $0<H'<1$.

Now, consider n and note that

$$\frac{\partial}{\partial n}\mathbb{E}\left[V-P_3|k=1\right]=\gamma\left(\frac{1}{\tau_A}+\frac{1}{\tau_\eta}\right)-\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\frac{\gamma}{\tau_\eta}H'\left(-\frac{\mu_\theta-c+\frac{\alpha\gamma\mu_Z}{\tau_\eta}-\frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right) \tag{85}$$

$$=\gamma\frac{1}{\tau_A}+\gamma\frac{1}{\tau_\eta}\left(1-\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}H'\left(-\frac{\mu_\theta-c+\frac{\alpha\gamma\mu_Z}{\tau_\eta}-\frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right) \tag{86}$$

$$\geq 0 \tag{87}$$

again because $0<H'<1$.

Considering α, following the proof of Lemma 1, let $A\equiv\mathbb{E}\left[s_p\right]=\mu_\theta+\frac{\alpha\gamma\mu_Z}{\tau_\eta}$ and $B\equiv\sqrt{\mathbb{V}\left[s_p\right]}=\sqrt{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}$. We can write

$$\frac{\partial}{\partial\alpha}\mathbb{E}\left[V-P_3|k=1\right]=\frac{\partial}{\partial\alpha}\left(-\frac{\gamma\alpha}{\tau_\eta}\left(\mu_Z+\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}{B}H\left(\frac{\overline{s}-A}{B}\right)\right)\right) \tag{88}$$

$$=\frac{\partial}{\partial\alpha}\left(\left(-\frac{\gamma\alpha}{\tau_\eta}\mu_Z-\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}BH\left(\frac{\overline{s}-A}{B}\right)\right)\right) \tag{89}$$

$$= -\frac{\gamma}{\tau_\eta}\mu_Z - \frac{\partial}{\partial\alpha}\left(\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\right) BH\left(\frac{\overline{s}-A}{B}\right) \tag{90}$$

$$- \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\frac{\partial}{\partial\alpha}\left(BH\left(\frac{\overline{s}-A}{B}\right)\right). \tag{91}$$

We clearly have $-\frac{\partial}{\partial\alpha}\left(\frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\right) BH\left(\frac{\overline{s}-A}{B}\right) < 0$ as long as $\alpha > 0$, so it remains to establish that the remaining terms are, collectively, negative. Note that we can express $-\frac{\gamma}{\tau_\eta}\mu_Z = -\frac{\partial}{\partial\alpha}A$, so we want to sign

$$-\frac{\partial}{\partial\alpha}A - \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\frac{\partial}{\partial\alpha}\left(BH\left(\frac{\overline{s}-A}{B}\right)\right). \tag{92}$$

Note that we have

$$\begin{aligned}
\frac{\partial}{\partial\alpha}BH\left(\frac{\overline{s}-A}{B}\right) &= \left(\frac{\partial}{\partial\alpha}B\right)H\left(\frac{\overline{s}-A}{B}\right) + BH'\left(\frac{\overline{s}-A}{B}\right)\frac{\partial}{\partial\alpha}\frac{\overline{s}-A}{B} \\
&= \left(\frac{\partial}{\partial\alpha}B\right)H\left(\frac{\overline{s}-A}{B}\right) + BH'\left(\frac{\overline{s}-A}{B}\right)\left[\frac{-B\frac{\partial}{\partial\alpha}A-(\overline{s}-A)\frac{\partial}{\partial\alpha}B}{B^2}\right] \\
&= \left(\frac{\partial}{\partial\alpha}B\right)\left(H\left(\frac{\overline{s}-A}{B}\right) - \frac{\overline{s}-A}{B}H'\left(\frac{\overline{s}-A}{B}\right)\right) - \left(\frac{\partial}{\partial\alpha}A\right)H'\left(\frac{\overline{s}-A}{B}\right)
\end{aligned}$$

and plugging in to eq. (92) now yields

$$-\frac{\partial}{\partial\alpha}A - \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\left(\left(\frac{\partial}{\partial\alpha}B\right)\left(H\left(\frac{\overline{s}-A}{B}\right) - \frac{\overline{s}-A}{B}H'\left(\frac{\overline{s}-A}{B}\right)\right) - \left(\frac{\partial}{\partial\alpha}A\right)H'\left(\frac{\overline{s}-A}{B}\right)\right) \tag{93}$$

$$= -\left(1 - \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}H'\right)\frac{\partial}{\partial\alpha}A - \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}\left(H\left(\frac{\overline{s}-A}{B}\right) - \frac{\overline{s}-A}{B}H'\left(\frac{\overline{s}-A}{B}\right)\right)\left(\frac{\partial}{\partial\alpha}B\right). \tag{94}$$

We know from the proof of Lemma 1 that $\frac{\partial}{\partial\alpha}A > 0$ if $\mu_Z > 0$ and $\frac{\partial}{\partial\alpha}B \geq 0$ for $\alpha \geq 0$. Furthermore, we always have $H' \in (0,1)$ $H(x) - xH'(x) \geq 0$. It follows therefore that the expression in eq. (92) is negative and hence that $\frac{\partial}{\partial\alpha}\mathbb{E}\left[V - P_3 | k = 1\right] < 0$ for $\mu_Z > 0$ and $\alpha \geq 0$.

Finally, consider τ_θ. It will be more convenient to study dependence on the variance $1/\tau_\theta$.

We have

$$\begin{aligned}
\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\mathbb{E}\left[V-P_3|k=1\right] &= -\frac{\gamma\alpha}{\tau_\eta}\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\mathbb{E}\left[Z|s_p\geq\bar{s}\right] \\
&= -\frac{\gamma\alpha}{\tau_\eta}\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\left(\mu_Z+\frac{\frac{1}{\beta}\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\frac{1}{\beta^2}\frac{1}{\tau_Z}}\left(\mathbb{E}\left[s_p|s_p\geq\bar{s}\right]-\mathbb{E}\left[s_p\right]\right)\right) \\
&= -\frac{\gamma\alpha}{\tau_\eta}\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\left(\mu_Z+\frac{\frac{1}{\beta}\frac{1}{\tau_Z}}{\frac{1}{\tau_\theta}+\frac{1}{\beta^2}\frac{1}{\tau_Z}}\sqrt{\mathbb{V}\left[s_p\right]}H\left(\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right)\right) \\
&= -\left(\frac{\gamma\alpha}{\tau_\eta}\right)^2\frac{1}{\tau_Z}\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\left(\frac{1}{\sqrt{\mathbb{V}[s_p]}}H\left(\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right)\right).
\end{aligned}$$

Hence, the derivative of expected returns with respect to $1/\tau_\theta$ has the opposite sign of $\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\left(\frac{1}{\sqrt{\mathbb{V}[s_p]}}H\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)\right)$. Applying the monotonic transformation $\log(\cdot)$ to $\frac{1}{\sqrt{\mathbb{V}[s_p]}}H\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)$ and differentiating yields

$$\begin{aligned}
\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\log\left(\frac{1}{\sqrt{\mathbb{V}[s_p]}}H\left(\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right)\right) &= -\frac{1}{2}\frac{1}{\mathbb{V}[s_p]}+\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\log\left(H\left(\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right)\right) \\
&= -\frac{1}{2}\frac{1}{\mathbb{V}[s_p]}+\frac{H'\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)}{H\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)}\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}} \\
&= -\frac{1}{2}\frac{1}{\mathbb{V}[s_p]}-\frac{1}{2}\frac{1}{\mathbb{V}[s_p]}\frac{H'\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)}{H\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)}\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}} \\
&= -\frac{1}{2}\frac{1}{\mathbb{V}[s_p]}\left(1+\frac{H'\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)}{H\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)}\frac{\bar{s}-\mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right).
\end{aligned}$$

Let $K<0$ be the unique root of the function $1+\frac{H'(K)}{H(K)}K$ and note that this function crosses zero from below as its argument increases, so that it is strictly negative for points below K and strictly positive for points above K.

We conclude that $\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\log\left(\frac{\frac{1}{\beta}\frac{1}{\tau_Z}}{\sqrt{\mathbb{V}[s_p]}}H\left(\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}\right)\right)>0$ if and only if $\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{\mathbb{V}[s_p]}}<K$. From this, it follows that $\frac{\partial}{\partial\left(\frac{1}{\tau_\theta}\right)}\mathbb{E}\left[V-P_3|k=1\right]<0$ and hence $\frac{\partial}{\partial\tau_\theta}\mathbb{E}\left[V-P_3|k=1\right]>0$. $\square$

A.6 Proof of Corollary 2

The difference between expected returns in the case $\alpha = 1$ vs. $\alpha = 0$ follows from plugging in the expressions from Proposition 3:

$$\Delta_{R+} = \mathbb{E}\left[V - P_3|k = 1; \alpha = 1\right] - \mathbb{E}\left[V - P_3|k = 1; \alpha = 0\right] \tag{95}$$

$$= \mathbb{E}\left[V - P_3|k = 1; \alpha = 1\right] - \gamma n\left(\frac{1}{\tau_A} + \frac{1}{\tau_\eta}\right) \tag{96}$$

$$= -\frac{\gamma\alpha}{\tau_\eta}\left(\mu_Z + \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}} H\left(-\frac{\mu_\theta - c + \frac{\alpha\gamma\mu_Z}{\tau_\eta} - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right)\Bigg|_{\alpha=1} \tag{97}$$

$$= -\frac{\gamma}{\tau_\eta}\left(\mu_Z + \frac{\left(\frac{\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}} H\left(-\frac{\mu_\theta - c + \frac{\gamma\mu_Z}{\tau_\eta} - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right) \tag{98}$$

where the next-to-last line follows from substituting from eq. (83). It is now immediate from inspection that if $\mu_Z > 0$ then $\Delta_{R+} < 0$. The claimed comparative static results follow directly from Proposition 3, applied to $\mathbb{E}\left[V - P_3|k = 1; \alpha = 1\right]$.

The difference between expected returns in the case $\alpha = 0$ vs. $\alpha = -1$ also follows from plugging in the expressions from Proposition 3:

$$\Delta_{R-} = \mathbb{E}\left[V - P_3|k = 1; \alpha = 0\right] - \mathbb{E}\left[V - P_3|k = 1; \alpha = -1\right] \tag{99}$$

$$= \gamma n\left(\frac{1}{\tau_A} + \frac{1}{\tau_\eta}\right) - \mathbb{E}\left[V - P_3|k = 1; \alpha = -1\right] \tag{100}$$

$$= \frac{\gamma\alpha}{\tau_\eta}\left(\mu_Z + \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}} H\left(-\frac{\mu_\theta - c + \frac{\alpha\gamma\mu_Z}{\tau_\eta} - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right)\Bigg|_{\alpha=-1} \tag{101}$$

$$= -\frac{\gamma}{\tau_\eta}\left(\mu_Z + \frac{-\left(\frac{\gamma}{\tau_\eta}\right)\frac{1}{\tau_Z}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}} H\left(-\frac{\mu_\theta - c - \frac{\gamma\mu_Z}{\tau_\eta} - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2\frac{1}{\tau_Z}}}\right)\right) \tag{102}$$

where the next-to-last line follows from substituting from eq. (83). Note that only the first μ_Z in the parentheses has the potential to make this overall expression negative, if μ_Z is sufficiently large. Hence, to establish the claim in the Corollary that $\Delta_{R-} > 0$ if and only if μ_Z is sufficiently small, it suffices to show that this expression (i) is monotonically decreasing in μ_Z and (ii) is strictly positive for $\mu_Z \to 0$. Differentiating with respect to μ_Z, we obtain:

$$\frac{\partial \Delta_{R-}}{\partial \mu_Z} = -\frac{\gamma^3 \tau_\theta \left(1 - H'\left(-\frac{\mu_\theta - c - \frac{\gamma \mu_Z}{\tau_\eta} - \frac{\gamma n}{\tau_\eta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}}\right)\right) + \gamma \tau_\eta^2 \tau_Z}{\gamma^2 \tau_\eta \tau_\theta + \tau_\eta^3 \tau_Z}, \tag{103}$$

which, since $H' \in (0,1)$, is strictly negative. Furthermore, it is immediate from inspection that $\lim_{\mu_Z \to 0} \Delta_{R-} > 0$. This establishes implies the claimed result. □

A.7 Proof of Proposition 4

$$\begin{aligned}
\Delta V &= \mathbb{E}\left[V|k=1\right] - \mathbb{E}\left[V|k=0\right] && (104)\\
&= \mathbb{E}\left[V|s_p > \bar{s}\right] - \mathbb{E}\left[V|s_p \leq \bar{s}\right] && (105)\\
&= \mathbb{E}\left[\theta - c|s_p > \bar{s}\right] && (106)\\
&= \mathbb{E}\left[\mu_\theta + \frac{\tau_\theta^{-1}}{\mathbb{V}\left[s_p\right]}\left(s_p - \mathbb{E}\left[s_p\right]\right)\middle| s_p > \bar{s}\right] - c && (107)\\
&= \mu_\theta - c + \frac{\tau_\theta^{-1}}{\mathbb{V}\left[s_p\right]}\sqrt{\mathbb{V}\left[s_p\right]}H\left(\frac{\bar{s} - \mathbb{E}\left[s_p\right]}{\sqrt{\mathbb{V}\left[s_p\right]}}\right) && (108)\\
&= \mu_\theta - c + \frac{\frac{1}{\tau_\theta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}}H\left(-\frac{\mu_\theta - c - \frac{\gamma}{\tau_\eta}\left(n - \alpha\mu_Z\right)}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}}\right) && (109)
\end{aligned}$$

where the fourth equality follows from the law of iterated expectations after conditioning on s_p, and the fifth equality substitutes in for $\mathbb{E}[s_p|s_p > \bar{s}]$ from Lemma 1.

When $\alpha = 0$, we have

$$\begin{aligned}
\Delta V &= \mu_\theta - c + \frac{1}{\sqrt{\tau_\theta}}H\left(-\sqrt{\tau_\theta}\left(\mu_\theta - c - \frac{\gamma}{\tau_\eta}n\right)\right) && (110)\\
&= \frac{1}{\sqrt{\tau_\theta}}\left(\sqrt{\tau_\theta}\left(\mu_\theta - c\right) + H\left(-\sqrt{\tau_\theta}\left(\mu_\theta - c\right) + \sqrt{\tau_\theta}\frac{\gamma}{\tau_\eta}n\right)\right) && (111)\\
&\geq \frac{1}{\sqrt{\tau_\theta}}\left(\sqrt{\tau_\theta}\left(\mu_\theta - c\right) + H\left(-\sqrt{\tau_\theta}\left(\mu_\theta - c\right)\right)\right) && (112)\\
&> 0 && (113)
\end{aligned}$$

since $x + H(-x) > 0$ for all x.

Next, supposing that $\alpha \neq 0$, consider the behavior of ΔV as $\mu_\theta - c$ becomes arbitrarily negative. We have

$$\lim_{\mu_\theta - c \to -\infty} \Delta V \tag{114}$$

$$= \lim_{\mu_\theta - c \to -\infty} \left(\mu_\theta - c + \frac{\frac{1}{\tau_\theta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} H \left(-\frac{\mu_\theta - c - \frac{\gamma}{\tau_\eta}(n - \alpha\mu_Z)}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \right) \right) \tag{115}$$

$$= \lim_{\mu_\theta - c \to -\infty} (\mu_\theta - c) \left(1 + \frac{\frac{1}{\tau_\theta}}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \frac{H \left(-\frac{\mu_\theta - c - \frac{\gamma}{\tau_\eta}(n - \alpha\mu_Z)}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \right)}{\mu_\theta - c} \right) \tag{116}$$

$$= \lim_{\mu_\theta - c \to -\infty} (\mu_\theta - c) \left(1 - \frac{\frac{1}{\tau_\theta}}{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}} \right) \tag{117}$$

$$= \frac{\left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}} \lim_{\mu_\theta - c \to -\infty} (\mu_\theta - c) \tag{118}$$

$$= -\infty \tag{119}$$

where the next-to-last equality follows from continuity and the fact that $\lim_{x\to-\infty} \frac{H(-a(x-b))}{x} = -\lim_{x\to\infty} \frac{H(a(x+b))}{x} = -a$ for any $a > 0, b \in \mathbb{R}$. Hence, if $\alpha \neq 0$, then for $\mu_\theta - c$ sufficiently negative, we have $\Delta V < 0$.

Further, differentiating yields

$$\frac{\partial}{\partial (\mu_\theta - c)} \Delta V = 1 - \frac{\frac{1}{\tau_\theta}}{\frac{1}{\tau_\theta} + \left(\frac{\gamma\alpha}{\tau_\eta}\right)^2 \frac{1}{\tau_z}} H' \left(-\frac{\mu_\theta - c - \frac{\gamma}{\tau_\eta}(n - \alpha\mu_Z)}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \right) \tag{120}$$

and since $H'(x) \in (0, 1)$, we have $\frac{\partial}{\partial(\mu_\theta - c)} \Delta V \in (0, 1)$.

Moreover,

$$\frac{\partial}{\partial \mu_Z} \Delta V = -\frac{\gamma\alpha}{\tau_\eta} \frac{\frac{1}{\tau_\theta}}{\frac{1}{\tau_\theta} + \left(\frac{\gamma\alpha}{\tau_\eta}\right)^2 \frac{1}{\tau_z}} H' \left(-\frac{\mu_\theta - c - \frac{\gamma}{\tau_\eta}(n - \alpha\mu_Z)}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \right) \tag{121}$$

which is negative for $\alpha > 0$ and positive for $\alpha < 0$.

Finally,

$$\frac{\partial}{\partial n} \Delta V = \frac{\gamma}{\tau_\eta} \frac{\frac{1}{\tau_\theta}}{\frac{1}{\tau_\theta} + \left(\frac{\gamma\alpha}{\tau_\eta}\right)^2 \frac{1}{\tau_z}} H' \left(-\frac{\mu_\theta - c - \frac{\gamma}{\tau_\eta}(n - \alpha\mu_Z)}{\sqrt{\frac{1}{\tau_\theta} + \left(\frac{\alpha\gamma}{\tau_\eta}\right)^2 \frac{1}{\tau_Z}}} \right) > 0. \tag{122}$$

$\square$

A.8 Proof of Proposition 5

If the unconditional NPV is positive $\mathbb{E}[s_p]-\overline{s}>0$, then a manager who does not condition on price optimally invests in all states of the world, leading to 'no feedback' investment $k_{NF}=1$. Hence, if $\mathbb{E}[s_p]>\overline{s}$, then to establish that feedback reduces welfare, it suffices to show that welfare is higher with $k_{NF}=1$ than with the equilibrium investment rule $k(s_p)=\mathbf{1}_{\{s_p>\overline{s}\}}$.

The small n limit in the Proposition is easier to establish by considering the limit of the s_p-conditional welfare in Proposition 7 state-by-state, while the small τ_ζ limit is easier to establish using the unconditional welfare expression in Proposition 8 directly.

We proceed first with the claim for $n\to 0$. Note that the difference in unconditional welfare can be written as the difference in the expected values of the conditional welfare expressions:

$$\begin{aligned}
&\mathcal{W}(\text{No Feedback})-\mathcal{W}(\text{Feedback})\\
&=\mathbb{E}\left[\mathcal{W}(k_{NF};s_p)-\mathcal{W}(k(s_p);s_p)\right]\\
&=\mathbb{E}\left[\mathcal{W}(1;s_p)-\mathcal{W}(k(s_p);s_p)\right]\\
&=\mathbb{E}\left[\mathbf{1}_{\{s_p\leq\overline{s}\}}\left(\mathcal{W}(1;s_p)-\mathcal{W}(k(s_p);s_p)\right)\right]
\end{aligned}\tag{123}$$

where the next-to-last line substitutes in $k_{NF}=1$, and the last line uses the fact that $k(s_p)=k_{NF}=1$ in states $s_p>\overline{s}$, which leads to identical conditional expected utilities in such states.

Now, for any state $s_p\leq\overline{s}$, consider

$$\begin{aligned}
\mathcal{W}(1;s_p)-\mathcal{W}(0;s_p)&=-D(1;s_p)\exp\left\{Q(1;s_p)\right\}-\left(-D(0;s_p)\exp\left\{Q(0;s_p)\right\}\right)\\
&=D(0;s_p)\exp\left\{Q(0;s_p)\right\}-D(1;s_p)\exp\left\{Q(1;s_p)\right\}.
\end{aligned}$$

If we can show that in the $n\to 0$ limit, this expression is positive, and can justify passing the limit through the expectation in eq.(123), this will establish that welfare is higher with no feedback than with feedback.

Because n does not enter $D(k;s_p)$ and $D(k;s_p)$ is strictly decreasing in k, it suffices to show that $Q(0;s_p)>Q(1;s_p)$ in the limit. We have

$$\begin{aligned}
&\lim_{n\to 0}Q\left(k;s_p\right)\\
&=\lim_{n\to 0}\Big\{-\gamma\left(\mu_A+k\left(\mathbb{E}_p[\theta]-c\right)\right)n\\
&\quad+\tfrac{1}{2}\gamma^2\left(\left(\tfrac{1}{\tau_A}+k^2\left(\mathbb{V}_p(\theta|z_i)+\tfrac{1}{\tau_\eta}\right)\right)n^2-2k\alpha\tfrac{1}{\tau_\eta}n\mathbb{E}_p[Z]+\tfrac{1}{\tau_\eta}\mathbb{E}_p^2[Z]\right)\\
&\quad+\tfrac{1}{2}\gamma^2\left(\frac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n+\gamma k\alpha\frac{1}{\tau_\eta}n-\gamma\frac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)'\\
&\qquad\times\left(\frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1}
\end{aligned}$$

$$\times\left(\frac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n+\gamma k\alpha\frac{1}{\tau_\eta}n-\gamma\frac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)\Bigg\}$$

$$=\frac{1}{2}\gamma^2\frac{1}{\tau_\eta}\mathbb{E}_p^2[Z]+\frac{1}{2}\gamma^2\left(\gamma\frac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)\left(\frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1}\left(\gamma\frac{1}{\tau_\eta}\mathbb{E}_p[Z]\right).$$

It is immediate from inspection of this expression that $\lim_{n\to 0}Q(1;s_p)<\lim_{n\to 0}Q(0;s_p)$. Interchanging the limit and expectation in eq. (123) follows from the dominated convergence theorem. To see this, note that

$$\begin{aligned}&\left|\mathbf{1}_{\{s_p\le\overline{s}\}}D(0;s_p)\exp\{Q(0;s_p)\}-\mathbf{1}_{\{s_p\le\overline{s}\}}D(1;s_p)\exp\{Q(1;s_p)\}\right|\\&\le\mathbf{1}_{\{s_p\le\overline{s}\}}D(0;s_p)\exp\{Q(0;s_p)\}+\mathbf{1}_{\{s_p\le\overline{s}\}}D(1;s_p)\exp\{Q(1;s_p)\}\end{aligned}$$

and for $s_p\le\overline{s}$ and all $n<\varepsilon$ we have

$$\begin{aligned}Q(k;s_p)&=-\gamma\left(\mu_A+k\left(\mathbb{E}_p[\theta]-c\right)\right)n\\&\quad+\frac{1}{2}\gamma^2\left(\left(\frac{1}{\tau_A}+k^2\left(\mathbb{V}_p(\theta|z_i)+\frac{1}{\tau_\eta}\right)\right)n^2-2k\alpha\frac{1}{\tau_\eta}n\mathbb{E}_p[Z]+\frac{1}{\tau_\eta}\mathbb{E}_p^2[Z]\right)\\&\quad+\frac{1}{2}\gamma^2\left(\frac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n+\gamma k\alpha\frac{1}{\tau_\eta}n-\gamma\frac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)'\\&\qquad\times\left(\frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1}\\&\qquad\times\left(\frac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n+\gamma k\alpha\frac{1}{\tau_\eta}n-\gamma\frac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)\\&=-\gamma\mu_A+\frac{1}{2}\gamma^2\frac{1}{\tau_A}n^2+\frac{1}{2}\gamma^2\left(\frac{1}{\tau_\theta}-\frac{1}{\tau_\eta}\right)n^2+\frac{1}{2}\gamma^2\frac{1}{\tau_\eta}\mathbb{E}_p^2[Z]-\gamma k\left(s_p-\overline{s}\right)n\\&\quad+\frac{1}{2}\gamma^2\left(\gamma^2\left(\frac{1}{\tau_\eta}\right)^2\mathbb{E}_p^2[Z]-2\gamma\frac{1}{\tau_\eta}\frac{k\frac{1}{\beta}\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n\mathbb{E}_p[Z]+\left(\frac{k\frac{1}{\beta}\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^2n^2\right)\\&\qquad\times\left(\frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1}\\&\le-\gamma\mu_A+\frac{1}{2}\gamma^2\frac{1}{\tau_A}\varepsilon^2+\frac{1}{2}\gamma^2\left|\frac{1}{\tau_\theta}-\frac{1}{\tau_\eta}\right|\varepsilon^2+\frac{1}{2}\gamma^2\frac{1}{\tau_\eta}\mathbb{E}_p^2[Z]-\gamma k\left(s_p-\overline{s}\right)\varepsilon\\&\quad+\frac{1}{2}\gamma^2\left(\gamma^2\left(\frac{1}{\tau_\eta}\right)^2\mathbb{E}_p^2[Z]+2\gamma\frac{1}{\tau_\eta}\frac{k\frac{1}{\beta}\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\varepsilon|\mathbb{E}[Z]|+\left(\frac{k\frac{1}{\beta}\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^2\varepsilon^2\right)\\&\qquad\times\left(\frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1}\end{aligned}$$

$$\equiv \hat{Q}(k; s_p, \varepsilon)$$

and therefore $\mathbb{E}\left[\mathbf{1}_{\{s_p \leq \bar{s}\}} D(k; s_p) \exp\left\{Q(k; s_p)\right\}\right] \leq \mathbb{E}\left[\mathbf{1}_{\{s_p \leq \bar{s}\}} D(k; s_p) \exp\left\{\hat{Q}(k; s_p, \varepsilon)\right\}\right] < \infty$. Hence, we have bounded $\mathbf{1}_{\{s_p \leq \bar{s}\}} D(k; s_p) \exp\left\{Q(k; s_p)\right\}$, for n sufficiently small, by an integrable function that does not depend on n, which allows us to apply the dominated convergence theorem.

Next, consider $\tau_\zeta \downarrow$, equivalently $1/\tau_\zeta \uparrow$. Note that in order for unconditional expected utility to exist, we must have $\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2 \frac{1}{\tau_\eta} > 0 \Leftrightarrow 0 \leq \frac{1}{\tau_\zeta} < \frac{\tau_\eta}{\gamma^2} - \frac{1}{\tau_Z}$. Hence, the relevant limit is $\frac{1}{\tau_\zeta} \uparrow \frac{\tau_\eta}{\gamma^2} - \frac{1}{\tau_Z}$. Using the unconditional welfare expression, welfare under the no-feedback investment level $k_{NF} = 1$ is higher than under the feedback policy $k(s_p) = \mathbf{1}_{\{s_p > \bar{s}\}}$ if and only if

$$\begin{aligned}
-D(1)\exp\left\{Q(1)\right\} > & -\Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\left\{Q(0)\right\} \\
& - \left(1 - \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right)\right) D(1)\exp\left\{Q(1)\right\} \\
\Leftrightarrow \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\left\{Q(0)\right\} > & \ \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right) D(1)\exp\left\{Q(1)\right\}.
\end{aligned}$$

Hence, to establish the claimed result, it suffices to show

$$\lim_{\frac{1}{\tau_\zeta}\uparrow\frac{\tau_\eta}{\gamma^2}-\frac{1}{\tau_Z}} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\left\{Q(0)\right\} > \lim_{\frac{1}{\tau_\zeta}\uparrow\frac{\tau_\eta}{\gamma^2}-\frac{1}{\tau_Z}} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right) D(1)\exp\left\{Q(1)\right\}.$$

We will show this by establishing that $\lim_{\frac{1}{\tau_\zeta}\uparrow\frac{\tau_\eta}{\gamma^2}-\frac{1}{\tau_Z}} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\left\{Q(0)\right\} = \infty$, while $\limsup_{\frac{1}{\tau_\zeta}\uparrow\frac{\tau_\eta}{\gamma^2}-\frac{1}{\tau_Z}} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right) D(1)\exp\left\{Q(1)\right\} < \infty$. Letting $a = \frac{\tau_\eta}{\gamma^2} - \frac{1}{\tau_Z}$ to reduce clutter, we have

$$\begin{aligned}
\lim_{\frac{1}{\tau_\zeta}\uparrow a} D(k) &= \lim_{\frac{1}{\tau_\zeta}\uparrow a} \sqrt{\frac{\frac{1}{\tau_A}+k^2\frac{1}{\tau_\eta}}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\eta}+\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}\right)}} \sqrt{\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}} \left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2 \frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} \frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}+\frac{1}{\tau_\eta}\right)} \frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1/2} \\
&= \sqrt{\frac{\frac{1}{\tau_A}+k^2\frac{1}{\tau_\eta}}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\eta}+\frac{1}{\beta^2(\tau_Z+1/a)}\right)}} \sqrt{\frac{1}{\frac{1}{\tau_Z}+a}} \lim_{\frac{1}{\tau_\zeta}\uparrow a} \left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2 \frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} \frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}+\frac{1}{\tau_\eta}\right)} \frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1/2} \\
&= \sqrt{\frac{\frac{1}{\tau_A}+k^2\frac{1}{\tau_\eta}}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\eta}+\frac{1}{\beta^2(\tau_Z+1/a)}\right)}} \sqrt{\frac{1}{\frac{1}{\tau_Z}+a}} \begin{cases} \infty & k=0 \\ \left(\frac{1}{\frac{1}{\tau_Z}+a} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2 \frac{a}{\frac{1}{\tau_Z}+a} \frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2(\tau_Z+1/a)}+\frac{1}{\tau_\eta}\right)} \frac{a}{\frac{1}{\tau_Z}+a}\right)^{-1/2} & k>0 \end{cases}.
\end{aligned}$$

Similarly,

$$\lim_{\frac{1}{\tau_\zeta}\uparrow a} Q(k) = \lim_{\frac{1}{\tau_\zeta}\uparrow a} \Bigg\{ -\gamma\left(\mu_A + k(\mu_\theta - c)\right) n + \tfrac{1}{2}\gamma^2 \left(\begin{smallmatrix} n \\ \mu_Z \end{smallmatrix}\right)' \begin{pmatrix} \frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta}+\frac{1}{\tau_\eta}\right) & -k\alpha\frac{1}{\tau_\eta} \\ -k\alpha\frac{1}{\tau_\eta} & \frac{1}{\tau_\eta} \end{pmatrix} \left(\begin{smallmatrix} n \\ \mu_Z \end{smallmatrix}\right)$$

$$\tfrac{1}{2}\gamma^2\left(\gamma\tfrac{1}{\tau_\eta}\mu_Z - \gamma k\alpha\tfrac{1}{\tau_\eta}n\right)'\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1}\left(\gamma\tfrac{1}{\tau_\eta}\mu_Z - \gamma k\alpha\tfrac{1}{\tau_\eta}n\right)\Bigg\}$$

$$= \begin{cases} \infty & k = 0 \\ \text{Finite} & k > 0 \end{cases}.$$

Because the function Φ is bounded, together these results imply that

$$\limsup_{\frac{1}{\tau_\zeta}\uparrow a} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right) D(1)\exp\{Q(1)\} < \infty.$$

It remains to show that $\lim_{\frac{1}{\tau_\zeta}\uparrow a} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\{Q(0)\} = \infty$. Considering $\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}$, if $1/\beta = 0$,then $\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}$ is constant in τ_ζ and we are done. Considering $1/\beta \neq 0$, we have

$$\begin{aligned}
\lim_{\frac{1}{\tau_\zeta}\uparrow a} \frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}} &= \lim_{\frac{1}{\tau_\zeta}\uparrow a} \frac{m(0)}{\sqrt{v(0)}} \\
&= \lim_{\frac{1}{\tau_\zeta}\uparrow a} \frac{-\gamma^2\frac{1}{\beta}\frac{1}{\tau_Z}\frac{1}{\tau_\eta}\mu_Z - \gamma^2\left(\frac{1}{\beta}\frac{1}{\tau_Z}\gamma\frac{1}{\tau_\eta}\right)\times\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta}\right)^{-1}\gamma\frac{1}{\tau_\eta}\mu_Z}{\sqrt{\mathbb{V}(s_p)+\gamma^2\frac{1}{\tau_\eta}\frac{1}{\beta^2}\frac{1}{\tau_Z^2}+\gamma^2\frac{1}{\beta}\frac{1}{\tau_Z}\frac{1}{\tau_\eta}\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta}\right)^{-1}\gamma^2\frac{1}{\beta}\frac{1}{\tau_Z}\frac{1}{\tau_\eta}}} \\
&= \begin{cases} -\infty & \frac{1}{\beta} > 0 \\ \infty & \frac{1}{\beta} < 0 \end{cases}
\end{aligned}$$

where the first equality follows from $v(0) \to \infty$ so $\frac{\bar{s}-\mathbb{E}[s_p]}{\sqrt{v(0)}} \to 0$, the second equality substitutes for $m(0)$ and $v(0)$, and the final equality evaluates the limit.

If $1/\beta < 0$, the proof is complete, since $Q(0) \to \infty$, $D(0) \to \infty$ and in this case $\lim_{\frac{1}{\tau_\zeta}\uparrow a} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right) > 0$, so that $\lim_{\frac{1}{\tau_\zeta}\uparrow a} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\{Q(0)\} = \infty$. If $1/\beta > 0$, then $\lim_{\frac{1}{\tau_\zeta}\uparrow a} \Phi\left(\frac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right) = 0$, so the limit is still indeterminate. Write

$\Phi\left(\frac{\overline{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\{Q(0)\}$ as

$$\Phi\left(\frac{\overline{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right) D(0)\exp\{Q(0)\} = \frac{\Phi\left(\frac{\overline{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right)}{\frac{1}{D(0)}\exp\{-Q(0)\}}$$

and note that the relevant limit ultimately depends on the relative rate at which the various terms grow as $x \equiv \left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta}\right)^{-1}$ approaches ∞ so that we can write

$$\lim_{\frac{1}{\tau_\zeta}\uparrow a} \frac{\Phi\left(\frac{\overline{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right)}{\frac{1}{D(0)}\exp\{-Q(0)\}} = \lim_{x\to\infty} \frac{\Phi\left(-\sqrt{x}\right)}{\frac{1}{\sqrt{x}}\exp\{-x\}}.$$

Using L'Hospital's rule yields

$$\begin{aligned}
\lim_{x\to\infty} \frac{\Phi\left(-\sqrt{x}\right)}{\frac{1}{\sqrt{x}}\exp\{-x\}} &= \lim_{x\to\infty} \frac{-\frac{1}{2}x^{-1/2}\phi\left(-\sqrt{x}\right)}{-\frac{1}{\sqrt{x}}\exp\{-x\} - \frac{1}{2}x^{-3/2}\exp\{-x\}} \\
&= \lim_{x\to\infty} \frac{\phi\left(-\sqrt{x}\right)}{2\exp\{-x\} + x^{-1}\exp\{-x\}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}x\right\}}{2\exp\{-x\} + x^{-1}\exp\{-x\}} \\
&= \infty,
\end{aligned}$$

which establishes $\lim_{\frac{1}{\tau_\zeta}\uparrow a} \frac{\Phi\left(\frac{\overline{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right)}{\frac{1}{D(0)}\exp\{-Q(0)\}} = \infty$ and completes the proof. $\square$

Internet Appendix

B Investor welfare

In this section, we characterize traders' expected utilities, which is a key step in establishing the welfare results in the text. Because it poses no additional difficulty, we characterize the expected utility for an arbitrary s_p-dependent investment rule and associated equilibrium asset price. In the material that follows, we will let I_n denote an $n \times n$ identity matrix and will follow the convention that all vectors are column vectors, with row vectors indicated explicitly, using $'$ to denote transposes.

The proof of Proposition 1 established that, in equilibrium, we have $P_1 \equiv P_3$ and traders are indifferent to any trading strategy with aggregate trade

$$X_{i3} + X_{i1} = \frac{\mathbb{E}_{i3}[V(k)] + \gamma \mathbb{C}_{i3}(V(k), \eta_C) z_i - P_3}{\gamma \mathbb{V}_{i3}(V(k))} - n \tag{124}$$

across the two rounds. Consequently, the following material proceeds, without loss of generality, under the assumption that $X_{i1} = 0$ and consequently only the $t = 3$ trading round contributes to the utility. Hence, to reduce notational clutter, we suppress the $t = 3$ dependence of the price function and other equilibrium objects where no confusion will result.

Proposition 6. *Consider an arbitrary s_p-dependent investment rule $k(s_p)$, with associated asset value $V = V(k(s_p))$ and pricing rule $P(s_p)$. Let $U = -\eta_C$ concisely denote the non-tradeable payoff. Consider an arbitrary investor i and define the 5×1 random vector*

$$Y = \begin{pmatrix} V - P \\ \vec{V} \\ \vec{Z}_i \end{pmatrix}, \tag{125}$$

where $\vec{V} = \begin{pmatrix} V(k) \\ U \end{pmatrix}$ and $\vec{Z}_i = \begin{pmatrix} n \\ z_i \end{pmatrix}$ are the sub-vectors of the tradeable and non-tradeable payoffs and the endowments, respectively. Let $\mathcal{F}_p = \sigma(s_p)$ be the information set given s_p. Finally, define the 5×5 matrix $\mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} I_2 \\ 0 & \frac{1}{\gamma} I_2 & 0 \end{pmatrix}$. The conditional expected utility of investor i given $\mathcal{F}_p$ is

$$-\left|\mathbb{V}_i(V-P)\mathbb{V}_p^{-1}(V-P)\right|^{1/2}\left|I_4 + \begin{pmatrix} 0 & \gamma I_2 \\ \gamma I_2 & 0 \end{pmatrix} \mathbb{V}_p((\vec{V}, \vec{Z}_i)|V-P)\right|^{-1/2} \exp\left\{-\frac{1}{2}\mathbb{E}_p[Y]'(\mathbb{V}_p(Y) + \mathcal{I})^{-1}\mathbb{E}_p[Y]\right\} \tag{126}$$

The following result expresses the s_p-conditional expected utility in a form that is more amenable to economic interpretation.

Proposition 7. *Consider an arbitrary s_p-dependent investment rule $k(s_p)$, with associated asset value $V = V(k(s_p))$ and pricing rule $P(s_p)$. The conditional expected utility of investor i given $\mathcal{F}_p = \sigma(s_p)$ can be written as*

$$\mathcal{W}(k; s_p) \equiv -D(k; s_p) \exp\{Q(k; s_p)\} \tag{127}$$

where the quadratic form Q is

$$
\begin{aligned}
Q\left(k;s_p\right) = & -\gamma\left(\mu_A + k\left(\mathbb{E}_p[\theta]-c\right)\right)n \\
& + \tfrac{1}{2}\gamma^2\left(\left(\tfrac{1}{\tau_A} + k^2\left(\mathbb{V}_p(\theta|z_i) + \tfrac{1}{\tau_\eta}\right)\right)n^2 - 2k\alpha\tfrac{1}{\tau_\eta}n\mathbb{E}_p[Z] + \tfrac{1}{\tau_\eta}\mathbb{E}_p^2[Z]\right) \\
& + \tfrac{1}{2}\gamma^2\left(\tfrac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n + \gamma k\alpha\tfrac{1}{\tau_\eta}n - \gamma\tfrac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)' \\
& \times\left(\tfrac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}} - \gamma^2\tfrac{1}{\tau_\eta} + \gamma^2\tfrac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\tfrac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\tfrac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1} \\
& \times\left(\tfrac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}n + \gamma k\alpha\tfrac{1}{\tau_\eta}n - \gamma\tfrac{1}{\tau_\eta}\mathbb{E}_p[Z]\right)
\end{aligned}
$$

and the determinant term D is

$$
\begin{aligned}
D(k;s_p) = & \sqrt{\frac{\frac{1}{\tau_A}+k^2\frac{1}{\tau_\eta}}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\eta}+\mathbb{V}_p\left(\theta|z_i\right)\right)}}\sqrt{\frac{1}{\beta^2\mathbb{V}_p\left(\theta\right)+\frac{1}{\tau_\zeta}}} \\
& \times\left(\tfrac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}} - \gamma^2\tfrac{1}{\tau_\eta} + \gamma^2\tfrac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\tfrac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\tfrac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1/2}.
\end{aligned}
$$

Taking the expectation of the s_p-conditional utility with respect to s_p delivers the unconditional welfare, which we record in the following Proposition.

Proposition 8. *The unconditional expected utility can be written as*

$$
\begin{aligned}
\mathcal{W} = & -\Phi\left(\tfrac{\bar{s}-\mathbb{E}[s_p]+m(0)}{\sqrt{v(0)}}\right)D(0)\exp\left\{Q(0)\right\} \\
& -\left(1-\Phi\left(\tfrac{\bar{s}-\mathbb{E}[s_p]+m(1)}{\sqrt{v(1)}}\right)\right)D(1)\exp\left\{Q(1)\right\}
\end{aligned}
$$

where

$$
D(k) = \sqrt{\frac{\frac{1}{\tau_A}+k^2\frac{1}{\tau_\eta}}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\eta}+\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}\right)}}\sqrt{\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}}\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1/2},
$$

$$
\begin{aligned}
Q(k) = & -\gamma\left(\mu_A + k(\mu_\theta - c)\right)n + \tfrac{1}{2}\gamma^2\binom{n}{\mu_Z}'\begin{pmatrix}\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta}+\frac{1}{\tau_\eta}\right) & -k\alpha\frac{1}{\tau_\eta} \\ -k\alpha\frac{1}{\tau_\eta} & \frac{1}{\tau_\eta}\end{pmatrix}\binom{n}{\mu_Z} \\
& + \tfrac{1}{2}\gamma^2\left(\gamma\tfrac{1}{\tau_\eta}\mu_Z - \gamma k\alpha\tfrac{1}{\tau_\eta}n\right)'
\end{aligned}
$$

$$\times\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1}$$
$$\times\left(\gamma\frac{1}{\tau_\eta}\mu_Z-\gamma k\alpha\frac{1}{\tau_\eta}n\right)$$

and

$$\begin{aligned}
m(k)={}&\gamma k\tfrac{1}{\tau_\theta}n-\gamma\tfrac{1}{\beta}\tfrac{1}{\tau_Z}\left(\gamma\tfrac{1}{\tau_\eta}\mu_Z-\gamma k\alpha\tfrac{1}{\tau_\eta}n\right)\\
&-\gamma^2\left(\tfrac{1}{\beta}\tfrac{1}{\tau_Z}\gamma\tfrac{1}{\tau_\eta}\right)\times\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1}\\
&\times\left(\gamma\tfrac{1}{\tau_\eta}\mu_Z-\gamma k\alpha\tfrac{1}{\tau_\eta}n\right);
\end{aligned}$$

$$v(k)=\mathbb{V}(s_p)+\gamma^2\tfrac{1}{\tau_\eta}\tfrac{1}{\beta^2}\tfrac{1}{\tau_Z^2}+\gamma^2\tfrac{1}{\beta}\tfrac{1}{\tau_Z}\tfrac{1}{\tau_\eta}\left(\frac{1}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}-\gamma^2\frac{1}{\tau_\eta}+\gamma^2\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\beta^2\left(\tau_Z+\tau_\zeta\right)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\frac{1}{\tau_Z}+\frac{1}{\tau_\zeta}}\right)^{-1}\gamma^2\tfrac{1}{\beta}\tfrac{1}{\tau_Z}\tfrac{1}{\tau_\eta}.$$

B.1 Proof of Proposition 6

To compute the expected utility, we will use the law of iterated expectations, first computing the expectation conditional on $\mathcal{F}_{i^+}=\sigma(\{\theta,z_i,s_p\})$, which is the trader information set augmented with s_p, and then conditional on $\mathcal{F}_p$. We emphasize that the initial step is, in principle, not identical to computing the expectation given the trader's information set $\mathcal{F}_i$ itself since s_p is only inferred by the trader in states in which investment is positive and the asset price has non-trivial dependence on s_p. However, as will be seen, the conditional expected utilities given $\mathcal{F}_i$ and $\mathcal{F}_{i^+}$ are identical.

Note that because $V=A$ in any state with zero investment (i.e., in any state in which the trader does not infer s_p from the price) and because the $\mathcal{F}_i$ and $\mathcal{F}_{i^+}$ information sets coincide in any states with positive investment (i.e., in any state in which the trader is able to infer s_p from the price), we necessarily have that $\vec{V}$ is conditionally jointly normally distributed under both $\mathcal{F}_{i^+}$ and $\mathcal{F}_i$ with conditional means

$$\begin{aligned}
\mathbb{E}_{i^+}[V]&=\begin{cases}\mathbb{E}[A|\theta,z_i,s_p] & k=0\\ \mathbb{E}[A+\theta-c+\alpha\eta_C+\sqrt{1-\alpha^2}\eta_I|\theta,z_i,s_p] & k>0\end{cases} &&(128)\\
&=\begin{cases}\mathbb{E}[A] & k=0\\ \mathbb{E}[A+\theta-c|\theta,z_i,s_p] & k>0\end{cases} &&(129)\\
&=\mathbb{E}_i[V] &&(130)
\end{aligned}$$

and

$$\mathbb{E}_{i^+}[U]=\mathbb{E}_{i^+}[U|\theta,z_i,s_p]=0=\mathbb{E}_i[U], \tag{131}$$

and with conditional variances and covariances

$$\mathbb{V}_{i^+}[V] = \begin{cases} \mathbb{V}[A|\theta, z_i, s_p] & k = 0 \\ \mathbb{V}[A + \theta - c + \alpha\eta_C + \sqrt{1-\alpha^2}\eta_I|\theta, z_i, s_p] & k > 0 \end{cases} \tag{132}$$

$$= \begin{cases} \mathbb{V}[A] & k = 0 \\ \mathbb{V}[A + \alpha\eta_C + \sqrt{1-\alpha^2}\eta_I|\theta, z_i, s_p] & k > 0 \end{cases} \tag{133}$$

$$= \mathbb{V}_i[V], \tag{134}$$

$$\mathbb{V}_{i^+}[U] = \mathbb{V}[U|\theta, z_i, s_p] = \tfrac{1}{\tau_\eta} = \mathbb{V}_i[U] \tag{135}$$

and

$$\mathbb{C}_{i^+}(V, U) = \begin{cases} \mathbb{C}(A, U|\theta, z_i, s_p) & k = 0 \\ \mathbb{C}(A + \theta - c + \alpha\eta_C + \sqrt{1-\alpha^2}\eta_I, U|\theta, z_i, s_p) & k > 0 \end{cases} \tag{136}$$

$$= \begin{cases} 0 & k = 0 \\ \mathbb{C}(\alpha\eta_C, U|\theta, z_i, s_p) & k > 0 \end{cases} \tag{137}$$

$$= \mathbb{C}_i(V, U). \tag{138}$$

Because the conditional distributions of payoffs are identical under both information sets, it follows that the expected utility given $\mathcal{F}_{i^+}$ and $\mathcal{F}_i$ are identical and given by the expresssion derived for the expected utility given $\mathcal{F}_i$ in Proposition 1.

$$\mathbb{E}_i[-e^{-\gamma W_{i4}^*}] = -e^{-\gamma n \mathbb{E}_i[V] + \gamma\left(n + z_i \frac{\mathbb{C}_i(V-P,U)}{\mathbb{V}_i(V-P)}\right)\mathbb{E}_i[V-P] - \frac{1}{2}\frac{\mathbb{E}_i^2[V-P]}{\mathbb{V}_i(V-P)} + \frac{1}{2}\gamma^2 \mathbb{V}_i(U|V-P) z_i^2} \tag{139}$$

where $h_i = \frac{\mathbb{C}_i(V-P,U)}{\mathbb{V}_i(V-P)}$ is the conditional regression coefficient of the endowment payoff U on the asset return $V - P$.

To complete the proof, we need to compute the conditional expectation of this quantity given $\mathcal{F}_p$. Let $\vec{h}_i = \left(1, \frac{\mathbb{C}_i(V-P,U)}{\mathbb{V}_i(V-P)}\right)$ be the 2×1 vector of conditional regression coefficients of (V, U) on $V - P$ and define the 5×5 block matrix

$$a_i = \begin{pmatrix} \mathbb{V}_i^{-1}(V-P) & 0 & -\gamma\vec{h}_i' \\ 0 & 0 & \gamma I_2 \\ -\gamma\vec{h}_i & \gamma I_2 & -\gamma^2 \mathbb{V}_i(\vec{V}|V-P) \end{pmatrix}. \tag{140}$$

With this notation, we can concisely write the $\mathcal{F}_{i^+}$ expected utility above as

$$\mathbb{E}_{i^+}\left[-e^{-\gamma\left(\left(\frac{\mathbb{E}_i[V-P] - \gamma\mathbb{C}_i(V,U)z_i}{\gamma\mathbb{V}_i(V)} - n\right)(V-P) + z_i U + nV\right)}\right] = -e^{-\frac{1}{2}\mathbb{E}_i[Y]' a_i \mathbb{E}_i[Y]}. \tag{141}$$

The random vector $\mathbb{E}_i[Y]$ is conditionally jointly normally distributed given $\mathcal{F}_p$ since the investment decision $k(s_p)$ is known given $\mathcal{F}_p$.[31]

We can now use standard formulas for expected exponential-quadratic forms of normal random vectors to compute

$$\mathbb{E}_R\left[-e^{-\frac{1}{2}\mathbb{E}_p[Y]' a_i \mathbb{E}_p[Y]}\right] \tag{142}$$

$$= -\left|a_i\right|^{-1/2}\left|\mathbb{V}_p(\mathbb{E}_i[Y]) + a_i^{-1}\right|^{-1/2}\exp\left\{-\frac{1}{2}\mathbb{E}_p[Y]'\left(\mathbb{V}_p(\mathbb{E}_i[Y]) + a_i^{-1}\right)^{-1}\mathbb{E}_p[Y]\right\} \tag{143}$$

where we use the law of iterated expectations to write $\mathbb{E}_p[\mathbb{E}_i[Y]] = \mathbb{E}_p[Y]$

This expression requires that the matrix a_i is invertible. However, using standard formulas for determinants of partitioned matrices (e.g., eq. (5) in Henderson and Searle (1981)) we can compute

$$|a_i| = \left|\mathbb{V}_i^{-1}(V-P)\right|\left|-\gamma^2 I_2\right| = \gamma^4\left|\mathbb{V}_i^{-1}(V-P)\right| > 0 \tag{144}$$

so that a_i is invertible and using standard formulas for inverses of partitioned matrices (e.g., eq. (8) in Henderson and Searle (1981)) we have

$$a_i^{-1} = \begin{pmatrix} \mathbb{V}_i(V-P) & \mathbb{C}_i(V-P,\vec{V}) & 0 \\ \mathbb{C}_i\left(\vec{V}, V-P\right) & \mathbb{V}_i\left(\vec{V}\right) & \frac{1}{\gamma}I_2 \\ 0 & \frac{1}{\gamma}I_2 & 0 \end{pmatrix}. \tag{145}$$

It now follows from the law of total variance, noting that $\vec{Z}_i$ is $\mathcal{F}_i$-measurable and so $\mathbb{V}_i(\vec{Z}_i) = 0$, that

$$\mathbb{V}_p(\mathbb{E}_i[Y]) + a_i^{-1} = \mathbb{V}_p(\mathbb{E}_i[Y]) + \begin{pmatrix} \mathbb{V}_i(V-P) & \mathbb{C}_i(V-P,\vec{V}) & 0 \\ \mathbb{C}_i\left(\vec{V}, V-P\right) & \mathbb{V}_i\left(\vec{V}\right) & \frac{1}{\gamma}I_2 \\ 0 & \frac{1}{\gamma}I_2 & 0 \end{pmatrix} \tag{146}$$

$$= \mathbb{V}_p(Y) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma}I_2 \\ 0 & \frac{1}{\gamma}I_2 & 0 \end{pmatrix} \tag{147}$$

$$\equiv \mathbb{V}_p(Y) + \mathcal{I} \tag{148}$$

where the final equality defines the 5×5 matrix $\mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma}I_2 \\ 0 & \frac{1}{\gamma}I_2 & 0 \end{pmatrix}$.

Putting together everything above, the conditional expected utility can be written

$$-\left|a_i\right|^{-1/2}\left|\mathbb{V}_p(\mathbb{E}_i[Y]) + a_i^{-1}\right|^{-1/2}\exp\left\{-\frac{1}{2}\mathbb{E}_p[Y]'\left(\mathbb{V}_p(\mathbb{E}_i[Y]) + a_i^{-1}\right)^{-1}\mathbb{E}_p[Y]\right\} \tag{149}$$

[31] Note that $\mathbb{E}_i[Y]$ follows a singular normal distribution since n is a constant. That is, the conditional variance matrix of $\mathbb{E}_i[Y]$ is only positive semidefinite. However, defining the random vector in this way causes no difficulties in the derivation below and simplifies the algebra by treating the endowment of shares and the nontradeable in a unified way.

$$= -\frac{1}{\gamma^2}\left|\mathbb{V}_i^{-1}(V-P)\right|^{-1/2}\left|\mathbb{V}_p(Y)+\mathcal{I}\right|^{-1/2}\exp\left\{-\frac{1}{2}\mathbb{E}_p(Y)'\left(\mathbb{V}_p(Y)+\mathcal{I}\right)^{-1}\mathbb{E}_p(Y)\right\}. \quad (150)$$

Finally applying eq. (5) of Henderson and Searle (1981) with $A = \mathbb{V}_p(V-P)$, $V = U' = \begin{pmatrix} \mathbb{C}_p(\vec{V},V-P) \\ \mathbb{C}_p(\vec{Z}_i,V-P) \end{pmatrix}$ and $D = \begin{pmatrix} \mathbb{V}_p(\vec{V}) & \mathbb{C}_p(\vec{V},\vec{Z}_i)+\frac{1}{\gamma}I_2 \\ \mathbb{C}_p(\vec{Z}_i,\vec{V})+\frac{1}{\gamma}I_2 & \mathbb{V}_p(\vec{Z}_i) \end{pmatrix}$ to compute the determinant of $|\mathbb{V}_p(Y)+\mathcal{I}|$ yields

$$\begin{aligned} |\mathbb{V}_p(Y)+\mathcal{I}| &= |A|\left|D - VA^{-1}U\right| && (151)\\ &= |\mathbb{V}_p(V-P)|\left|\mathbb{V}_p((\vec{V},\vec{Z}_i)|V-P) + \begin{pmatrix} 0 & \frac{1}{\gamma}I_2 \\ \frac{1}{\gamma}I_2 & 0 \end{pmatrix}\right| && (152)\\ &= \frac{1}{\gamma^2}|\mathbb{V}_p(V-P)|\left|I_4 + \begin{pmatrix} 0 & \gamma I_2 \\ \gamma I_2 & 0 \end{pmatrix}\mathbb{V}_p((\vec{V},\vec{Z}_i)|V-P)\right|. && (153) \end{aligned}$$

Plugging this expression into the most recent expression for the expected utility now yields the expression in the statement of the Proposition. □

B.2 Proof of Proposition 7

From Proposition 6, we know that the expected utility is

$$-\left|\mathbb{V}_i(V-P)\mathbb{V}_p^{-1}(V-P)\right|^{1/2}\left|I_4+\begin{pmatrix} 0 & \gamma I_2 \\ \gamma I_2 & 0\end{pmatrix}\mathbb{V}_p((\vec{V},\vec{Z}_i)|V-P)\right|^{-1/2}\exp\left\{-\tfrac{1}{2}\mathbb{E}_p[Y]'(\mathbb{V}_p(Y)+\mathcal{I})^{-1}\mathbb{E}_p[Y]\right\}. \quad (154)$$

Focus first on the quadratic form in the exponential, which we will denote $Q(s_p)$. Use the law of total variance, conditioning on z_i, to decompose the matrix in the quadratic form as

$$\mathbb{V}_p(Y)+\mathcal{I} = \mathbb{V}_p(Y|z_i)+\mathcal{I}+\mathbb{C}_p(Y,z_i)\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,Y),$$

which allows us to use standard matrix inversion results (e.g., eq. (17) in Henderson and Searle (1981)) to express the matrix inverse as

$$\begin{aligned} &(\mathbb{V}_p(Y)+\mathcal{I})^{-1}\\ &= (\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\\ &\quad - (\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{C}_p(Y,z_i)\left(\mathbb{V}_p(z_i)+\mathbb{C}_p(z_i,Y)(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{C}_p(Y,z_i)\right)^{-1}\mathbb{C}_p(z_i,Y)(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\\ &= (\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\\ &\quad - (\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{C}_p(Y,z_i)\mathbb{V}_p^{-1}(z_i)\\ &\qquad \times\left(\mathbb{V}_p^{-1}(z_i)+\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,Y)(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{C}_p(Y,z_i)\mathbb{V}_p^{-1}(z_i)\right)^{-1}\\ &\qquad\quad \times \mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,Y)(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}, \end{aligned}$$

so that the overall quadratic form is

$$\begin{aligned} &-\frac{1}{2}\mathbb{E}_p[Y]'(\mathbb{V}_p(Y)+\mathcal{I})^{-1}\mathbb{E}_p[Y] && (155)\\ &= -\frac{1}{2}\mathbb{E}_p[Y]'\left(\mathbb{V}_p(Y|z_i)+\mathcal{I}\right)^{-1}\mathbb{E}_p[Y] && (156) \end{aligned}$$

$$+\frac{1}{2}\mathbb{E}_p[Y]'(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{C}_p(Y,z_i)\mathbb{V}_p^{-1}(z_i) \tag{157}$$

$$\times\left(\mathbb{V}_p^{-1}(z_i)+\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,Y)(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{C}_p(Y,z_i)\mathbb{V}_p^{-1}(z_i)\right)^{-1} \tag{158}$$

$$\times\ \mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,Y)(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\mathbb{E}_p[Y]. \tag{159}$$

Let us first focus on the term $-\frac{1}{2}\mathbb{E}_p[Y]'\left(\mathbb{V}_p(Y|z_i)+\mathcal{I}\right)^{-1}\mathbb{E}_p[Y]$. We have

$$\begin{aligned}
&(\mathbb{V}_p(Y|z_i)+\mathcal{I})^{-1}\\
&=\begin{pmatrix} \mathbb{V}_p(V-P|z_i) & \mathbb{C}_p(V-P,\vec{V}|z_i) & 0\\ \mathbb{C}_p(\vec{V},V-P|z_i) & \mathbb{V}_p(\vec{V}|z_i) & \frac{1}{\gamma}I_2\\ 0 & \frac{1}{\gamma}I_2 & 0\end{pmatrix}^{-1}\\
&=\begin{pmatrix} \mathbb{V}_p^{-1}(V-P|z_i) & 0 & -\gamma\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V}|z_i)\\ 0 & 0 & \gamma I_2\\ -\gamma\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i) & \gamma I_2 & -\gamma^2\mathbb{V}_p(\vec{V}|V-P,z_i)\end{pmatrix}.
\end{aligned}$$

Furthermore,

$$\mathbb{E}_p[Y]=\begin{pmatrix}\mathbb{E}_p[V-P]\\ \mathbb{E}_p[\vec{V}]\\ \mathbb{E}_p[\vec{Z}_i]\end{pmatrix} \tag{160}$$

$$=\begin{pmatrix}\gamma\mathbb{C}_i(V,\vec{V})\mathbb{E}_p[\vec{Z}]\\ \mathbb{E}_p[\vec{V}]\\ \mathbb{E}_p[\vec{Z}]\end{pmatrix} \tag{161}$$

$$=\begin{pmatrix}\gamma\mathbb{C}_p(V-P,\vec{V})\mathbb{E}_p[\vec{Z}]\\ \mathbb{E}_p[\vec{V}]\\ \mathbb{E}_p[\vec{Z}]\end{pmatrix} \tag{162}$$

where the final equality uses the fact that the law of total covariance implies

$$\mathbb{C}_p(V-P,\vec{V})=\mathbb{C}_p\left(V-\mathbb{E}_i[V]+\gamma\mathbb{C}_i(V,\vec{V})\vec{Z},\vec{V}\right) \tag{163}$$

$$=\mathbb{C}_p\left(V-\mathbb{E}_i[V],\vec{V}\right) \tag{164}$$

$$=\mathbb{C}_i(V,\vec{V}). \tag{165}$$

It follows that the first term in eq. (159) is

$$-\frac{1}{2}\mathbb{E}_p[Y]'\left(\mathbb{V}_p(Y|z_i)+\mathcal{I}\right)^{-1}\mathbb{E}_p[Y] \tag{166}$$

$$=-\frac{1}{2}\begin{pmatrix}\gamma\mathbb{C}_p(V-P,\vec{V})\mathbb{E}_p[\vec{Z}]\\ \mathbb{E}_p[\vec{V}]\\ \mathbb{E}_p[\vec{Z}]\end{pmatrix}' \tag{167}$$

$$\times\begin{pmatrix} \mathbb{V}_p^{-1}(V-P|z_i) & 0 & -\gamma\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V}|z_i)\\ 0 & 0 & \gamma I_2\\ -\gamma\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i) & \gamma I_2 & -\gamma^2\mathbb{V}_p(\vec{V}|V-P,z_i)\end{pmatrix}\begin{pmatrix}\gamma\mathbb{C}_p(V-P,\vec{V})\mathbb{E}_p[\vec{Z}]\\ \mathbb{E}_p[\vec{V}]\\ \mathbb{E}_p[\vec{Z}]\end{pmatrix} \tag{168}$$

$$= -\frac{1}{2}\begin{pmatrix} 0 \\ \gamma\mathbb{E}_p[\vec{Z}] \\ -\gamma^2\mathbb{V}_p(\vec{V}|z_i)\mathbb{E}_p[\vec{Z}]+\gamma\mathbb{E}_p[\vec{V}] \end{pmatrix}' \begin{pmatrix} \gamma\mathbb{C}_p(V-P,\vec{V})\mathbb{E}_p[\vec{Z}] \\ \mathbb{E}_p[\vec{V}] \\ \mathbb{E}_p[\vec{Z}] \end{pmatrix} \tag{169}$$

$$= -\gamma\mathbb{E}_p[\vec{Z}]'\mathbb{E}[\vec{V}] + \tfrac{1}{2}\gamma^2\mathbb{E}_p[\vec{Z}]'\mathbb{V}_p(\vec{V}|z_i)\mathbb{E}_p[\vec{Z}] \tag{170}$$

where the second equality uses the relation $\mathbb{V}_p(\vec{V}|V-P, z_i) = \mathbb{V}_p(\vec{V}|z_i) - \mathbb{C}_p(\vec{V}, V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P, \vec{V}|z_i)$ to simplify the third element of the first vector.

Now consider the second term in eq. (159). We have that

$$\mathbb{C}_p(Y, z_i)\mathbb{V}_p^{-1}(z_i) = \begin{pmatrix} \mathbb{C}_p(V-P, z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{V}, z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{Z}_i, z_i)\mathbb{V}_p^{-1}(z_i) \end{pmatrix} \tag{171}$$

is a vector of conditional regression coefficients.

It follows that

$$\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i, Y)\left(\mathbb{V}_p(Y|z_i) + \mathcal{I}\right)^{-1}\mathbb{E}_p[Y] \tag{172}$$

$$= \begin{pmatrix} \mathbb{C}_p(V-P,z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{V},z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i) \end{pmatrix}' \begin{pmatrix} \mathbb{V}_p^{-1}(V-P|z_i) & 0 & -\gamma\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V}|z_i) \\ 0 & 0 & \gamma I_2 \\ -\gamma\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i) & \gamma I_2 & -\gamma^2\mathbb{V}_p(\vec{V}|V-P,z_i) \end{pmatrix} \begin{pmatrix} \gamma\mathbb{C}_p(V-P,\vec{V})\mathbb{E}_p[\vec{Z}] \\ \mathbb{E}_p[\vec{V}] \\ \mathbb{E}_p[\vec{Z}] \end{pmatrix} \tag{173}$$

$$= \begin{pmatrix} \mathbb{C}_p(V-P,z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{V},z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i) \end{pmatrix}' \begin{pmatrix} 0 \\ \gamma\mathbb{E}_p[\vec{Z}] \\ -\gamma^2\mathbb{V}_p(\vec{V}|z_i)\mathbb{E}_p[\vec{Z}]+\gamma\mathbb{E}_p[\vec{V}] \end{pmatrix} \tag{174}$$

$$= \gamma\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i, \vec{V})\mathbb{E}_p[\vec{Z}] + \mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i, \vec{Z}_i)\left(\gamma\mathbb{E}_p[\vec{V}] - \gamma^2\mathbb{V}_p(\vec{V}|z_i)\mathbb{E}_p[\vec{Z}]\right) \tag{175}$$

$$= \gamma\begin{pmatrix} \mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,k\theta) \\ 0 \end{pmatrix}'\mathbb{E}_p\left[\vec{Z}_i\right] + \begin{pmatrix} 0 \\ 1 \end{pmatrix}'\left(\gamma\mathbb{E}_p\left[\vec{V}\right] - \gamma^2\mathbb{V}_p(\vec{V}|z_i)\mathbb{E}_p[\vec{Z}]\right) \tag{176}$$

$$= \gamma\frac{k\mathbb{C}_p(Z,\theta)}{\mathbb{V}_p(z_i)}n + \gamma\mathbb{E}_p[U] - \gamma^2\mathbb{C}_p(U,V)n - \gamma^2\mathbb{V}_p(U)\mathbb{E}_p[z_i] \tag{177}$$

$$= \gamma\frac{-k\beta\mathbb{V}_p(\theta)}{\beta^2\mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}}n + \gamma^2 k\alpha\tfrac{1}{\tau_\eta}n - \gamma^2\tfrac{1}{\tau_\eta}\mathbb{E}_p[Z] \tag{178}$$

where the next-to-last line performs the matrix multiplication explicitly, and the final line simplifies using $\mathbb{C}_p(Z,\theta) = \beta\mathbb{C}_p(\theta + \frac{1}{\beta}Z - \theta, \theta) = \beta\mathbb{C}_p(s_p - \theta, \theta) = -\beta\mathbb{V}_p(\theta)$, and similarly for $\mathbb{V}_p(z_i)$, and uses $\mathbb{E}_p[U] = \mathbb{E}_p[-\eta_C] = 0$.

To complete the simplification of the second term in eq. (159), we need to compute

$$\left(\mathbb{V}_p^{-1}(z_i) + \mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i, Y)\left(\mathbb{V}_p(Y|z_i) + \mathcal{I}\right)^{-1}\mathbb{C}_p(Y, z_i)\mathbb{V}_p^{-1}(z_i)\right)^{-1}. \tag{179}$$

We have

$$\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i, Y)\left(\mathbb{V}_p(Y|z_i) + \mathcal{I}\right)^{-1}\mathbb{C}_p(Y, z_i)\mathbb{V}_p^{-1}(z_i) \tag{180}$$

$$
= \begin{pmatrix} \mathbb{C}_p(V-P,z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{V},z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i) \end{pmatrix}' \begin{pmatrix} \mathbb{V}_p^{-1}(V-P|z_i) & 0 & -\gamma\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V}|z_i) \\ 0 & 0 & \gamma I_2 \\ -\gamma\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i) & \gamma I_2 & -\gamma^2\mathbb{V}_p(\vec{V}|V-P,z_i) \end{pmatrix} \begin{pmatrix} \mathbb{C}_p(V-P,z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{V},z_i)\mathbb{V}_p^{-1}(z_i) \\ \mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i) \end{pmatrix} \tag{181}
$$

$$
= \left(\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,V-P)\mathbb{V}_p^{-1}(V-P|z_i)-\gamma\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z}_i)\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\right)\mathbb{C}_p(V-P,z_i)\mathbb{V}_p^{-1}(z_i) \tag{182}
$$

$$
+ 2\gamma\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z}_i)\mathbb{C}_p(\vec{V},z_i)\mathbb{V}_p^{-1}(z_i) \tag{183}
$$

$$
+ \left(-\gamma\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,V-P)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V}|z_i)-\gamma^2\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z}_i)\mathbb{V}_p(\vec{V}|V-P,z_i)\right)\mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i). \tag{184}
$$

Note that

$$
\mathbb{C}_p(V-P,z_i) = \mathbb{C}_p(V-\mathbb{E}_i[V]+\gamma\mathbb{C}_i(V,\vec{V})\vec{Z},z_i) \tag{185}
$$

$$
= \gamma\mathbb{C}_i(V,\vec{V})\mathbb{C}_p(\vec{Z},z_i) \tag{186}
$$

$$
= \gamma\mathbb{C}_p(V-P,\vec{V})\mathbb{C}_p(\vec{Z},z_i) \tag{187}
$$

and furthermore, $\mathbb{C}_p(\vec{Z}_i,z_i) = (0,\mathbb{V}_p(z_i))$ and $\mathbb{C}_p(\vec{V},z_i) = (\mathbb{C}_p(V,z_i),0)$, so that the term $\mathbb{C}_p(z_i,\vec{Z}_i)\mathbb{C}_p(\vec{V},z_i) = 0$.

It follows that the previous displayed equation can be written, after grouping terms, as

$$
\gamma^2\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z})\mathbb{C}_p(\vec{V},V-P)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V})\mathbb{C}_p(\vec{Z},z_i)\mathbb{V}_p^{-1}(z_i) \tag{188}
$$

$$
-\gamma^2\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z}_i)\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V})\mathbb{C}_p(\vec{Z},z_i)\mathbb{V}_p^{-1}(z_i) \tag{189}
$$

$$
-\gamma^2\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z})\mathbb{C}_p(\vec{V},V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,\vec{V})\mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i) \tag{190}
$$

$$
-\gamma^2\mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i,\vec{Z}_i)\mathbb{V}_p(\vec{V}|V-P,z_i)\mathbb{C}_p(\vec{Z}_i,z_i)\mathbb{V}_p^{-1}(z_i). \tag{191}
$$

Furthermore, $\mathbb{C}(\vec{Z}_i,z_i) = \left(\begin{smallmatrix} 0 \\ \mathbb{V}_p(z_i) \end{smallmatrix}\right)$ and $\mathbb{C}(\vec{Z},z_i) = \left(\begin{smallmatrix} 0 \\ \mathbb{V}_p(Z) \end{smallmatrix}\right)$ which further simplifies the previous expression to

$$
\gamma^2\tfrac{\mathbb{V}_p(Z)}{\mathbb{V}_p(z_i)}\mathbb{C}_p(U,V-P)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,U)\tfrac{\mathbb{V}_p(Z)}{\mathbb{V}_p(z_i)} \tag{192}
$$

$$
-\gamma^2\mathbb{C}_p(U,V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,U)\tfrac{\mathbb{V}_p(Z)}{\mathbb{V}_p(z_i)} \tag{193}
$$

$$
-\gamma^2\tfrac{\mathbb{V}_p(Z)}{\mathbb{V}_p(z_i)}\mathbb{C}_p(U,V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,U) \tag{194}
$$

$$
-\gamma^2\mathbb{V}_p(U|V-P,z_i). \tag{195}
$$

We can write

$$
\mathbb{V}_p(U|V-P,z_i) = \mathbb{V}_p(U|z_i) - \mathbb{C}_p(U,V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,U|z_i), \tag{196}
$$

which after plugging in and simplifying yields

$$
-\gamma^2\mathbb{V}_p(U|z_i) + \gamma^2\left(1-\tfrac{\mathbb{V}_p(Z)}{\mathbb{V}_p(z_i)}\right)\mathbb{C}_p(U,V-P|z_i)\mathbb{V}_p^{-1}(V-P|z_i)\mathbb{C}_p(V-P,U|z_i)\left(1-\tfrac{\mathbb{V}_p(Z)}{\mathbb{V}_p(z_i)}\right) \tag{197}
$$

$$= -\gamma^2 \mathbb{V}(U) + \gamma^2 \tfrac{\mathbb{V}_p(\zeta_i)}{\mathbb{V}_p(z_i)} \mathbb{C}_p(V-P, U|z_i) \mathbb{V}_p^{-1}(V-P|z_i) \mathbb{C}_p(V-P, U|z_i) \tfrac{\mathbb{V}_p(\zeta_i)}{\mathbb{V}_p(z_i)} \tag{198}$$

$$= -\gamma^2 \tfrac{1}{\tau_\eta} + \gamma^2 \frac{\frac{1}{\tau_\zeta}}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}} \frac{k^2 \alpha^2 \left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A} + k^2 \left(\frac{1}{\tau_\theta + \beta^2(\tau_Z + \tau_\zeta)} + \frac{1}{\tau_\eta}\right)} \frac{\frac{1}{\tau_\zeta}}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}}. \tag{199}$$

Putting everything together above, we have established that the quadratic form in the exponential is

$$Q(k; s_p) = -\gamma \mathbb{E}_p[\vec{Z}]' \mathbb{E}[\vec{V}] + \tfrac{1}{2} \gamma^2 \mathbb{E}_p[\vec{Z}]' \mathbb{V}_p(\vec{V}|z_i) \mathbb{E}_p[\vec{Z}] \tag{200}$$

$$- \left(\gamma \frac{-k\beta \mathbb{V}_p(\theta)}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}} n + \gamma^2 k \alpha \tfrac{1}{\tau_\eta} n - \gamma^2 \tfrac{1}{\tau_\eta} \mathbb{E}_p[Z] \right) \tag{201}$$

$$\left(\frac{1}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}} - \gamma^2 \tfrac{1}{\tau_\eta} + \gamma^2 \frac{\frac{1}{\tau_\zeta}}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}} \frac{k^2 \alpha^2 \left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A} + k^2 \left(\frac{1}{\tau_\theta + \beta^2(\tau_Z + \tau_\zeta)} + \frac{1}{\tau_\eta}\right)} \frac{\frac{1}{\tau_\zeta}}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}} \right)^{-1} \tag{202}$$

$$\left(\gamma \frac{-k\beta \mathbb{V}_p(\theta)}{\beta^2 \mathbb{V}_p(\theta) + \frac{1}{\tau_\zeta}} n + \gamma^2 k \alpha \tfrac{1}{\tau_\eta} n - \gamma^2 \tfrac{1}{\tau_\eta} \mathbb{E}_p[Z] \right) \tag{203}$$

where the inverse term is guaranteed to be finite for $k \in \{0, 1\}$ under the maintained parameter restriction $\left(\frac{1}{\tau_Z} + \frac{1}{\tau_\zeta}\right)^{-1} - \gamma^2 \frac{1}{\tau_\eta} > 0$ After performing the matrix multiplication in the the first line and factoring γ^2 out of the final term, this yields the expression for Q in the Corollary.

Now, consider the determinant term in the expected utility. From eq. (150) in the proof of Proposition 6, we can write the determinant term as

$$-\frac{1}{\gamma^2} \left| \mathbb{V}_i^{-1}(V-P) \right|^{-1/2} \left| \mathbb{V}_p(Y) + \mathcal{I} \right|^{-1/2}. \tag{204}$$

Note that

$$|\mathbb{V}_p(Y) + \mathcal{I}| \tag{205}$$

$$= \left| \mathbb{V}_p(Y|z_i) + \mathcal{I} + \mathbb{C}_p(Y, z_i) \mathbb{V}_p^{-1}(z_i) \mathbb{C}_p(z_i, Y) \right| \tag{206}$$

$$= |\mathbb{V}_p(Y|z_i) + \mathcal{I}| \left| I + (\mathbb{V}_p(Y|z_i) + \mathcal{I})^{-1} \mathbb{C}_p(Y, z_i) \mathbb{V}_p^{-1}(z_i) \mathbb{C}_p(z_i, Y) \right| \tag{207}$$

$$= |\mathbb{V}_p(Y|z_i) + \mathcal{I}| \left| I + (\mathbb{V}_p(Y|z_i) + \mathcal{I})^{-1} \mathbb{C}_p(Y, z_i) \mathbb{V}_p^{-1}(z_i) \mathbb{V}_p(z_i) \mathbb{V}_p^{-1}(z_i) \mathbb{C}_p(z_i, Y) \right| \tag{208}$$

$$= |\mathbb{V}_p(Y|z_i) + \mathcal{I}| \left| I + \mathbb{V}_p^{-1}(z_i) \mathbb{C}_p(z_i, Y) (\mathbb{V}_p(Y|z_i) + \mathcal{I})^{-1} \mathbb{C}_p(Y, z_i) \mathbb{V}_p^{-1}(z_i) \mathbb{V}_p(z_i) \right| \tag{209}$$

$$= |\mathbb{V}_p(Y|z_i) + \mathcal{I}| \, |\mathbb{V}_p(z_i)| \left| \mathbb{V}_p^{-1}(z_i) + \mathbb{V}_p^{-1}(z_i) \mathbb{C}_p(z_i, Y) (\mathbb{V}_p(Y|z_i) + \mathcal{I})^{-1} \mathbb{C}_p(Y, z_i) \mathbb{V}_p^{-1}(z_i) \right| \tag{210}$$

$$= \tfrac{1}{\gamma^4} |\mathbb{V}_p(V-P|z_i)| \, |\mathbb{V}_p(z_i)| \left| \mathbb{V}_p^{-1}(z_i) + \mathbb{V}_p^{-1}(z_i) \mathbb{C}_p(z_i, Y) (\mathbb{V}_p(Y|z_i) + \mathcal{I})^{-1} \mathbb{C}_p(Y, z_i) \mathbb{V}_p^{-1}(z_i) \right| \tag{211}$$

where the first equality follows from the law of total variance, the second equality uses the multiplicative property of the determinant, the third equality multiplies and divides by

$\mathbb{V}_p(z_i)$ in the second determinant, the fourth equality uses the Matrix Determinant Lemma (e.g., eq. (6) in Henderson and Searle (1981)), and the fourth line computes $|\mathbb{V}_p(Y|z_i) + \mathcal{I}|$.

Since we established

$$\mathbb{V}_p^{-1}(z_i) + \mathbb{V}_p^{-1}(z_i)\mathbb{C}_p(z_i, Y)\left(\mathbb{V}_p(Y|z_i) + \mathcal{I}\right)^{-1}\mathbb{C}_p(Y, z_i)\mathbb{V}_p^{-1}(z_i) \tag{212}$$

$$= \frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}} \tag{213}$$

as part of deriving the expression for $Q(k; s_p)$ above, we can now plug back into eq. (204) to yield the overall determinant term

$$-D(k; s_p) = -\sqrt{\frac{\mathbb{V}_i(V-P)}{\mathbb{V}_p(V-P|z_i)}}\sqrt{\frac{1}{\mathbb{V}_p(z_i)}} \tag{214}$$

$$\times\left(\frac{1}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}} - \gamma^2\frac{1}{\tau_\eta} + \gamma^2\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\frac{k^2\alpha^2\left(\frac{1}{\tau_\eta}\right)^2}{\frac{1}{\tau_A}+k^2\left(\frac{1}{\tau_\theta+\beta^2(\tau_Z+\tau_\zeta)}+\frac{1}{\tau_\eta}\right)}\frac{\frac{1}{\tau_\zeta}}{\beta^2\mathbb{V}_p(\theta)+\frac{1}{\tau_\zeta}}\right)^{-1/2}, \tag{215}$$

which matches the expression in the statement of the Corollary after substituting the explicit expressions for the conditional variances in the first two square root terms. □

B.3 Proof of Proposition 8

Using the law of iterated expectations, the unconditional expected utility can be represented as the unconditional expectation as the s_p-conditional expected utility from Proposition 6, evaluated at the equilibrium investment rule $k(s_p) = \mathbf{1}_{\{s_p > \bar{s}\}}$:

$$\mathcal{W} = \mathbb{E}\left[\mathcal{W}(s_p)\right] \tag{216}$$

$$= \mathbb{P}(s_p > \bar{s})\mathbb{E}[\mathcal{W}(s_p)|s_p > \bar{s}] + \mathbb{P}(s_p \le \bar{s})\mathbb{E}[\mathcal{W}(s_p)|s_p \le \bar{s}]. \tag{217}$$

From Proposition 6, we have that the conditional expected utilities are of the form

$$\mathcal{W}(s_p) = -\left|\mathbb{V}_i(V-P)\mathbb{V}_p^{-1}(V-P)\right|^{1/2}\left|I_4 + \left(\begin{smallmatrix} 0 & \gamma I_2 \\ \gamma I_2 & 0 \end{smallmatrix}\right)\mathbb{V}_p((\vec{V}, \vec{Z}_i)|V-P)\right|^{-1/2} \tag{218}$$

$$\times \exp\left\{-\tfrac{1}{2}\mathbb{E}_p[Y(k)]'(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{E}_p[Y(k)]\right\} \tag{219}$$

where the asset payoff $V = V(k)$ and price function $P = P(k)$ are those associated with the a particular investment decision $k \in \{0, 1\}$, and where the vector

$$Y(k) = \begin{pmatrix} V(k) - P \\ \vec{V} \\ \vec{Z}_i \end{pmatrix}, \tag{220}$$

with $\vec{V} = \begin{pmatrix} V(k) \\ -\eta_C \end{pmatrix}$ and $\vec{Z}_i = \begin{pmatrix} n \\ z_i \end{pmatrix}$ the sub-vectors of the tradeable and non-tradeable payoffs and the endowments, respectively.

To evaluate the expected utility, eq. (217), it is straightforward to calculate the probabilities of the two regions. It remains to to calculate the the conditional expectation of $\mathcal{W}(s_p)$ given $s_p > \overline{s}$ and $s_p \leq \overline{s}$. Given that the determinant terms in eq. (219) are constant within each region, this reduces to computing the expectation of the exponential term.

To proceed, note that using standard normal-normal updating we can represent

$$\mathbb{E}_p[Y(k)] = \mathbb{E}[Y(k)] + \mathbb{C}(Y(k), s_p)\mathbb{V}^{-1}(s_p)(s_p - \mathbb{E}[s_p]). \tag{221}$$

Hence, the expression in the exponential in eq. (219) can be written as

$$\begin{aligned}
&-\frac{1}{2}\mathbb{E}_p[Y(k)]'(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{E}_p[Y(k)] && (222)\\
&= -\frac{1}{2}\mathbb{E}[Y(k)]'(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{E}[Y(k)] && (223)\\
&\quad - \mathbb{E}[Y(k)]'(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{C}(Y(k), s_p)\mathbb{V}^{-1}(s_p)(s_p - \mathbb{E}[s_p]) && (224)\\
&\quad\quad - \tfrac{1}{2}(s_p - \mathbb{E}[s_p])\mathbb{V}^{-1}(s_p)\mathbb{C}(s_p, Y(k))(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{C}(Y(k), s_p)\mathbb{V}^{-1}(s_p)(s_p - \mathbb{E}[s_p]), && (225)
\end{aligned}$$

which is in the quadratic form $d + a'X + X'AX$ with $X = s_p - \mathbb{E}[s_p]$ and

$$\begin{aligned}
d &= -\frac{1}{2}\mathbb{E}[Y(k)]'(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{E}[Y(k)] && (226)\\
a &= -\mathbb{V}^{-1}(s_p)\mathbb{C}(s_p, Y(k))(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{E}[Y(k)] && (227)\\
A &= -\tfrac{1}{2}\mathbb{V}^{-1}(s_p)\mathbb{C}(s_p, Y(k))(\mathbb{V}_p(Y(k)) + \mathcal{I})^{-1}\mathbb{C}(Y(k), s_p)\mathbb{V}^{-1}(s_p). && (228)
\end{aligned}$$

We can now compute the two conditional expectations, corresponding to the investment and no-investment regions, using Lemma 2 from Appendix B.4, which provides a closed-form expression for the expected exponential-quadratic of a truncated normally distributed random variable.

A large amount of tedious algebra, analogous to that in the proof of the conditional welfare expression in Proposition 7, then delivers the expression in the Proposition. From inspection of the determinant term in the expression, the maintained parameter restriction $\left(\frac{1}{\tau_Z} + \frac{1}{\tau_\zeta}\right)^{-1} - \gamma^2\frac{1}{\tau_\eta} > 0$ further further ensures that the expected utility is finite. □

B.4 Exponential-quadratic form of truncated normal random vector

The following result computes the unconditional expectation of an exponential-quadratic form of a truncated normal random variable, which is used to characterize the unconditional welfare under the equilibrium investment rule.

Lemma 2. *Suppose $X \in \mathbb{R}^n$ is distributed $N(\mu, \Sigma)$ with positive definite variance matrix Σ. Consider the quadratic form $d + a'X + X'AX$, for conformable d, a and A. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be* *an arbitrary set.*

Suppose that $I - 2\Sigma A$ *is positive definite and define the composite parameters*

$$\hat{\mu} = (I - 2\Sigma A)^{-1}(\mu + \Sigma a) \tag{229}$$

$$\hat{\Sigma} = (\Sigma^{-1} - 2A)^{-1} = (I - 2\Sigma A)^{-1}\Sigma \tag{230}$$

$$\widehat{\mathcal{C}} = \{B\,(x - \hat{\mu}) : x \in \mathcal{C}\}\,, \tag{231}$$

where B *is the invertible, symmetric* $n \times n$ *matrix square root that factorizes the positive definite* $\hat{\Sigma}^{-1}$ *as* $BB = \hat{\Sigma}^{-1}$.

We have

$$\mathbb{E}\left[\exp\{d + a'X + X'AX\}\,\middle|\,X \in \mathcal{C}\right] \tag{232}$$

$$= \frac{\int_{\widehat{\mathcal{C}}} \phi(y)dy}{\int_{\mathcal{C}} \phi(y)dy} \frac{1}{|I - 2\Sigma A|^{1/2}} \exp\left\{d - \tfrac{1}{2}\mu'\Sigma^{-1}\mu + \tfrac{1}{2}(\mu + \Sigma a)'\Sigma^{-1}(\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right\} \tag{233}$$

B.5 Proof of Lemma 2

Writing the expectation explicitly as an integral, we have

$$\mathbb{E}\left[\exp\{d + a'X + X'AX\}\,\middle|\,X \in \mathcal{C}\right] \tag{234}$$

$$= \int_{\mathbb{R}^n} \exp\{d + a'x + x'Ax\}\,\mathbb{P}(X \in dx | X \in \mathcal{C}) \tag{235}$$

$$= \int_{\mathcal{C}} \exp\{d + a'x + x'Ax\}\, \frac{\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu)\right\}}{\mathbb{P}(X \in \mathcal{C})} dx. \tag{236}$$

By completing the square, we can group the terms in the exponentials as

$$d + a'x + x'Ax - \tfrac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu) \tag{237}$$

$$= d - \tfrac{1}{2}\mu'\Sigma^{-1}\mu + \tfrac{1}{2}\left[(\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right]'(\Sigma^{-1} - 2A)\left[(\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right] \tag{238}$$

$$- \tfrac{1}{2}\left(x - (\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right)'(\Sigma^{-1} - 2A)\left(x - (\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right) \tag{239}$$

$$= d - \tfrac{1}{2}\mu'\Sigma^{-1}\mu + \tfrac{1}{2}(\mu + \Sigma a)'\Sigma^{-1}(\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a) \tag{240}$$

$$- \tfrac{1}{2}\left(x - (I - 2\Sigma A)^{-1}(\mu + \Sigma a)\right)'(\Sigma^{-1} - 2A)\left(x - (I - 2\Sigma A)^{-1}(\mu + \Sigma a)\right). \tag{241}$$

So plugging back in to eq. (236) yields

$$= \int_{\mathcal{C}} \exp\{d + a'x + x'Ax\}\, \frac{\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu)\right\}}{\mathbb{P}(X \in \mathcal{C})} dx \tag{242}$$

$$= \exp\left\{d - \tfrac{1}{2}\mu'\Sigma^{-1}\mu + \tfrac{1}{2}(\mu + \Sigma a)'\Sigma^{-1}(\Sigma^{-1} - 2A)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right\} \tag{243}$$

$$\times \tfrac{1}{\mathbb{P}(X\in\mathcal{C})} \int_{\mathcal{C}} \tfrac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\tfrac{1}{2}\big(x-(I-2\Sigma A)^{-1}(\mu+\Sigma a)\big)'(\Sigma^{-1}-2A)\big(x-(I-2\Sigma A)^{-1}(\mu+\Sigma a)\big)\right\}.dx \quad (244)$$

Letting

$$\hat{\mu} = (I - 2\Sigma A)^{-1}(\mu + \Sigma a) \quad (245)$$

$$\hat{\Sigma} = (\Sigma^{-1} - 2A)^{-1} = (I - 2\Sigma A)^{-1}\Sigma \quad (246)$$

$$\hat{\mathcal{C}} = \left\{B\left(x - (I - 2\Sigma A)^{-1}(\mu + \Sigma a)\right) : x \in \mathcal{C}\right\} \quad (247)$$

where B is the invertible, symmetric $n \times n$ matrix square root that factorizes the positive definite matrix $\hat{\Sigma}^{-1}$ as $\hat{\Sigma}^{-1} = BB$, we can further express the integral in eq. (244) as

$$\int_{\mathcal{C}} \tfrac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\tfrac{1}{2}(x-\hat{\mu})'\hat{\Sigma}^{-1}(x-\hat{\mu})\right\} dx \quad (248)$$

$$= \int_{\hat{\mathcal{C}}} \tfrac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \tfrac{1}{|B|} \exp\left\{-\tfrac{1}{2}y'y\right\} dy \quad (249)$$

$$= \tfrac{1}{|I-2\Sigma A|^{1/2}} \int_{\hat{\mathcal{C}}} \phi(y)dy \quad (250)$$

where the first equality changes variables $y = B^{-1}\left(x - (I - 2\Sigma A)^{-1}(\mu + \Sigma a)\right)$ and the final line uses $|B| = |BB|^{1/2} = \left|\hat{\Sigma}^{-1}\right|^{1/2} = |\Sigma^{-1}(I - 2\Sigma A)|^{1/2}$ and simplifies notation using the n-dimensional standard normal cdf $\phi(x) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}y'y\right\}$. Plugging this expression for the integral back into eq. (244) delivers the expression in the Lemma. □